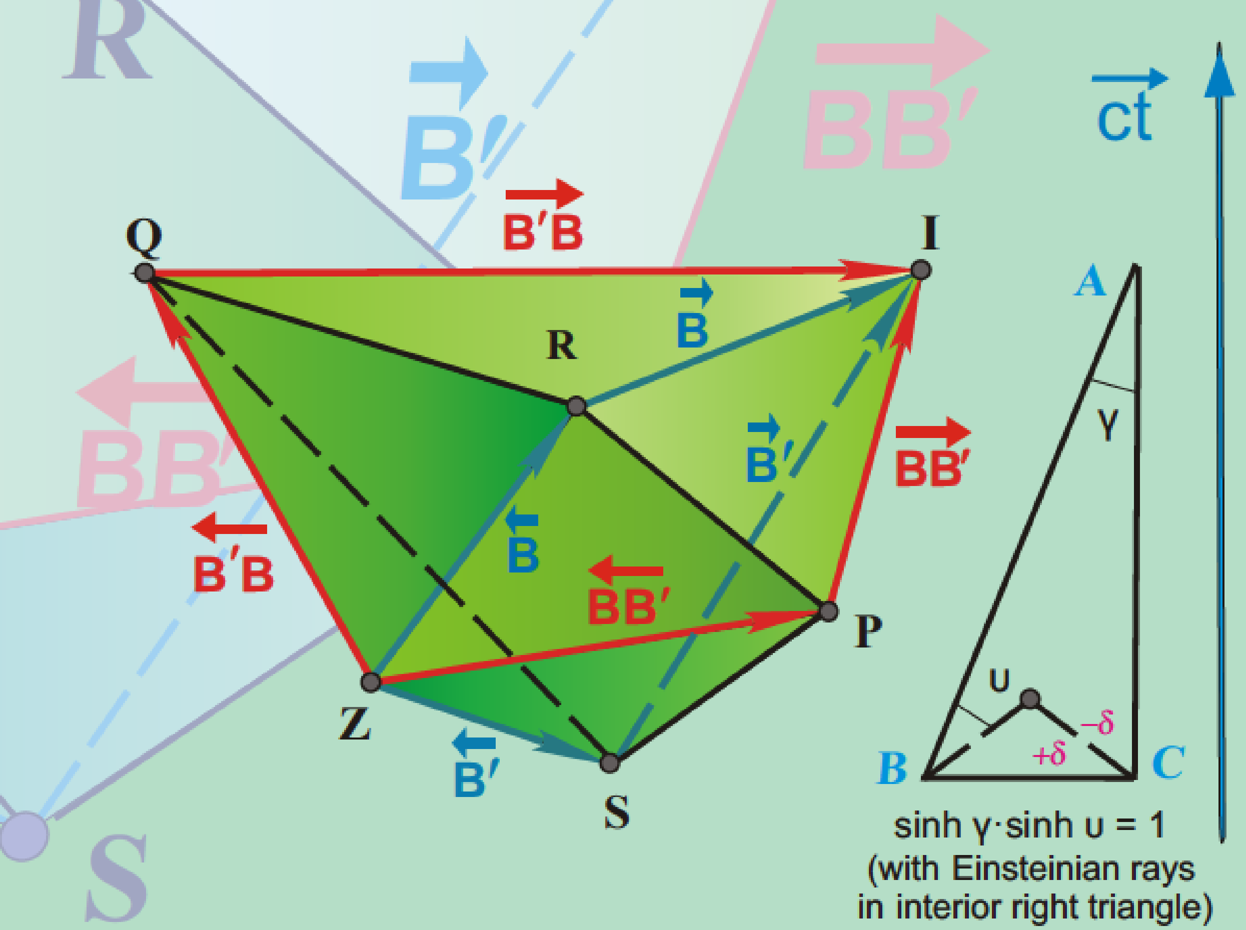
A.S. NINUL

TENSOR TRIGONOMETRY



Publisher FIZMATKNIGA

Ninul A. S.

TENSOR TRIGONOMETRY

The 3-rd edition
last from author-himself
(updated and added)

Publisher "Fizmatkniga" – 2025

ББК 22.143
Н 60
UDC 514.1/512.64/530.12
MSC2020: 15A/51F/53B/83 (of AMS and zbMATH)

Ninul A. S. Tensor Trigonometry. – Moscow: Publisher "Fizmatkniga", 2025, 320 p., 8 ill. Third ed., updated and added in English from original ed. Ninul A. S. Tensor Trigonometry. *Theory and Applications.* – Moscow: "Mir", 2004, 336 p. (ISBN 10: 5-03-003717-9)

The Tensor Trigonometry, with revealing a tensor nature of the angles and their functions and added by differential trigonometry, is developed for wide applications in various fields.

Planimetry includes metric part and trigonometry. In geometries of metric spaces from the end of XIX age their tensor forms are widely used. Trigonometry was remaining in its *scalar* flat forms. Tensor Trigonometry is its development from Leonard Euler classic forms into spatial k-dimensional (at $k \geq 2$) *tensor* forms with vector and scalar orthoprojections, with step by step increasing a complexity and opportunities. Described in the book are fundamentals of this new mathematical subject with many initial examples of applications.

In theoretic plan, Tensor Trigonometry complements naturally Analytic Geometry and Linear Algebra. In practical plan, it gives the clear tools for analysis and solutions of various geometric and physical problems in homogeneous isotropic spaces, as Euclidean, quasi- and pseudo-Euclidean ones, on perfect surfaces of constant radius embedded into them with n-D non-Euclidean Geometries, and in Theory of Relativity. So, it gives classic projective models of non-Euclidean Geometries as trigonometric ones, general laws of summing two-steps and polysteps motions in complete differential and integral forms with polar decomposition of the sum into principal and induced orthospherical motions. The applications were developed till the differential tensor trigonometry of world lines and curves in 3D and 4D pseudo- and quasi-Euclidean spaces, in addition, to the classic Frenet–Serret theory, with absolute and relative differential-geometric parameters of curves, main kinematic and dynamical characteristics of a body moving in space-time along a world line with **4**-velocity of Poincaré. Due to our tensor trigonometric approach, clear explanations of all well-known and new STR and GR relativistic effects are given with physical interpretations in full agreement with the Law of Energy-Momentum conservation, Quantum Mechanics, Noether Theorem and Higgs Theory.

The Tensor Trigonometry can be useful in various domains of mathematics and physics. It is intended to researchers in the fields of analytic geometry of any dimension, linear algebra with matrix theory, non-Euclidean geometries, theory of relativity, quantum mechanics and to all those who is interested in new knowledges and applications, given by exact sciences. It may be useful for educational purposes with this new math subject in the university and graduate schools departments of algebra, geometry and physics – relativistic and classical.

The 1st edition of the Tensor Trigonometry was published by the main Russian scientific publishing house "MIR" in October 2004 – ISBN 10: 5030037179 (for instance, A. S. Ninul "Tenzornaja trigonometrija" in SUB.Uni-Goettingen.de and WorldCat OCLC 255128609). It was reviewed by the Moscow State University professor, Dr.Sc. M. M. Postnikov and by the Moscow Regional University professor, Dr.Sc. O. V. Manturov.
This 3rd edition is a renovation of previous two in 2004 (by MIR) and in 2021 (by Fizmatlit).

ISBN-13: 978-5-89155-429-0 (for this paper book)
DOI 10.29039/978-5-89155-429-0-320-01-2025

The first edition of the book in September 2004 in Russian was devoted by the author to the 175ys anniversary of the first publications on non-Euclidean Geometry, to the 100ys anniversary of the first publications on Theory of Relativity and to their great creators – Lobachevsky, Bolyai, Lorentz, Poincaré, Einstein

To the readers

The author brings to your attention the 3-rd edition of Tensor Trigonometry, significantly renovated and expanded by him, essentially in its applications.

Originated in antiquity the Trigonometry completed own development and obtained its modern form at the end of the 18th century in the works of great Leonard Euler. Meanwhile Geometry, from the historically initial Euclidean forms, passed far ahead for the last two centuries. Furthermore, its various multi-dimensional and non-Euclidean tensor forms were discovered and studied.

In the monograph, we undertook in 2004 constructing general and various useful particular forms of the new mathematical subject Tensor Trigonometry in k-dimensional homogeneous and isotropic spaces with their quadratic metric, as Euclidean, quasi-Euclidean and pseudo-Euclidean. The binary angle between two lines or vectors, between two subspaces or lineors in linear spaces (at $k \geq 2$) has a nature of bivalent tensors with properties, determined by reflector tensor of such a space. A kind of the space is determined by its quadratic metric. In such metric spaces, the tensor angle and its trigonometric functions are respectively either orthogonal, or quasi-orthogonal, or pseudo-orthogonal bivalent tensors.

In order to obtain all the arising tensor constructions, it was necessary to preliminary highly thoroughly consider and supplement a number of concepts in the Theory of Exact Matrices, which is a part of Linear Algebra. Our efforts were rewarded by attainments of interesting and unexpected results in Algebra, Geometry and in Theoretical Physics with the Theory of Relativity.

Tensor Trigonometry point of view gives such advantages, that some rather difficult and not easily perceivable mathematical or physical theories became quite transparent and natural for understanding. So, we exposed this on more descriptive examples of trigonometric modelling different motions with the use of their polar representations in the quasi-, pseudo-Euclidean and non-Euclidean geometries (with the globe) and in Theory of Relativity. Thus, the measureless hyperbolic tensor of motion with certain scalar multipliers produces all main dynamic tensor, vector and scalar physical characteristics of relativistic moving material body and gives the general law of summing velocities. The measureless hyperbolic tensor of deformation produces all seeming us geometric parameters of relativistic moving object. Under this Tensor Trigonometric approach, we opened wide opportunities for application of the relativistic Poincaré–Minkowski space-time with the Higgs field without its curving in the field of gravitation.

Contents of the book are at the joint of problems studied in multi-dimensional Geometry and Linear Algebra. Since its exposition required many of additional notations and terms, the author tried to give them the most convenient and logical forms with the full matrix alphabet based on wide-spreading literature.

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"Without exaggeration,
I put into this symphony
the whole of my soul ..."
P. I. Tchaikovsky

Introduction

The 3rd and last edition of this book from author-himself, with most full consequential exposition of this new subject of mathematics and its various applications, has been prepared by him with numerous updates and innovations aimed at improving presentation of its very extensive contents, and also with the goal of making this math subject yet more accessible to users including in the higher mathematical and physical education.

In Theory of Matrices such usual concepts as a singular matrix, its rank, eigenvalues, eigenvectors or eigensubspaces, annuling polynomial, etc, have a sense only for exact matrices and at exact computations. We distinguish in our mathworks, for instance, [15] and [17], the exact theory of notions and the approximating theory of notions' estimates. Each of them places its own important role. The notions connected with exact characteristics are used not only for constructing and analysis of abstractions, but they are important for objects from applied problems because the characteristics of objects are always exact and only their various estimates are approximate. Such creative approach was most vividly confessed in the works of the great mathematician, physicist and philosopher of science Henri Poincaré.

The main two parts of the monograph, in twelve chapters, contain both the results of our investigations in Theory of Exact Matrices (Part I, Chs. $1 \div 4$) and developed on this platform Tensor Trigonometry (Part II, chapters $5 \div 12$). The latter is a constituent division of the corresponding to it k-dimensional Geometry with a certain quadratic metric and a certain reflector tensor in the basis homogeneous and isotropic arithmetic and physical spaces.

The historical roots of Scalar Trigonometry, as a constituent part of two-dimensional Geometry, refer to far-away times. So, yet in the Euclid's "Elements" some trigonometric formulations were be found. Much later, in II age Claudius Ptolemy of Alexandria widely used in "Almagest" sine-cosine invariant as a trigonometric equivalent of the Pythagorean Theorem. Some spherical functions were used in IX–X ages by Arabian mathematicians. It is of interest that the Trigonometry on a sphere became developed much earlier than one on a plane. It was, due to the fact, that it was needed in the practical astronomy. So, in 1603, Th. Harriot connected the angular excess of a spherical triangle with area and radius. Though some trigonometric elements were introduced into the European science by R. Wallingford yet in early XIY age. He used them in solving of a right triangle on a plane.

Hyperbolic functions were discovered by A. Moivre (1722) and obtained in complete set by V. Riccati from a unity hyperbola (1757). First these functions were used really also in geometry, but as if on the "hypothetical sphere of an imaginary radius" with hyperbolic arcs–segments, by J. Lambert and F. Taurinus in their pioneer investigations. (Now we may named this object as the top sheet of the Minkowski hyperboloid II – see this in Ch. 12.) So, in 1763, J. Lambert, using the specific analogy between spherical and hyperbolic angles with their functions, connected the angular defect of a hyperbolic triangle on this sphere with its area and radius [36]. Later, in 1825, F. Taurinus inferred in first that a sum of angles in the such hyperbolic triangle less π [38]. I. e., they did the pioneer steps in creation of the non-Euclidean planimetry. The great creators of the hyperbolic non-Euclidean geometry, as based on the holistic axiomatic system, N. Lobachevsky and J. Bolyai used such specific analogy in the small with the spherical geometry as a mathematical instrument for inferences of the hyperbolic geometry metric relations.

In addition to the non-Euclidean geometry with affine topology, identified descriptively by H. Jancen [52] in 1909 on the Minkowski hyperboloid II, we revealed the *hyperbolic-elliptical non-Euclidean geometry* with cylindrical topology on the Minkowski hyperboloid I, which, as was proved, is *one-step isometric* to the geometry on the Beltrami pseudosphere!

These non-Euclidean geometries of spherical, hyperbolic and hyperbolic-elliptical types, realized on own curvilinear hypersurfaces of the constant radius-parameter R and can be embedded into their $(n+1)D$ enveloping homogeneous and isotropic binary spaces $\langle \mathcal{Q}^{n+1} \rangle$ and $\langle \mathcal{P}^{n+1} \rangle$, have such an essential feature. Each has a group of nD rotations limiting by one degree of freedom from constancy of R around frame axes in these spaces, isomorphic with a group nD motions on these hypersurfaces with non-Euclidean geometries. In the quasi-Euclidean binary space $\langle \mathcal{Q}^{n+1} \rangle$, this hypersuface is real-valued (the Special hyperspheroid), but in the pseudo-Euclidean binary space $\langle \mathcal{P}^{n+1} \rangle$, this hypersurface is either real-valued at imaginary R (the hyperboloid II) or it has one the imaginary dimension at real-valued R (the hyperboloid I). As was established in our work, only such hypersurfaces have angular metric forms and they may be represented by the angular Absolute Pythagorean theorems with three principal differential arcs. For a systematic, we classified such hypersurfaces of the constant Gaussian curvature, but of constant radius-parameter with enveloping spaces as "perfect surfaces and spaces". From here we infer next of the main our results, that the rotational Tensor Trigonometry (i. e., with $R = 1$) – quasi-Euclidean and pseudo-Euclidean is isometric with motions on such hypersurfaces with the exactness till factor R!!!

The term “Trigonometry” was raised thanks to Bartholomaus Pitiscus and appeared in 1595 in his book [2]. Within the framework of the term “Tensor Trigonometry” introduced by the author of the book of 2004, we singled it out as a new and useful subject of Mathematics, in which were presented both many new concepts, formulae and theorems and some of known notions related to this area, but which have not yet been explicitly attributed to the subject. Historically the modern perfect form of Scalar Trigonometry was given by L. Euler [1], who realized also its complexification. On the other hand, Geometry continued to develop and essentially violently according to the appeared idea of a multi-dimensional space.

Multi-dimensional space was arisen apparently at the middle of XIX age in classical work of H. Grassmann ”Die lineale Ausdehnungslehre” [3]. H. Grassmann and, independently of him, W. Hamilton laid the foundation of Vector Analysis in similar spaces. Before (in 1808) J.-G. Garnier emits Analytical Geometry as the whole division of Geometry. Outstanding contribution in justification of such an algebraic approach to the Geometry of objects in arithmetic spaces was realized by the famous ”Cantor–Dedekind Axiom about Continuum”.

About of that time appearance of Linear Algebra and its following development in the works of F. Frobenius, G. Cramer, L. Kronecker, A. Capelli, J. Sylvester, L. Hesse, C. Jordan, Ch. Hermite and other mathematicians led, with time, to its larger filling by geometric content. That is why, Linear Algebra found effective applications in the theory of vector Euclidean spaces and also, after the well-known works of H. Poincare and H. Minkowski, in the theory of new pseudo-Euclidean spaces. This process was activated thanks to algebraic definitions of notions connected with metric properties of arithmetic spaces and of their geometric objects (the lengths of vectors and the values of scalar angles between them). For the basic algebraic definitions of measures mathematicians used the Pythagorean Theorem and the algebraic cosine Inequality of Cauchy or sine Inequality of Hadamard.

Besides, for the strict algebraic approach to the geometry in arithmetic spaces, it is impossible to realize it completely without Theory of Exact Matrices. For example, E. Moore and later R. Penrose proposed the general methods of quasi-inversion of singular matrices. R. Courant developed the large parameter optimization method with penalty functions, useful in such algebraic applications too. A. Tichonov gave the small parameter method of regularization with the limit method for normal solving degenerated systems of linear equations. Results of these investigations had also a big geometric importance and, to some degree, served for initiating the present work.

The main aims of this monograph were (as 1st) to develop with further applications a number of algebraic and geometric notions in Theory of Exact Matrices (Part I, Chs. 1 ÷ 4), and then (as 2nd) on the platform to work out the basic aspects of the Tensor Trigonometry for binary tensor angles formed by two linear subspaces or formed by rotation of a linear subspace in linear enveloping spaces (Part II, Chs. 5 ÷ 12). Since the Tensor Trigonometry has a lot of applications in other mathematical and in some physical domains, the largest examples of which are exposed in the book's Appendix.

First of all, the structure of matrix characteristic coefficients in the explicit form was installed by us with our special differential method (though historically els in early 1981). They appeared in Theory of Exact Matrices in middle of XX age in the works of J.-M. Souriau and D. K. Faddeev in addition to scalar characteristic coefficients with their well-known structure. The latters were used yet in XIX age by U. Le Verrier at his famous prediction of Neptune. We express all eigenprojectors and quasi-inverse matrices in explicit form, in terms of the scalar and matrix coefficients. And the minimal annulling polynomial for $n \times n$-matrix in explicit form is identified with the connections of all matrix singularity parameters.

In passing, the general inequality for all average values is inferred, and hierarchical invariants for the spectrally positive matrix are installed for the justification of the stated geometric norms. The new global limit method for step by step calculating all roots of a real algebraic equation is proposed, and the more strict necessary condition for all its roots reality and positivity, than the classical Descartes condition, is gotten.

The particular (of order t) and general (of order r) quadratic norms are introduced for the geometric objects *lineors*, determined by $n \times r$-matrices A, where $1 \leq r < n$ (at $r = 1$ they are vectors), and for the *tensor angles* between them or between their images in the n-dimensional arithmetic spaces. In particular, at $t = 1$ they are Euclidean and Frobenius norms (measures). The theoretical basis for these particular and general norms is the hierarchical general inequality for all average positive values. Also the specific multiplications of cosine and sine types are defined for a pair of these lineors with inferring the so-called *general cosine and general sine inequalities* through the *especial matrix trigonometric spectra* with a binary nature (as all the tensor angles too). Their elementary algebraic and trigonometric cases are the cosine Inequality of Cauchy and the sine Inequality of Hadamard.

Tensor Trigonometry, as the main new content of this monograph, is exposed then with two types of its tensor angles – projective and motive ones. Projective tensor angle acts in the projective tensor trigonometric functions as their argument and in the different eigen-reflectors – symmetrical and oblique, orthogonal and affine, spherical and hyperbolic. Motive tensor angle acts in the motive rotational (sine–cosine) and deformational (tangent–secant) tensor trigonometric functions as their argument. Both these types of principal tensor angles are connected in one-to-one correspondence by clear matrix formulae. The principal tensor angles are added by induced or free secondary orthospherical tensor angles, which we reveal either by polar decomposition of general motions or through differentiation of vector functions of motive angles. Thus, any general mixed rotations or motions are presented by polar decomposition in matrix formulae in the principal and secondary orthospherical parts.

Under introducing a reflector tensor to the affine (or arithmetic) homogeneous isotropic space with a quadratic metric, all concepts above with two binary spaces are divided into quasi-Euclidean and pseudo-Euclidean ones. Two pairs of rotations (spherical, orthospherical), (hyperbolic, orthospherical) in these two binary spaces form two noncommutative groups. The first is the new homogeneous group of quasi-Euclidean rotations. The second is the well-known in the Theory of Relativity and the hyperbolic geometry homogeneous group of pseudo-Euclidean motions or rotations (Lorentz group). The intersection of these two groups in the so-called universal base is a subgroup of orthospherical rotations. So, in the Minkowski space-time, it is a subgroup of Euclidean rotations of the external cavity of dividing isotropic (light) cone (in each k-th Euclidean subspaces). The set of reflections in the same binary space is generated by the same reflector tensor, and it is not a group.

As a bright novelty, we gave solution of pseudo-Euclidean right triangles in a pseudoplane with connections of complementary hyperbolic angles and proposed an updated concept of the parallel angle in the hyperbolic non-Euclidean geometry, true in any admitted bases.

Our binary quasi-Euclidean space with its geometry filled a previously unnoticed gap that existed in the theory of homogeneous isotropic spaces. It is a natural and useful addition to the Minkowski pseudo-Euclidean space. Though the latter with its Lorentz group was introduced back in 1905 by Henry Poincaré as the complex binary space of the Theory of Relativity (named so later by Max Planck). In 1907 this binary space as the $4D$ space-time was realificated by Herman Minkowski and added by his real-valued hyperboloids I and II. In Chs. 7A and 10A, we use the Minkowski space-time with its unity trigonometric (at $|R| = 1$) hyperboloids I and II for tensor trigonometric modeling geometric motions in hyperbolic and hyperbolic–elliptical non-Euclidean geometries with affine and cylindrical topologies and respectively in Theory of Relativity with the Minkowski space-time, in accordance with the fundamental Mach Principle and the Higgs Theory, confirmed this space-time in our time!

In Appendix (Chs. 1A÷10A) – see in the Preface to it, as the rather important case, we considered tensor trigonometric transformations in the so-called elementary forms, i. e., with single principal and single orthospherical eigen angles of motions, and hence with single frame axis for them. The new interesting possibilities are discovered for the very clear study of various types of in all non-Euclidean geometries with the same reflector tensor, but with own quadratic metrics; in all non-Euclidean geometries of constant radius; and in Theory of Relativity. The general law of summing non-collinear segments, principal spherical or hyperbolic geometric motions or velocities in STR is established in the trigonometric matrix, vector, scalar (tvs) forms with identification of the orthospherical rotation. In non-Euclidean geometries and STR, we gave this law for two-steps motions also in the noncommutative biorthogonal form with the Big and Small Pythagorean theorems; and added to them the General Law of polysteps motions summation in its hyperbolic and spherical kinds.

In the *Kunstkammer* of the book to end, the readers may test themselves in solving of the suggested by the author questions and interest tasks near to this work's topics.

In conclusion, it is necessary to clarify the new subject name on the Titul. Why tensor? We see that usual angles are binary as between two linear geometric objects. They and their tensor functions are determined by square matrices how for any bivalent tensors. In the presence of some from two quadratic metric, the tensors are orthogonal; in the absence of metric, they are affine. This new math subject deals with orthogonal and affine tensors, their projections and invariants. On a quasiplane these tensors are spherically orthogonal, in a quasi-Euclidean space they are quasi-Euclidean orthogonal. On a pseudoplane they are hyperbolically orthogonal, and in a pseudo-Euclidean space they are pseudo-Euclidean orthogonal. In addition, they may be symmetric and anti-symmetric, real, imaginary and complex. For tensor trigonometric functions of the binary tensor angles we use by analogy with scalar ones, as most convenient here, the classical notations of J. Lagrange and K. Scherffer.

The date of the Tensor Trigonometry birth is October 4, 2004, when its first edition exited in the world by the "MIR" Publisher [15] thanks to a bright review of the eminent and encyclopedically versatile mathematician Postnikov M.M., well-known as author of a large number of valuable monographs and textbooks in various mathematical fields. In January of 2021 the 2-nd, but English edition of Tensor Trigonometry was issued by "Fizmatlit" [16]. This significantly renovated, widen and optimized by design 3-rd edition is being released with corrections of all found minor inaccuracies and typos, with new textual commentaries and preservation of principal theorems, corollaries, formulae, pictures, and with presenting of the most developed *tensor differential trigonometry* as one else mathematical subject.

New methods of Tensor Trigonometry can be used in the various domains of mathematics and physics. The author hopes that readers will find a lot of interesting contents and of new knowledge. I'll welcome, if somebody wishes to dare in this new direction for its following development with surprising results! However I'll post adherents of plagiarisms on web-site.

Notations

1. Notations of matrices (Matrices alphabet)

A – rectangular $n \times m$- or $m \times n$-matrix, or $n \times r$-*lineor* in a space (at $r = 1$ – $n \times 1$-vector $\mathbf{a}$),

A^+ – spherically orthogonal quasi-inverse matrix of Moore–Penrose,

B – quadratic $n \times n$-matrix or external multiplication $B = A_1 A_2'$ of $n \times r$-*lineors* A_1, A_2;

B^- – affine or oblique (or hyperbolically orthogonal) quasi-inverse matrix,

B^V – adjoint matrix for nonsingular B ($B^{-1} = B^V/detB$),

$B_i = B - \mu_i I$ – i-th singular eigenmatrix for B,

B (as Bp) – *null-prime* singular matrix: $\langle ker\ B\rangle \cap \langle im\ B\rangle \equiv \langle \mathbf{0}\rangle$,

B (as Bm and Bn) – *adequately and Hermitian null-normal* matrices: $\langle ker\ B\rangle \perp \langle im\ B\rangle$,

B (as Bc) – *null-cell (two-block-diagonal) form* of Bp, Bm, Bn,

$\overrightarrow{B}$ (as $\overrightarrow{Bp}$) – affine or oblique eigenprojector into $\langle ker\ B\rangle$ parallel to $\langle im\ B\rangle$,

$\overleftarrow{B}$ (as $\overleftarrow{Bp}$) – affine or oblique eigenprojector into $\langle im\ B\rangle$ parallel to $\langle ker\ B\rangle$,

$\overleftarrow{B}$ (as $\overleftarrow{Bm}$) – spherically orthogonal eigenprojector into $\langle im\ B\rangle \equiv \langle im\ B'\rangle$,

$\overrightarrow{B}$ (as $\overrightarrow{Bm}$) – spherically orthogonal eigenprojector into $\langle ker\ B\rangle \equiv \langle ker\ B'\rangle$,

$\overleftarrow{BB'}$ (as $\overleftarrow{Bm}$) – spherically orthogonal eigenprojector into $\langle im\ B\rangle$,

$\overrightarrow{B'B}$ (as $\overrightarrow{Bm}$) – spherically orthogonal eigenprojector into $\langle ker\ B\rangle$,

C – free cellular matrix multiplier or internal multiplication $C = A_1' A_2$ of these $n \times r$-*lineors*,

$C_\mu(B)$ – basic (q-block-diagonal) form of the matrix B (q – quantity of the eigenvalues of B),

D – diagonal matrix,

$\tilde{E}_k$ – certain unity coordinates base (frame of reference),

$\tilde{E}_1$ – unity base of the diagonal cosine or *universal base* for the spherical-hyperbolic analogy,

$F(\dots)$ – matrix function of $(\dots)$,

$\{G^+\}(\mathbf{x})$, $\{G^\pm\}(\mathbf{u})$ and $\hat{G}$, – metric tensors (positive, sign-indefinite and mutual with G),

$H = H^*$ – Hermitean complex matrix, $H^\oplus$ – positively definite Hermitean complex matrix,

I – unity matrix, I^+ and I^- – metric reflector tensors of Euclidean and anti-Euclidean spaces,

$I^\pm$, $I^\mp$ or $\{R_W' I^\pm R_W\} = \{\sqrt{I}\}_S$ – reflector tensors of quasi- and pseudo-Euclidean spaces,

It– *totally-unity* matrix: all the elements of which are equal to 1,

$J_\mu\ (B)$ – canonic Jordan form of a matrix B,

K – anti-symmetric real or complex matrix,

$K_B(\epsilon)$ – matrix characteristic polynomial of the parameter ε for a matrix B,

$K_1(B,t)$ and $K_2(B,t)$ – first matrix characteristic coefficients for a matrix B of order t,

$K_2(B,t)$ and $K_2(B,t)$ – second matrix characteristic coefficients for a matrix B of order t,

$\pm Ref\{B\}$ – eigenreflectors for matrices B (affine, oblique),

$\pm Ref\{Bm\}$ – eigenreflectors for matrices Bm (spherically orthogonal),

$\pm Ref\{Bp\}$ – eigenreflectors for matrices Bp (affine or oblique or hyperbolically orthogonal),

$\pm Ref\{AA'\}$ – eigenreflectors for a matrices AA' (spherically orthogonal),

$L_\mu(B)$ – q-block-triangular form of a matrix B (q - quantity of eigenvalues of B),

$M\ (MM' = M'M)$ – normal (real-valued or adequately complex) normal matrix,

$N\ (NN^* = N^*N$ – Hermitean complex normal matrix,

O – nilpotent matrix,

P – prime matrix,

Q – anti-Hermitean complex matrix,

$Q_B(\epsilon)$ – reduced matrix characteristic polynomial of the parameter ε for a matrix B,

$Q_1(B,t)$ and $Q_2(B,t)$ – reduced matrix characteristic coefficients for a matrix B of order t

$R\ (RR' = I)$ – orthogonal (real or adequately complex) matrix, Rq – *quasi-orthogonal* matrix,

R_W – orthogonal modal matrix for transformation of a prime matrix P into its W-form,

$S = S'$ – symmetric real or complex matrix, $S^\oplus$ – positively definite symmetric real matrix,

T – matrix of the rotational trigonometric modal transformation (active or passive),

$U\ (UU^* = I)$ – unitary (*Hermitean orthogonal*) complex matrix,

V – matrix of the general linear modal transformation (active or passive),

$W(P)$ – mono-binary form of a prime matrix P,

X – matrix argument,

Y – matrix function, connected one-to-one two spaces in their direct sum in a basis space,

Z – zero matrix.

2. Notations of binary tensor angles and their functions

$\tilde{\Phi} = \tilde{\Phi}'$ – principal tensor spherical projective angle between two planars and in reflectors,

$\Phi = -\Phi'$ – principal tensor spherical motive angle in rotations and deformations,

$\tilde{\Xi}$ and Ξ – complementary tensor spherical angles till the tensor spherical right angle $\Pi/2$,

$\tilde{\Gamma} = -\tilde{\Gamma}'$ – principal tensor hyperbolic projective angle between two planars and in reflectors,

$\Gamma = \Gamma'$ – principal tensor hyperbolic motive angle in rotations and deformations,

$\tilde{\Upsilon}$ and Υ – complementary tensor hyperbolic angles with angles $\tilde{\Gamma}$ and Γ till the right angle Δ,

$\tilde{\Theta} = \tilde{\Theta}'$ – tensor orthospherical projective angle (additional to the angle $\tilde{\Phi}$ or the angle $\tilde{\Gamma}$),

$\Theta = -\Theta'$ – tensor orthospherical motive angle (additional to the angle Φ or the angle Γ),

$\tilde{\Psi} = \tilde{\Phi} + i\tilde{\Gamma}$, $\Psi = \Phi + i\Gamma$ – complex adequate tensor projective and motive spherical angles,

$\tilde{\mathcal{H}} = \tilde{\Phi} + i\tilde{\Gamma} = \tilde{\mathcal{H}}^*$ – Hermitean tensor projective spherical angle, $\tilde{\Phi} = \tilde{\Phi}^*, \tilde{\Gamma} = -\tilde{\Gamma}^*$

$\mathcal{H} = \Phi + i\Gamma = -\mathcal{H}^*$ – skew-Hermitean tensor motive spherical angle, $\Phi = -\Phi^*, \Gamma = \Gamma^*$
(all the tensor angles correspond to the set reflector tensor of the space – see in item 2),

$Rot\ \Phi$ and $rot\ \Phi$ – principal spherical rotation at the angle Φ (and elementary one),

$Roth\ \Gamma$ and $roth\ \Gamma$ – principal hyperbolic rotation at the angle Γ (and elementary one),

$Rot\ \Theta$ and $rot\ \Theta$ – secondary orthospherical rotation at the angle Θ (and elementary one),

$Def\ \Phi$ and $def\ \Phi$ – spherical deformation at the angle Φ (and elementary one),

$Defh\ \Gamma$ and $defh\ \Gamma$ – hyperbolic deformation at the angle Γ (and elementary one).

3. Notations of spaces and sub-spaces

$\langle\mathcal{A}^n\rangle$ – arithmetic affine n-dimensional space,

$\langle\mathcal{E}^n\rangle$ – Euclidean n-dimensional space, $\langle\mathcal{C}^n\rangle$ – Euclidean cylindrical n-dimensional space,

$\langle\mathcal{E}^{n+q}\rangle$ – complex binary Euclidean $(n+q)$-dimensional space of the index q ($q \leq n$),

$\langle\mathcal{Q}^{n+q}\rangle$ – real binary quasi-Euclidean $(n+q)$-dimensional space of the index q ($q \leq n$),

$\langle\mathcal{P}^{n+q}\rangle$ – real binary pseudo-Euclidean $(n+q)$-dimensional space of the index q ($q \leq n$),

$\langle\mathcal{Q}^{n+q}\rangle_c$ – complex binary quasi-Euclidean $(n+q)$-dimensional space of the index q ($q \leq n$),

$\langle\langle\mathcal{E}^n\rangle\rangle$ – projective flat hyperplane, $\langle\langle\mathcal{C}^n\rangle\rangle$ – projective cylindrical hyperplane),

$\langle\mathcal{E}^n\rangle^{(k)}$, $\langle\mathcal{E}^q\rangle^{(k)}$ – Euclidean subspaces in $\langle\mathcal{Q}^{n+q}\rangle$ or $\langle\mathcal{P}^{n+q}\rangle$ with respect to the base $\tilde{E}_k$,

$\langle\mathcal{P}_i\rangle$, $\langle\mathcal{P}_{ij}\rangle$ – trigonometric subspaces of the tensor angle.

4. Other notations

$a, b, ...$ and $\mathbf{a}$, $\mathbf{b}$, ... – scalar and $n \times 1$-vector elements, $||\mathbf{a}||_E$ – Euclidean norm for $\mathbf{a}$,

$||A||_F = ||A||_1$ – Frobenius norm (first order's quadratic norm) for the $n \times m$-matrix A,

$||A||_t$ – particular quadratic of order t norms for the $n \times m$-matrix or $n \times r$-lineor A,

$\overline{||A||_t}$ – trimmed particular quadratic of order t or algebraic norms (algebraic medians),

$||A||_r$ – general quadratic of order r or geometric norm (geometric median),

C_n^t – binomial Newtonian coefficients, $det\ B$ – determinant of the matrix B,

$\mathbf{d}(\mathbf{x})$ – residual of the linear algebraic equation of $\mathbf{x}$,

$\mathcal{D}l(r)B$ – *dianal* of the singular $n \times n$-matrix B, i. e. the full sum of its basis principal minors,

$\langle im\ A\rangle$ or $\langle im\ B\rangle$ – image of the matrix A or of the matrix B,

$\langle ker\ A'\rangle$ and $\langle ker\ B\rangle$ – kernel of the matrix A' or of the matrix B,

$k_B(\epsilon) = det(B + \epsilon I)$ – scalar characteristic polynomial of parameter ϵ for the matrix B,

$k_B(-\mu) = det(B - \mu_i I) = 0$ – secular equation for the matrix B,

$k(B, t)$ – scalar characteristic coefficient for the matrix B of order t,

l – Euclidean and quasi-Euclidean length, λ – pseudo-Euclidean length,

$\overline{m_t}$ – algebraic mean (small median) of order t, $\overline{M_\theta}$ – power mean (large median) of order θ,

$\mathcal{M}t(r)A$ – *minorant* of singular A (the square root of the full sum of quadric basis minors A),

n – dimension of the space,

q – index of the quasi- or pseudo-Euclidean space,

$\mathrm{q}_B(\epsilon)$ – reduced scalar characteristic polynomial of parameter ϵ for the matrix B,

$q_B(-\mu) = 0$ – reduced secular equation for the matrix B,

$q(B, t)$ – reduced scalar characteristic coefficient of the matrix B of order t,

$r = rankB$ ($r = rankA$) – rank of the matrix,

r' – 1st *rock* of the singular matrix B, i. e. maximal order of non-zero $k(B, t)$,

r'' – 2sd *rock* of the singular matrix B, i. e. maximal order of non-zero $K(B, t)$,

s and s' – geometric and algebraic multiplicities of the zero eigenvalue of a singular matrix B,

$s_i^0 = r_i'' - r_i' + 1$ – annulling multiplicity of the i-th eigenvalue of a quadratic matrix B,

t – order of matrices characteristics, dimension of submatrices and minors,

trB – trace of the matrix B,

$\overline{v_t}$ – reversive algebraic mean (reversive small median) of order t,

$\overline{V_\theta}$ – reversive power mean (reversive large median) of order θ,

$\mathbf{x}$, $\mathbf{y}$ – real-number vectorial arguments (variables),

$\mathbf{z}$ and $\overline{\mathbf{z}}$ – complex-number vectorial arguments (conjugate variables),

sin, **sinh**, **cos**, **cosh**, **tan**, **tanh**, **sec**, sech, **cot**, **coth**, **cosec**, csch – trigonometric functions, **arcsin**, arsinh, **arccos**, arcosh, **arctan**, artanh, arcsec, arsech – reverse to them functions.

Greek some notations :

φ – principal scalar spherical angle, γ – principal scalar hyperbolic angle,

θ – secondary scalar orthospherical angle (respectively to the principal angles φ or γ).

ξ – complementary spherical angle with φ (relatively to the right spherical angle $\pi/2$),

υ – complementary hyperbolic angle with γ in some right pseudo-Euclidean triangle,

δ – infinite hyperbolic angle in some right pseudo-Euclidean triangle,

η – scalar Hermitean spherical angle,

π – Archimedes Number and an open spherical angle,

$\omega = arsh\ 1$ – especial hyperbolic angle (and number) as analog of the spherical angle $\pi/4$

μ_i – i-th eigenvalue of a quadratic matrix with its quantity q_i,

σ_j – j-th eigenvalue of multiplicative matrices AA' and $A'A$,

2τ – trigonometric rank of the binary tensor angle of projective or motive type,

ν' – dimension of the sub-space of the intersection $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ (i. e., of zero sine),

ν'' – dimension of the sub-space of the intersection $\langle im\ A_1\rangle$ and $\langle ker\ A_2'\rangle$ (i. e., of zero cosine).

5. Using symbols

$'$ – mark of simple transposing, $*$ – mark of Hermitean transposing,

$\ldots \subset \ldots$ – set ... belong to set ... , $\ldots \subseteq \ldots$ – set ... belong or is identical to set ... ,

$\ldots \in \ldots$ – element ... belong to set ... , $\ldots \notin \ldots$ – element ... no belong to set ... ,

$\ldots \cup \ldots$ – mark of summing (joining) two sets, $\ldots \cap \ldots$ – mark of intersecting two sets,

$\ldots \equiv \ldots$ – mark for the identity of the two sets,

$\ldots \oplus \ldots$ – mark of direct summing two sets, $\ldots \boxplus \ldots$ and $\ldots \boxtimes \ldots$ – marks of spherical and hyperbolic orthogonal direct summing two sets, $\ldots \uplus \ldots$ – mark of geometric summing two angles,

$\overset{\angle}{\Phi}$ and $\overset{\angle}{\Gamma}$ – mark over the summarized tensor angles in the case of reverse order of two- or multistep rotations (particular motions), and in the case of reverse angular shifting.

Part I

Theory of Exact Matrices: some of general questions

The main aims of this monograph in 2004 [15] were, in first, to develop in necessary us degree a number of algebraic and geometric notions in the Theory of Exact Matrices in Part I (Chs. 1÷4), and then, in Part II, on such platform to work out the fundamental of the new mathematical subject under general name "Tensor Trigonometry" with its following numerous applications in mathematical—physical fields, mainly in the big Appendix.

In Chapter 1, structures and properties of the scalar and matrix characteristic coefficients of $n\times n$-matrix B are found and studied. The fundamental relation and inequality for basic parameters of singularity for the matrix B are established. As additional result, from the highest orders r', r'' of these scalar and matrix characteristic coefficients for eigenmatrices B_i a minimal annulling polynomial of the matrix B is identified in its explicit form. The general inequality for average values (means) is formulated and proved in a whole form, including the chain of particular inequalities for algebraic means as a basis of hierarchical algebraic norms entered subsequently. Its opportunities are shown in the theory and technique for solutions of real algebraic equations, in that number, of secular ones. In the case of equation's positive roots (e. g., of the eigenvalues for positively definite matrices), the limit method and formulae for calculating of maximal and minimal roots are gotten in terms of the equation coefficients (with following sequential calculation of all the roots). (Note the fact of inferring here the classic Theorem of Hamilton–Cayley in one line, and many of interesting other.)

In Chapter 2, the explicit formulae for two characteristic eigenprojectors and the quasi-inverse matrix for a null-prime singular $n\times n$-matrix B in terms of its matrix and scalar characteristic coefficients of the highest order $r = rankB$ are established. (The simplest case of null-prime matrices is a $n\times n$-matrix B consisting from r of basis columns and $n-r$ of zero columns.) As a very important especial case, the *null-normal* singular $n\times n$-matrices B, whose image and kernel form a direct orthogonal sum, are entered and studied. (Their considered separately important particular cases are symmetric S and multiplicative matrices $AA', A'A$.) Besides, the modal matrices for transformations of these null-prime and null-normal matrices into the two-cell block-diagonal canonic form are gotten. And as additional applications of the eigenprojectors and quasi-inverse matrices, the general formulae for solutions of vector and matrix linear equations are gotten.

In Chapter 3, the more general linear geometric objects in linear spaces than $n\times 1$-vectors and lines are entered additionally into consideration, as $n\times m$-lineors A and planars $\langle im\ A\rangle$ and $\langle ker\ A'\rangle$, i. e., given by the matrix A, where $1 \le m \le n$ (in particular, if $m = 1$ they are vector $\mathbf{a}$, lines $\langle im\ \mathbf{a}\rangle$ and hyperspace $\langle ker\ \mathbf{a}'\rangle$). The scalar invariant relations for matrices or matrix geometric objects with corresponding to them inequalities having cosine or sine nature (relations generalizing the well-known algebraic norms for a cosine and a sine of an angle between vectors or lines in Euclidean spaces) are defined. As an additional result, the limit explicit formulae for the eigenprojectors and quasi-inverse matrices are gotten by algebraic and functional manners. (Note the fact of inferring here the classic Theorem of Kronecker–Capelli in one line, and also many of interesting other.)

In Chapter 4, the main alternative complexification's variants of different mathematical notions are considered upon transition from the initial real arithmetic spaces into various complex ones. It is important, in particular, for following constructing similar complex variants of the new concepts of Tensor Trigonometry in all its kinds. A number of the specific complexification's examples in different mathematical regions, including arithmetic, algebraic, geometric and functional ones, are given.

Chapter 1

Coefficients of characteristic polynomials

1.1 Simultaneous definition of scalar and matrix coefficients

In Theory of Exact Matrices, especial attention is paid to characteristic polynomials. They are studied from algebraic and geometric points of view. Detailed analysis of the question is necessary for further construction of Tensor Trigonometry foundation.

As it is known, for each $n \times n$-matrix there is its own secular equation determined by the *scalar characteristic polynomial* (a polynomial with scalar coefficients) depending on a certain parameter μ. The roots μ_i of this polynomial (the roots of the secular equation) for a given square matrix B are the eigenvalues of the matrix. The matrix B has also the *matrix characteristic polynomial* (a polynomial with matrix coefficients).

For the next, introduce simultaneously two kinds of the characteristic polynomials and their coefficients, following mainly to D. K. Faddeev [29, p. 311–316]. Consider a nonzero $n \times n$-matrix B of rank r with the unity matrix I. *The resolvent* of B is transformation of the type:

$$(B + \epsilon I)^{-1} = \frac{(B + \epsilon I)^V}{det(B + \epsilon I)} = \frac{K_B(\epsilon)}{k_B(\epsilon)}. \tag{1}$$

In fact, it is the usual formula for the inverse matrix of $(B + \epsilon I)$: the numerator is the adjoint matrix, the denominator is its determinant, ϵ is an arbitrary scalar parameter. This operation determines two characteristic polynomials: scalar one of order n as the denominator and matrix one of order $n-1$ as the numerator of the fraction:

$$K_B(\epsilon) = \sum_{t=0}^{n-1} K_1(B,t) \cdot \epsilon^{n-t-1} = \epsilon^{n-1} + K_1(B,1) \cdot \epsilon^{n-2} + \cdots + K_1(B, n-1),$$

$$k_B(\epsilon) = \sum_{t=0}^{n} k(B,t) \cdot \epsilon^{n-t} = \epsilon^n + k(B,1) \cdot \epsilon^{n-1} + \cdots + k(B,n) = \epsilon^n + tr\ B \cdot \epsilon^{n-1} + \cdots + det\ B.$$

The formulae of the polynomials contain so-called the *scalar characteristic coefficients* $k(B,t)$ and the *matrix characteristic coefficients of 1-st kind* $K_1(B,t)$, where we have $K_1(B,0) = I$, $K_1(B,n) = Z$ (see in sect. 1.4), coefficient of the 2-nd kind $K_2(B,t)$ will be defined later. The sequential-increasing number t is the *order* of such scalar and matrix coefficients.

In this book, we consider both characteristic polynomials of B with all their coefficients, as a rule, in the *sign-constant form* as polynomials with the scalar parameter $\epsilon = -\mu$. The opposite parameters $\mu = -\epsilon$ are the *eigenvalues* μ of the matrix B. The scalar polynomial of μ is zero and determines the *sign-alternating* secular algebraic equation for matrices B:

$$k_B(-\mu) = (-\mu)^n + k_1 \cdot (-\mu)^{n-1} + \cdots + k_n = (-\mu)^n + tr\ B \cdot (-\mu)^{n-1} + \cdots + det\ B = 0.$$

Thus the scalar coefficients of order t are the Viète sums of μ_i and the sums of all principal $t \times t$-minors, but with the summands of constant sign. They may be computed by Le Verrier's method [27, 29] with use of the recurrent Waring formula [21, p. 38], where the Viète sums are changed by the scalar characteristic coefficients, and the Waring sums are replaced by the characteristic traces (of the same order t):

$$k(B,t) = \frac{1}{t} \cdot \sum_{\theta=1}^{t} (-1)^{\theta-1} k(B, t-\theta) \cdot tr\ B^{\theta}. \tag{2}$$

It is the recurrent Waring–Le Verrier *direct* formula. Note, that the equivalent explicit expressions

$$k(B,t) = \frac{1}{t!} \cdot det \begin{bmatrix} tr\ B & 1 & 0 & \cdots & 0 \\ tr\ B^2 & tr\ B & 2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ tr\ B^{t-1} & tr\ B^{t-2} & tr\ B^{t-3} & \cdots & t-1 \\ tr\ B^t & tr\ B^{t-1} & tr\ B^{t-2} & \cdots & tr\ B \end{bmatrix} \qquad (3)$$

are of more theoretical interest [21, p. 38]. Formulae (2) and (3) are obtained from the Newton system of linear equations for n unknown coefficients with n given roots as the result of the change described above. The sequence of the scalar coefficients (the Viète sums) is, due to the Newton system of equations, in the one-to-one correspondence with the sequence of the characteristic traces (the Waring sums) up to the special order, what has the following property

$$t = r' = \min\{rankB^h\} \le r$$

and all the scalar coefficients of greater orders are equal to 0. Here the number r' is called *the* 1*-st rock* of the matrix B (*the* 2*-nd rock* r'' is the greatest order of the nonzero matrix characteristic coefficients). All problems concerning the scalar coefficients for equations may be expressed in terms of the Waring sums, and ones for the matrices may be analyzed in terms of the characteristic traces.

1.2 The general inequality of means (average values)

In main part II, we often deal with the positively (semi)definite symmetric and Hermitian matrices of fixed rank and their scalar invariants. Suppose that B is such a matrix. Consider the secular equation for B in the usual sign-alternating form and its scalar coefficients. All these coefficients of orders up to $r' = r = rankB$ are positive real numbers. Moreover, all the roots μ_i of the secular equation (the eigenvalues of the matrix B) are nonnegative real numbers.

Let μ_i be n nonnegative numbers and exactly r of them ($r \le n$) are nonzero. Special characteristics of the set $\langle\mu_i\rangle$, *the small medians* $\overline{m_1}, \overline{m_t}$ (*the algebraic means*) and *the large medians* $\overline{M_1}, \overline{M_\theta}$ (*the power means*), are defined as follows:

$$\overline{m_1} = \overline{M_1} = \sum_{i=1}^{n} \frac{\mu_i}{n}, \qquad (4)$$

$$\overline{m_t} = \sqrt[t]{s_t(\mu_i)/C_n^t} = \sqrt[t]{k(B,t)/C_n^t}, \qquad (5)$$

$$\overline{M_\theta} = \sqrt[\theta]{S_\theta(\mu_i)/n} = \sqrt[\theta]{tr\ B^\theta/n}, \qquad (6)$$

where $s_t(\mu_i)$ are the Viète sums, $S_\theta(\mu_i)$ are the Waring sums, n is the size of the set $\langle\mu_i\rangle$ or of the quadratic matrix, t and θ are orders of the corresponding means, C_n^t are the Newton binomial coefficients. (The arithmetic mean $\overline{m_1} = \overline{M_1}$ is the intersection of the set of all small medians and the set of all large ones.) Therefore formulae (5) express the algebraic medians not only in terms of the Viète sums, but also in terms of the equation coefficients, and formula (6) represents the power medians in terms of the Waring sums as well as in terms of the matrix traces. If there are zeros among μ_i and $t > r$, then $\overline{m_t} = 0$.

Otherwise the analogous reverse medians are defined as follows:

$$\overline{v_1} = \overline{V_1} = \left(\sum_{i=1}^{n} \frac{\mu_i^{-1}}{n}\right)^{-1}, \tag{7}$$

$$\overline{v_t} = \sqrt[-t]{s_t(\mu_i^{-1})/C_n^t} = \sqrt[-t]{k(B^{-1},t)/C_n^t}, \tag{8}$$

$$\overline{V_\theta} = \sqrt[-\theta]{S_\theta(\mu_i^{-1})/n} = \sqrt[-\theta]{tr\ B^{-\theta})/n}. \tag{9}$$

They too play the role of average values, i. e., the reverse means of the numbers $1/\mu_i$. Notice that the geometric mean $\overline{m_n} = \overline{v_n}$ is the intersection of the set of all small medians and the set of all their reverse analogs; but $\overline{v_1} = \overline{V_1}$ is the harmonic mean.

For a set of n positive real numbers $\langle\mu_i\rangle$ containing at least two distinct ones, the following general inequality of means does hold on all the interval in $\mathbb{R}$ containing $\langle\mu_i\rangle$:

$$\max\langle\mu_i\rangle = \overline{M_\infty} > \cdots > \overline{M_\theta} > \cdots > \overline{M_1} = \tag{10}$$

$$= \overline{m_1} > \cdots > \overline{m_t} > \cdots > \overline{m_n} = \tag{11}$$

$$= \overline{v_n} > \cdots > \overline{v_t} > \cdots > \overline{v_1} = \tag{12}$$

$$= \overline{V_1} > \cdots > \overline{V_\theta} > \cdots > \overline{V_\infty} = \min\langle\mu_i\rangle \tag{13}$$

$$(t = 1, \ldots, n; \quad \theta = 1, \ldots, \infty).$$

The equality for all the means simultaneously does hold iff $\mu_1 = \cdots = \mu_n$. If there are exactly $n - r$ zeros among μ_i, then $\overline{m_1} \cdots \overline{m_r} \neq 0$ and $\overline{m_t} = 0$ for all $t > r$. Moreover, if under this condition all nonzero μ_i are equal, then the medians are expressed as the functions

$$\overline{m_t} = \mu \cdot \sqrt[t]{C_r^t/C_n^t}, \quad \overline{M_\theta} = \mu \cdot \sqrt[\theta]{r/n}.$$

Note, that in the general inequality middle chains (11) and (12) of means are connected by one-to-one functional bound. The same relates to any continuous chains of it from n means iff all the original n numbers are different. This bond is interpreted obviously as direct and back n-vector-function of n-vector-argument. The fact will be used in the next section.

Special cases of the general inequality are the Cauchy inequality for arithmetic and geometric means and its reverse analog for harmonic and geometric means, the Maclaurin inequality for algebraic means and its reverse analog, the Hölder inequality for power means and its reverse analog [23]. Suppose B is a spectrally positive (all $\mu_i > 0$) matrix. The arithmetic, geometric, and harmonic medians are defined as follows:

$$\overline{m_1} = tr\ B/n = \overline{M_1}, \tag{14}$$

$$\overline{m_n} = \sqrt[n]{det\ B} = \overline{v_n}, \tag{15}$$

$$\overline{v_1} = \left(tr\ B^{-1}/n\right)^{-1} = \overline{V_1}. \tag{16}$$

Let A be an $m \times n$-matrix (in particular, $A = \mathbf{a}$ may be an $n \times 1$-vector), $B = AA'$. Then the arithmetic median is expressed in terms of the Frobenius and Euclidean norms:

$$n \cdot \overline{m_1(B)} = tr\ B = \begin{cases} ||A||_F^2, \\ ||\mathbf{a}||_E^2. \end{cases}$$

Since B is a spectral-positive matrix, the chain of simplest inequalities–estimations

$$\max\langle\mu_i^n\rangle \geq tr\ B^n/n \geq (tr\ B/n)^n \geq det\ B \geq$$

$$\geq \left(tr\ (B^{-1})/n\right)^{-n} \geq \left(tr\ (B^{-n})/n\right)^{-1} \geq \min\langle\mu_i^n\rangle \qquad (17)$$

follows from (10)–(13). Closer to each other are the eigenvalues, less are all the defects in (17). The equality holds iff the matrix B is proportional to the unit matrix I.

Clearly, the limit medians for B in the general inequality are the extremal eigenvalues of B:

$$\max\langle\mu_i^n\rangle = \lim_{n\to\infty} \overline{M_n}, \qquad (18)$$

$$\min\langle\mu_i^n\rangle = \lim_{n\to\infty} \overline{V_n}. \qquad (19)$$

Further we prove the general inequality and analyze it with use of differentiation to explore extrema.

Consider n positive numbers x_i as the vector $\mathbf{x} = (x_1, \ldots, x_n)$ in the 1-st quadrant (the basis is standard) and the scalar functions expressing the differences and the ratios of the corresponding means:

$$r\begin{bmatrix} t \\ t+1 \end{bmatrix}(\mathbf{x}) = \overline{m_t(\mathbf{x})} - \overline{m_{t+1}(\mathbf{x})},$$

$$r\begin{bmatrix} 1 \\ n \end{bmatrix}(\mathbf{x}) = \overline{m_1(\mathbf{x})} - \overline{m_n(\mathbf{x})},$$

$$f\begin{bmatrix} t \\ t+1 \end{bmatrix}(\mathbf{x}) = \overline{m_t(\mathbf{x})}/\overline{m_{t+1}(\mathbf{x})},$$

$$f\begin{bmatrix} 1 \\ n \end{bmatrix}(\mathbf{x}) = \overline{m_1(\mathbf{x})}/\overline{m_n(\mathbf{x})},$$

$$R\begin{bmatrix} \theta+1 \\ \theta \end{bmatrix}(\mathbf{x}) = \overline{M_{\theta+1}(\mathbf{x})} - \overline{M_\theta(\mathbf{x})},$$

$$R\begin{bmatrix} \theta \\ 1 \end{bmatrix}(\mathbf{x}) = \overline{M_\theta(\mathbf{x})} - \overline{M_1(\mathbf{x})},$$

$$F\begin{bmatrix} \theta+1 \\ \theta \end{bmatrix}(\mathbf{x}) = \overline{M_{\theta+1}(\mathbf{x})}/\overline{M_\theta(\mathbf{x})},$$

$$F\begin{bmatrix} \theta \\ 1 \end{bmatrix}(\mathbf{x}) = \overline{M_\theta(\mathbf{x})}/\overline{M_1(\mathbf{x})}.$$

Each of the functions r, R, and f, F has the only and common stationary value corresponding to $\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is the bisectrix of the 1-st quadrant. These functions have the zero gradients at all points of $\mathbf{b}$. Therefore,

$$r'(\mathbf{b}) = R'(\mathbf{b}) = f'(\mathbf{b}) = F'(\mathbf{b}) = \mathbf{0}, \quad x_1 = \cdots x_n = b,$$

$$r(\mathbf{b}) = R(\mathbf{b}) = 0, \quad f(\mathbf{b}) = F(\mathbf{b}) = 1,$$

and $\mathbf{b}$ is the region of minimum. This is true because the corresponding Hesse matrices are positively semi-definite.

Their rank is $n-1$):

$$r''\left[\begin{array}{c}1\\n\end{array}\right](\mathbf{b})=(n-1)r''\left[\begin{array}{c}t\\t+1\end{array}\right](\mathbf{b})=$$

$$=bf''\left[\begin{array}{c}1\\n\end{array}\right](\mathbf{b})=b(n-1)f''\left[\begin{array}{c}t\\t+1\end{array}\right](\mathbf{b})=$$

$$=R''\left[\begin{array}{c}\theta+1\\\theta\end{array}\right](\mathbf{b})=\frac{1}{\theta-1}R''\left[\begin{array}{c}\theta\\1\end{array}\right](\mathbf{b})=$$

$$=bF''\left[\begin{array}{c}\theta+1\\\theta\end{array}\right](\mathbf{b})=\frac{b}{\theta-1}F''\left[\begin{array}{c}\theta\\1\end{array}\right](\mathbf{b})=\frac{nI-It}{n^2b}=G,$$

where It is the *totally-unity* matrix, all its elements are equal to 1. The matrix G has the positive principal minors of orders r, $r<n$, they are equal to

$$\left(\frac{1}{nb}\right)^r\cdot\frac{n-r}{n}.$$

The Hesse matrix is degenerated at all points of the bisectrix, the one-dimensional linear subspace. The stationary values computed above lead to the following equalities

$$r''\left[\begin{array}{c}t\\t+m\end{array}\right](\mathbf{b})=mr''\left[\begin{array}{c}t\\t+1\end{array}\right](\mathbf{b}),$$

$$f''\left[\begin{array}{c}t\\t+m\end{array}\right](\mathbf{b})=mf''\left[\begin{array}{c}t\\t+1\end{array}\right](\mathbf{b}),$$

$$R''\left[\begin{array}{c}\theta+m\\\theta\end{array}\right](\mathbf{b})=mR''\left[\begin{array}{c}\theta+1\\\theta\end{array}\right](\mathbf{b}),$$

$$F''\left[\begin{array}{c}\theta+m\\\theta\end{array}\right](\mathbf{b})=mF''\left[\begin{array}{c}\theta+1\\\theta\end{array}\right](\mathbf{b}).$$

Therefore, on the bisectrix $\mathbf{b}$, these facts give us the following logical corollaries.

1. *The Hesse matrices of the adjacent means ratio do not depend on their orders.*

2. *These matrices vary as additive functions of the difference between the orders.*

3. *The Hesse matrices for all adjacent power means ratios are equal to the Hesse matrix for the ratio of the arithmetic and geometric means.*

4. *The Hesse matrices for all adjacent algebraic means ratios consist of $n-1$ identical parts of the matrix from Corollary 3.*

But two next corollaries seem surprising and paradoxical. Namely:

5. *The Hesse matrix for the ratio of the power and arithmetic means is unlimited at all points of the bisectrix, it increases as proportional to θ. Though the same function F, in accordance to* (18), *tends to x_{max}/M_1 as $\theta\to\infty$, it is continuous and takes the minimal value* 1 *at all points of the bisectrix.*

6. *The Hesse matrix for the adjacent power means ratio is constant at all points of the bisectrix even as $\theta\to\infty$. Though, according to* (18), *the same function F tends to* 1 *at all points of the bisectrix, its limit value is the constant for which the gradient and the Hesse matrix are zero.*

These conclusions seem contradictory, but they can be explained by correlation between the infinitely small deviation of $\mathbf{x}$ from the bisectrix and the infinitely large parameter θ. The Hesse matrix is discontinuous and becomes zero in the neighborhood of the bisectrix.

The function $F\left[\begin{array}{c}\theta\\1\end{array}\right](\mathbf{x})$, in its turn, tends to 1 as $\theta\to\infty$, but it depends up to infinitesimal on $\mathbf{x}$ and takes the minimal value 1 at points of $\mathbf{b}$. Contrary, the function $F\left[\begin{array}{c}\theta+1\\\theta\end{array}\right](\mathbf{x})$ takes the value 1 there at once.

Interpret these facts on the model functions of one scalar variable:

$$F_1\left[\begin{array}{c}\theta+1\\\theta\end{array}\right](x)=\sqrt[\theta+1]{\frac{1+x^{\theta+1}}{2}}\Bigg/\sqrt[\theta]{\frac{1+x^{\theta}}{2}},$$

$$F_2\left[\begin{array}{c}\theta\\1\end{array}\right](x)=\sqrt[\theta]{\frac{1+x^{\theta}}{2}}\Bigg/\frac{1+x}{2},\quad (x>0,\qquad \theta\geq 2).$$

Suppose, for certain conditions of the task, that $x\geq 1$, then it is the greatest element of the model set $\langle 1,x\rangle$.

If θ is *finite*, then

$$F_1(1)=F_2(1)=1=\min,\quad 1<F_1(x)<F_2(x);$$

$$\frac{dF_1}{dx}(1)=\frac{dF_2}{dx}(1)=0;$$

$$\frac{d^2F_1}{dx^2}(1)=\frac{1}{4},\quad \frac{d^2F_2}{dx^2}(1)=\frac{\theta-1}{4},\quad \frac{d^2F_2}{dx^2}(x)\geq\frac{d^2F_1}{dx^2}(x)>0.$$

If θ is *infinite*, then

$$F_1(x)=1+\beta(x),\ \beta(x)\to 0,\ \beta(1)=0,\ F_2(1)=1=\min,$$

$$F_2(x)=\begin{cases}2x/(1+x) & \text{if } x>1,\\ 2/(1+x) & \text{if } x<1,\end{cases}$$

$$\frac{dF_1}{dx}(x)=\frac{dF_2}{dx}(1)=0,\ \frac{dF_2}{dx}(1\pm\alpha)=\pm\frac{1}{2}\ \ (\alpha\to 0);$$

$$\frac{d^2F_1}{dx^2}(1)=\frac{1}{4},\qquad \frac{d^2F_1}{dx^2}(x)=0 \text{ provided that } x\neq 1,$$

$$\frac{d^2F_2}{dx^2}(1)=\frac{\theta-1}{4}\to\infty,\quad \frac{d^2F_2}{dx^2}(1\pm\alpha)=0\ \ (\alpha\to 0).$$

The Hesse matrix is also discontinuous in the neighborhood of $\langle\mathbf{b}\rangle$, that is why the trivalent symmetric matrix of third derivatives tends to infinite one as $\theta\to\infty$ and is negatively semi-definite at all points of the bisectrix. Notice that for the analogous functions of the reverse means, all these facts do hold, the only difference is that the Hesse matrix changes the sign. The same transformation of the Hesse matrix takes place under inverting the ratios.

These arguments as well as limit formulae (18) and (19) complete our proof and analysis of the general inequality of means (or average values). Now we consider some applications of the general inequality in the theory and techniques for solving algebraic equations, particularly, secular ones. smallskip

1.3 Serial method of solving algebraic equations with real roots

The small and large medians are connected by the system of modified Newton equations and the modified Waring–Le Verrier formulae, for example, of the *direct type*. These *direct* formulae are similar to (2) provided that $t > r$ and $\overline{m_t} = 0$:

$$C_{n-1}^{t-1}(\overline{m_t})^t = C_n^{t-1}(\overline{m_{t-1}})^{t-1}(\overline{M_1})^1 - C_n^{t-2}(\overline{m_{t-2}})^{t-2}(\overline{M_2})^2 + \cdots +$$

$$+(-1)^{t-2}C_n^1(\overline{m_1})^1(\overline{M_{t-1}})^{t-1} + (-1)^{t-1}(\overline{M_t})^t.$$

If all the coefficients of a secular equation are the same, then the well known particular formula for binomial coefficients

$$C_{n-1}^{t-1} = C_n^{t-1} - C_n^{t-2} + \cdots + (-1)^{t-2}C_n^1 + (-1)^{t-1}$$

follows from one above.

Limit formulae (18) and (19) allow one to compute consequently all the roots of an algebraic equation provided that all its roots are real numbers. Multiplicity of the roots may be found in the process of reducing, but it is worth to separate the roots before solving with use of the 1-st derivative and Euclidean algorithm. Sturm's method [28, p. 225–229] and the prior boundaries of the roots ($\mp\infty$) ensure one that the roots are real numbers. Other useful criterions for identification of the roots reality follow from the inequalities for the real roots of an algebraic equation presented here in its sign-alternating form [21, p. 40]:

$$-1 - \sqrt[h_1]{-\min k_j} = \Delta^{(-)} < \mu_i < \Delta^{(+)} = 1 + \sqrt[h_2]{-\min(-1)^j k_j},$$

where $\Delta^{(-)}$ and $\Delta^{(+)}$ are the boundaries of the negative and positive real roots, h_1 and h_2 are the indexes of the first negative coefficients, respectively k_j and $(-1)^j k_j$. Maclaurin's Theorem is used for inferring these inequalities [28, p. 223].

The serial limit method for solving an algebraic (may be secular) equation is the following

It is supposed to be already known that all the equation roots are real nonnegative numbers, in particular, they may be the eigenvalues of a nonnegatively definite matrix AA' or $A'A$.

The first step is computing the Viète sums and the Waring sums up to order r. For example, the Waring–Le Verrier recurrent formula of the direct type (such as (2)) is used for matrices, and the following Waring–Le Verrier recurrent formula of the *reverse type* [16, p. 38] is used for an arbitrary algebraic (polynomial) equation:

$$S_\theta = s_1 S_{\theta-1} - s_2 S_{\theta-2} + \cdots + (-1)^{r-2} s_{r-1} S_{\theta-r+1} + (-1)^{r-1} s_r S_{\theta-r} =$$

$$= F_\theta(S_1, \ldots, S_r) = f_\theta(s_1, \ldots, s_r), \qquad \theta = r+1, r+2, \ldots.$$

Next step is consequent computing the power medians

$$\overline{M_\theta} = \sqrt[\theta]{s_\theta / r}.$$

Due to (10), the sequence of the fixed root approximations increases. Clearly, more different are the roots, faster is the process. The recurrent formula with limit value (18), being divided by $x^{\theta-n}$, is the original equation as an identity. Hence on a certain iteration computing should be finished in order to avoid a round-off error for the maximal root.

The minimal root may be found according to (19) by the similar way with use of the equation inverse form in $(-1/x)$ obtained by dividing original equation $y(-x) = k_B(-x) = 0$ (where $x = -\mu$) by $(-x)^n$ and by the highest coefficient k_n. (For matrices B: $k_n = det\ B$.)

Approximate computing a *rational root* induces a periodic sequence starting with some significant digit, that is why the precise value of this root should be checked in the original equation. *Irrational roots* are computed up to a given precision. Thus the algorithm results in all the real roots of an algebraic equation. This method has the common limit idea with classic Lobachevsky–Greffe's method (1834) [21, p. 657] (see detailed comparison of both these methods in other our monograph [17, p. 162–163]).

If all the equation roots are real numbers of arbitrary signs, then its variable x should be substituted for $x + C$, where the constant $C > 0$ shifts the variable into the positive semi-axis. In order to faster convergence, this shift should be as small as possible.

It is known that all the eigenvalues of real symmetric matrices $S = S'$ and imaginary anti-symmetric ones $(iK)' = -iK$, where $K = -K'$ is a real matrix, are real valued numbers. In particular, these matrices are characteristic ones for a real-valued matrix B:

$$S = (B + B')/2, \quad K = (B - B')/2 \ \rightarrow \ B = S + K.$$

Here condition of the commutativity $SK = KS$ means that $B \in \langle M \rangle$ is a real-valued normal matrix, which has some double complex conjugated roots. These matrices may be transformed into their diagonal forms simultaneously. Then the double complex eigenvalues of such a normal matrix B are the sums of the summand matrices eigenvalues. Thus separated solving the secular equations for S and $-iK$ (the secular equation for $-iK$ is biquadratic) result in the real and imaginary parts of the normal matrix B complex eigenvalues. Further, the values obtained should be paired by checking in the secular equation for B.

This approach may be extended on complex matrices by use of the Hermitean and skew-Hermitean conjugations. All eigenvalues of Hermitean matrices are real numbers. Take advantage of the following complex Hermitean normal matrix decomposition:

$$H = (B + B^*)/2, \quad Q = (B - B^*)/2 \ \rightarrow \ B = H + Q = H + iH_Q,$$

$$HQ = QH \Leftrightarrow HH_Q = H_QH \Leftrightarrow B \in \langle N \rangle, \text{ where } NN^* = N^*N,$$

and so on.

Thus the serial method represented here is also applicable to real-valued normal matrices and complex Hermitean normal ones.

Suppose that all the roots of the secular equation for a some matrix are real numbers and shifting described above is used. Then, for the equation in alternating-sign form, the lower boundary of the negative roots satisfies the following inequality:

$$\min\langle \mu_i \rangle > \Delta^{(-)} = -1 - \sqrt[k]{-\min k_j}.$$

Substitution $x = y + \Delta^{(-)}$ results in the equation with the positive coefficients and roots, this may be checked by Sturm's method on $(0; +\infty)$. This shift leads to the matrix transformation $B \rightarrow (B - \Delta^{(-)}I)$.

There exists another way as alternative to shifting. If all the eigenvalues of a some matrix B are real numbers of arbitrary signs, then the following sequence of actions may be performed instead of shifting:

1) squaring B,

2) computing the squared eigenvalues,

3) choosing the signs of the eigenvalues by checking in the equation.

If all the roots of an algebraic equation are real positive numbers, then the theoretical value of its greatest root is in the explicit form.

Below, in the most general matrix form, we obtain maximal and minimal roots of an algebraic equation of any extent as limits.

$$\max\langle\mu_i\rangle = \lim_{\theta\to\infty} \sqrt[\theta]{det\ K^{(1)}/r}, \tag{20}$$

where $K^{(1)}$ *is the following* $(r+\theta)\times(r+\theta)$*-matrix of the equation coefficients:*

$$K^{(1)} =$$

$$= \begin{bmatrix} k_1 & -1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ -2k_2 & k_1 & -1 & \dots & 0 & 0 & \dots & 0 & 0 \\ 3k_3 & -k_2 & k_1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{r-1}rk_r & (-1)^{r-2}k_{r-1} & (-1)^{r-3}k_{r-2} & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & (-1)^{r-1}k_r & (-1)^{r-2}k_{r-1} & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & (-1)^{r-1}k_r & \dots & \dots & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & (-1)^{r-2}k_{r-1} & (-1)^{r-3}k_{r-2} & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & (-1)^{r-1}k_r & (-1)^{r-2}k_{r-1} & \dots & k_1 & -1 \\ 0 & 0 & 0 & \dots & 0 & (-1)^{r-1}k_r & \dots & -k_2 & k_1 \end{bmatrix}.$$

All zero elements of the matrix are only in the two triangles of sizes θ and $n+\theta-2$, i. e., for lower and upper ones, other elements are nonzero. Here $det\ K^{(1)} = S_\theta$ is the Waring sum of order θ (see above), according Waring–Le Verrier *reverse explicit formula* [21, p. 38].

By similar arguments and due to (9),

$$\min\langle\mu_i\rangle = \lim_{\theta\to\infty} \sqrt[-\theta]{det\ (K^{(2)}/k_n)/r},$$

where $K^{(2)}$ is the following $(r+\theta)\times(r+\theta)$-matrix of the same equation coefficients considered in the inverse form:

$$K^{(2)} =$$

$$= \begin{bmatrix} k_{r-1} & -k_r & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ -2k_{r-2} & k_{r-1} & -k_r & \dots & 0 & 0 & \dots & 0 & 0 \\ 3k_{r-3} & -k_{r-2} & k_{r-1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{r-1}r & (-1)^{r-2}k_1 & (-1)^{r-3}k_2 & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & (-1)^{r-1} & (-1)^{r-2}k_1 & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & (-1)^{r-1} & \dots & \dots & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & (-1)^{r-2}k_1 & (-1)^{r-3}k_2 & \dots & -k_r & 0 \\ 0 & 0 & 0 & \dots & (-1)^{r-1} & (-1)^{r-2}k_1 & \dots & k_{r-1} & -k_r \\ 0 & 0 & 0 & \dots & 0 & (-1)^{r-1} & \dots & -k_{r-2} & k_{r-1} \end{bmatrix}$$

By Sylvester's criterion, a symmetric or Hermitian matrix is positively definite iff all its principal minors are positive. The minor of the highest order is the determinant, so Sylvester's condition also means that the matrix is nonsingular. Besides, a singular symmetric or Hermitian matrix is positively semi-definite iff all its sign-alternating secular equation's coefficients up to order r are positive, and ones of orders $t > r$ are equal to 0, as all the roots here are real numbers. Thus the elements of normal matrices contain sufficient information for finding all the eigenvalues provided that all the roots of the secular equation are real numbers, and then the serial method is applicable.

Solvability of the same problem for more general matrices as well as the similar one for an arbitrary algebraic equation of degree $n > 4$ depends on the answer to the question: *whether a given algebraic equation has complex conjugate roots?* We showed above that the answer can be found by Sturm's method. However this method does not give necessary and sufficient conditions on the equation coefficients under which all the roots are real numbers and, due to shifting, positive.

One well known necessary condition follows from the Descartes sign Rule [21, p. 40]: all the coefficients of an equation in the sign-alternating form must be positive. Unfortunately even under this condition pairs of conjugate complex roots are possible. If the shift parameter is greater than noted above, for example, it is equal to $1 + \max|k_j|$, then only the real parts of the roots are necessarily positive [21, p. 39].

Inequalities (11) have the following corollary.

If all the roots of an algebraic equation are real positive numbers, then all its medians in $(10) - (13)$ *are equal to each other iff the equation has the binomial form*

$$(x - \mu)^n = 0.$$

This means also that $\overline{m_t} = \mu$.

If an equation in the sign-alternating form has at least two distinct roots, then its coefficients do not form the binomial sequence and then inequalities (11) do hold. For example, if there exist two adjacent medians equal to each other or some of the equation coefficients of order less than r are equal to zero, or the median hierarchy is violated, then there exist complex conjugate roots.

The following conditions are necessary and stronger than Descartes' one given above.

For all n the roots of an algebraic equation of degree n represented in the sign-alternating form to be positive real numbers it is necessary that all the equation coefficients-medians (5) *satisfy the following two conditions*:

(i) *they are positive real numbers*
(according to the Descartes sign Rule),
(ii) *all of n inequalities* (11) *do hold.*

For an $n \times n$-matrix to be positively definite it is necessary that all the matrix traces-medians (6) *of orders $1, 2, \ldots, n$ satisfy two conditions*:
(i) *they are positive real numbers,*
(ii') *first n inequalities* (10) *do hold.*

For any real algebraic equation and any real quadratic matrix condition (i) may be satisfied by use of shifting. For real symmetric or complex Hermitian matrix Sylvester's criterion gives the necessary and sufficient condition for all the roots of the secular equation (its eigenvalues) to be positive real numbers. If a real matrix is of the form AA', then all its eigenvalues are a priori real and nonnegative. The *necessary and sufficient conditions* for all the roots of an algebraic (polynomial) equation of degree n to be positive real numbers are inferred in our monograph [17, p. 165–191] with the use of the *Special diagrams*.

Note that for any algebraic median,

$$\sqrt[p]{\overline{m_i}(x_1^p + \cdots + x_n^p)} < \sqrt[q]{\overline{m_i}(x_1^q + \cdots + x_n^q)}$$

provided that

$$1 \le p < q, \qquad i = 1, \ldots, n-1,$$

there exist at least two distinct elements, and the quantity of the nonzero elements is greater than i. This follows from (10).

1.4 Structures of scalar and matrix characteristic coefficients

For a given square matrix B, its scalar characteristic coefficients of any order t may be represented according to (5) as the Viète sums of the eigenvalues μ_i. The eigenvalues are invariant under all linear transformations of the matrix and the bases; therefore, the scalar coefficients are invariant under such transformations too.

For any matrix B there exists a unique pair of matrices (P_B, O_B) such that P_B is a prime matrix, O_B is a nilpotent matrix, and

$$B = P_B + O_B. \tag{21}$$

The matrices P_B and O_B are determined by the Jordan form J_B or the triangle form of B.

As it is known, a matrix O is nilpotent iff all its scalar characteristic coefficients are equal to zero. Evaluate the nilpotency degree j of the matrix O_B. Let $j(i)+1$ be the maximal size of the Jordan subcell in J_B with the eigenvalue μ_i at the diagonal. Then

$$j = \max_{(\mu_i)}\{j(i)\}.$$

Not only O_B but also $O_B P_B$ and $P_B O_B$ are nilpotent matrices, and the matrices B and P_B have the same secular equation as well as the same eigenvalues with the same algebraic multiplicities. Thus the scalar coefficients for the matrix B possess the following additional properties:

$$k(P_B + O_B, t) = k(P_B, t) = k(B, t). \tag{22}$$

$$k(P_B \cdot O_B, t) = k(O_B \cdot P_B, t) = k(O_B, t) = 0. \tag{23}$$

From the structural point of view, any scalar coefficient $k(B,t)$ is the sum of all diagonal (principal) $t \times t$-minors of B [5, p. 78].

Further, consider most important properties of scalar and matrix characteristic coefficients, establish also the structure of the latters and all connections of them.

At first, resolvent's formula (1) is equivalent to each of the following identities:

$$det\ (B + \epsilon I)I = (B + \epsilon I)(B + \epsilon I)^V.$$

$$k_B(\epsilon)I = (B + \epsilon I)K_B(\epsilon), \tag{24}$$

$$\sum_{t=0}^{n} \epsilon^{n-t}\left[k(B,t)I - BK_1(B,t-1) - K_1(B,t)\right] = Z,$$

where Z is the zero matrix (all the polynomials are here in the constant-sign form with ϵ).

These formulae give, in particular, the following corollaries.

1. *The scalar parameter ϵ in* (24) *may be changed for a matrix one E commuting with B*:

$$k_B(E) = (B + E)K_B(E).$$

2. Along the way, with relation (24) at $E = -B$, through this graceful formula, we prove clear **in one line** the classic *Hamilton–Cayley Theorem*:

$$k_B(-B)I = (B - B)K_B(-B) \ \Rightarrow \ k_B(-B) = Z.$$

Contrary, if $E = +B$, then $k_B(B) = 2BK_B(B)$.

3. *The recurrent matrix formula of Jean-Marie Souriau* from 1948 [27]

$$K_1(B,t) = -BK_1(B,t-1) + k(B,t)I \tag{25}$$

is valid because the parameter ϵ in (24) is arbitrary. *The initial values*

$$k(B,0) = 1, \ K_1(B,0) = I$$

follow from (1). Note, that $k(B,1) = tr\ B, \ k(B,n) = det\ B$.

4. Define additionally *the matrix characteristic coefficients $K_2(B,t)$ of the 2-nd kind* as

$$K_2(B,t) = BK_1(B,t-1).$$

The initial value is $K_2(B,0) = Z$. Clarity, $K_2(B,1) = B$. Taking this into account, one may transform (25) into

$$K_1(B,t) + K_2(B,t) = k(B,t)I. \tag{26}$$

Repeating application of the recurrent formula (25) with the initial values leads to the following *representation of the matrix characteristic coefficients as polynomials in B*:

$$\left.\begin{array}{lcl} K_1(B,t) & = & \sum_{\theta=0}^{t} k(B,t-\theta)(-B)^{\theta}, \\ K_2(B,t) & = & -\sum_{\theta=1}^{t} k(B,t-\theta)(-B)^{\theta}. \end{array}\right\} \tag{27}$$

Hence, the matrix coefficients $K_1(B,t)$ and $K_2(B,t)$ commute with each other and with B.

5. *The Jean-Marie Souriau scalar binding formulae* [27] for both types of these coefficients

$$k(B,t) = \frac{1}{t} \cdot tr\ K_2(B,t), \ \ \{k(B,t) = \frac{1}{n-t} \cdot tr\ K_1(B,t)\} \tag{28}$$

follow from (27) and (2), i. e., using Le Verrier' method (see above).

6. In order to inverse non-singular matrix $B \to B^{-1}$ through its coefficients, J.-M. Souriau suggested in 1948 the very elegant algorithm with successive calculating all these coefficients of order $t \geq 1$. This algorithm was based on his formulae (25) and (28). Unfortunately, his article [20] in the Proceedings of the French Academy of Sciences was published as very brief paper, without details. Though he was then by very eminent French mathematician in many fields, in particular, he is well-known as a pioneer in symplectic geometry and as a high level analyst. The same results were repeated later, probably independently, by D. K. Faddeev [29], with a reference onto this Souriau's work. Faddeev's approach was based on the use of a resolvent B (1) for definition of matrix and scalar characteristic coefficients.

A year after this Souriau's publication, similar brief article from Frame J. S., with the same algorithm, was published in "AMS Bulletin", 1949, v. 55, n. 11, p. 1045 without a reference onto [27]. But the more unknown Frame became famous in American mathematical circles by this *single publication*, where it is cited, while Souriau's original article is for some reason entirely ignored, which is very strange from the point of view of scientific ethics!?
(P. S.: Maybe someday our readers will see and a "new author" of the Tensor Trigonometry?)

Further, the first formula in (27) and the Hamilton–Cayley Theorem lead to equalities

$$K_1(B,n) = k_B(-B) = Z,$$

and from (25) we infer: $K_2(B,n) = BK_1(B,n-1) = k(B.n)I = (det B)I = BB^V$.

If B is nonsingular, then multiplying these equalities by B^{-1} gives us the following:

$$B^{-1} = \frac{K_1(B,n-1)}{k(B,n)} = \frac{B^V}{det\ B}.$$

This is *the Souriau algorithmic method (and only his one!) for inverting a matrix and of joint computing all these characteristic coefficients* $k(B,t)$ *and* $K_1(B,t)$, $t = 1,\dots,n$.

7. Therefore, *all the values of the matrix coefficients* computed above are the following:

$$\left.\begin{array}{ll} K_1(B,0) = I, & K_2(B,0) = Z, \\ K_1(B,1) = (trB)I - B, & K_2(B,1) = B, \\ \dots & \dots \\ K_1(B,n-1) = B^V, & K_2(B,n-1) = (tr\ B^V)I - B^V, \\ K_1(B,n) = k_B(-B) = Z, & K_2(B,n) = (det\ B)I. \end{array}\right\} \quad (29)$$

The formulae of $K_1(B,n-1)$ and $K_2(B,n-1)$ are yet inferred only for non-singular matrices B, but they are true and for singular ones – see further in their structures.

Further, find *the greatest order* r'' *of the nonzero matrix coefficients* in (29). Due to (28) it is equal to the terminating order of the Souriau algorithm in formula (25). It does exist, due to (26) and (28), and is called here as *the* 2*-nd rock of the matrix* B. Moreover, $r'' \geq r'$, where r' is the greatest order of the nonzero scalar coefficients, i. e., the 1-st rock of the matrix B (see in sect. 1.1). If B is a nonsingular $n \times n$ matrix, then $r' = r'' = n$.

Inequality $r' < r$, where $r = rank\ B$, may be inferred only from *the structure of scalar coefficients*: they are *the sums of all diagonal minors of order* t. Similarly to it, only *the structures of matrix B coefficients* determine the 2-nd rock and its connection with other numerical parameters of the square matrix singularity (in that number, the annulling eigenvalues multiplicities in its minimal annulling polynomial) as well as its matrix characteristics, such as eigenprojectors, quasi-inverse matrices and modal matrices.

In order to clear *the structure of these matrix characteristic coefficients* of the 1-st and 2-nd kinds, we apply our *Special differential method* for establishing the structures of scalar and matrix coefficients simultaneously. (These complete structures was established by the author in the beginning of 1981 with introduction of both rocks for a matrix B.)

Although, for the scalar coefficients, the standard (direct) method for exploring their structure is well known (see, for example, in [5, p. 78]).

Consider an $n \times n$-matrix B and an arbitrary set of its m *generating elements* $\{b_{i_k,j_k},\ k = 1,\dots,m,\ 1 \leq m \leq n\}$, i. e., if $p \neq q$, then $i_p \neq i_q$ and $j_p \neq j_q$. The coefficient at $\prod_{k=1}^{m} b_{i_k,j_k}$ in expansion of $det\ B$ is

$$\frac{\partial^m det\ B}{\partial b_{i_1,j_1} \cdots \partial b_{i_m,j_m}} (-1)^{\sum_{k=1}^{m}(i_k+j_k)} \left\{ \frac{\begin{array}{c} i_1,\dots,i_m \notin \\ j_1,\dots,j_m \notin\ B \end{array}}{\text{minor(n-m)}} \right\}. \quad (30)$$

In the partial differentiation, the variables for all the elements of B are supposed to be distinct. The order of partial differentiation execution doesn't influence the end result. The minors of order $t = n - m$ in (30) is the adjunct of the minor, determined by the set of m generating elements; i_k, j_k are all their indexes of rows and columns.

Our pure differential formula (30) is the result of successive partial differentiating $det\ B$ with respect to $b_{i_1,j_1} \ldots, b_{i_m,j_m}$. Further, apply differential formula (30) for evaluating the resolvent of B in (1), i. e.,

$$(B + \epsilon I)^{-1} = \frac{(B + \epsilon I)^V}{det(B + \epsilon I)} = \frac{K_B(\epsilon)}{k_B(\epsilon)}.$$

Expand the numerator and the denominator in powers of ϵ.

The denominator is the scalar polynomial in ϵ of order n. According to (30) with $m = n - t$, the coefficient at $\prod_{k=1}^{n-t}(b_{i_k,I_k} + \epsilon)$ is

$$\left\{ \frac{(i_1, i_1) \ldots, (i_{n-t}, i_{n-t}) \notin}{\text{D-minor(t)}} (B + \epsilon I) \right\}.$$

It is the diagonal t-minor of $B + \epsilon I$ no containing indicated generating elements, the quantity of such minors (and multiplications) is C_n^t. Only diagonal entries of the minor contain ϵ. Put $\epsilon = 0$ in all these minors. We obtain the expression of the coefficient at ϵ^{n-t} in the scalar polynomial $det\ (B + \epsilon I)$ as *the sum of all its diagonal minors of order* t and its initial mean as $k(B, 0) = 1$.

The numerator is the following matrix. Its diagonal entries are polynomials in ϵ of degree $n - 1$, other entries are polynomials of degree $n - 2$. The matrix is represented by the following polynomial in ϵ:

$$(B + \epsilon I)^V = \sum_{t=0}^{n} K_1(B, t)\epsilon^{n-1-t}, \qquad K_1(B, 0) = I.$$

We wish to compute $K_1(B, t)$. For this aim it is necessary to consider the (p, p)- and (p, q)-entries of $(B + \epsilon I)^V$. Find the (p, p)-entry. It is equal to

$$\frac{\partial det\ (B + \epsilon I)}{\partial(b_{p,p} + \epsilon)} = Ad_{p,p}(B + \epsilon I) = \left\{ \frac{(p, p) \notin}{\text{D-minor(n-1)}} (B + \epsilon I) \right\},$$

where $Ad_{p,p}$ is the adjunct of the (p, p)-entry $b_{p,p} + \epsilon$. Similarly to arguments above, the coefficient at ϵ^{n-t-1} (as $n-t-1 = (n-1)-t = (n-(t+1))$ in expansion of this determinant is the (p, p)-entry of the matrix $K_1(B, t)$:

$$(p, p)K_1(B, t) = \sum_{(C_{n-1}^t \text{ terms})} \left\{ \frac{(p, p) \notin}{\text{D-minor(t)}} B \right\} =$$

$$= \sum_{(C_{n-1}^t \text{ terms})} Ad_{p',p'} \left\{ \frac{(p, p) \in}{\text{D-sub(t+1)}} B \right\}$$

(here D-sub stands for a diagonal $(t + 1) \times (t + 1)$-submatrix of B).

These are sums of D-minors. Both the sums consist of C_{n-1}^t terms, as one generating element, $b_{p,p}$, among n ones takes part in the first differentiation, i. e., in forming the first (main) adjunct. Here p' are the new indexes of the rows and the columns in D-minors of order $t + 1$.

Then find the (p,q)-entry of $(B+\epsilon I)^V$. It is equal to

$$\frac{\partial det\ (B+\epsilon I)}{\partial b_{q,p}} = Ad_{q,p}(B+\epsilon I) = (-1)^{p+q}\left\{\frac{(p,q)\in}{\text{Dh-minor(n-t)}}(B+\epsilon I)\right\},$$

(here Dh-minor stands for *hypodiagonal minor*).

It contains only one nondiagonal generating element, $b_{p,q}$, and thus, after the first partial differentiation with respect to $b_{q,p}$ (although the order of partial differentiation executions is of no importance) does not contain $b_{q,p}$, $b_{p,p}+\epsilon$, and $b_{q,q}+\epsilon$. Due to (30), the coefficient at $\prod_{k=1}^{n-t-1}(b_{i_k,i_k}+\epsilon)$ in expansion of the determinant is

$$\frac{\partial^{n-t-1}\left\{\frac{(p,q)\in}{\text{Dh-minor(n-t)}}(B+\epsilon I)\right\}}{\partial(b_{i_1,i_1}+\epsilon)\cdots\partial(b_{i_{n-t-1},i_{n-t-1}}+\epsilon)} = \frac{\partial\left[\frac{\partial^{n-t-1}det(B+\epsilon I)}{\partial(b_{i_1,i_1}+\epsilon)\cdots\partial(b_{i_{n-t-1},i_{n-t-1}}+\epsilon)}\right]}{\partial b_{q,p}} =$$

$$= Ad_{q',p'}\left\{\frac{(i_1,i_1),\ldots,(i_{n-t-1},i_{n-t-1})\notin}{\text{D-sub(t+1)}}(B+\epsilon I)\right\}.$$

Put here $\epsilon=0$, obtain the coefficient at ϵ^{n-t-1}, i. e. the (p,q)-entry of $K_1(B,t)$:

$$(p,q)K_1(B,t) = \sum_{(C_{n-2}^{t-1}\text{ terms})}(-1)^{p''+q''+1}\left\{\frac{(p,q)\in}{\text{Dh-minor(t)}}B\right\} =$$

$$= \sum_{(C_{n-2}^{t-1}\text{ terms})} Ad_{q',p'}\left\{\frac{(p,q)\in}{\text{D-sub(t+1)}}B\right\}.$$

Here D-sub stands for a diagonal $(t+1)\times(t+1)$-submatrix of B. Both the sums consist of C_{n-2}^{t-1} terms, as two generating elements $b_{p,q}$ and $b_{q,p}$ are used in forming the first (main) adjunct. The (p,q)-element has indexes p',q' in the diagonal minor and p'',q'' in the hypodiagonal one, $p'+q'=p''+q''+1$.

These two parts are the complete formula for $K_1(B,t)$. From it and formula (26), the expressions for $K_2(B,t)$ follow. *The structure of matrix coefficients is completely specified.* These structural properties of all the characteristic coefficients confirms formulae (29), (28), and, taking (27) into account, the Waring–Le Verrier recurrent formula (2).

Note the *corollary* of these transformations: for a quadratic matrix B, the adjunct of $b_{p,p}$ or $b_{p,q}$ may be interpreted as the partial derivative of $det\ B$ with respect to $b_{p,p}$ or $b_{p,q}$ according to (30), and conversely, the reverse operation, convolution of given adjuncts into $det\ B$, may be interpreted as their partial integrating on $b_{p,p}$ or $b_{p,q}$.

Compare the scalar and matrix coefficients structures. Both kinds of the coefficients are expressed with the use of minors sums. For scalar coefficients the summands are exactly all diagonal minors. Unlike them, the summands of matrix coefficients are diagonal minors and hypodiagonal ones, other minors cannot be the summands. Moreover, other r-minors can exist only under condition $1<r<n-1$. These facts specify *relationship between the 1-st and 2-nd rocks*, and also the rank of a matrix:

(1) $r'\le r''$ (see (28);

(2) $r'<r''\le r$ *if there exists a unique nonzero hypodiagonal minor of order* r'';

(3) $r'<r''<r$ *if there exists a unique nonzero minor of order* r'' *and this minor is not diagonal, nor hypodiagonal.*

Thus the structures of scalar and matrix characteristic coefficients specifies the following *fundamental inequalities for the principal singularity parameters of a square matrix*:

$$0\le r'\le r''\le r\le n. \qquad (31)$$

The structure of matrix characteristic coefficients in addition to the well-known structure of scalar coefficients (thanks to Le Verrier) was established by the author using his differential method (30) in early 1981, moreover with the use of resolvent (1) for the *limit specific introduction* of coefficients and eigenprojectors. (See about this on the author's web-site.)

The game of these three parameters within the boundaries allowed by inequality (31) determines the entire variety of singular matrices with the identification of their special and important special cases. Note the following special cases.

1. $r' = 0 \;\Leftrightarrow\;$ matrix B is nilpotent.
2. $r'' = 0 \;\Leftrightarrow\; B = Z$. (As well $r'' > 0$ iff $K_2(B,1) = B \neq Z$.)
3. $r = 1 \;\Leftrightarrow\; r'' = 1$. (By the same argument).
4. $r = n-1 \;\Leftrightarrow\; r'' = n-1$. ($K_1(B,n-1) = B^V$ contains all minors of rank $n-1$).

The value $t = r''$ is final in the Souriau algorithm [27]. The 1-st and 2-nd rocks are extremely important singularity parameters of a square matrix B and for eigenmatrices $B_i = B - \mu_i I$. The latters are always singular matrices. The 1-st and 2-nd rocks are invariant parameters under linear transformations as well as the rank and others. In particular, namely these principal parameters $r'_i \leq r''_i \leq r_i$ determine the exact and explicit formula for the minimal annulling polynomial of a given square matrix, as well as the cellular and subcellular structure of its Jordan form with all accompanying parametric equalities and inequalities, starting with the fundamental inequality (31) for the matrix itself and for all its eigenmatrices.

1.5 The minimal annulling polynomial of a matrix in explicit form

As it is well known – see, for instance, in [3, 4], the exact, but non-explicit formula for the minimal annulling polynomial of a matrix B was traditionally determined only after the mathematical operation of reduction with the decrease of the degrees of all the eigenmatrices in its characteristic polynomial $k_B(-B) = Z$ up to the minimum values. As a result, it does not give the explicit connection for these minimum degrees of eigenmatrices with the above principal parameters of eigenmatrices r'_i, r''_i, r_i, determined exactly and explicitly by the elements of the matrix B, based on the structures of its scalar and matrix characteristic coefficients.

The results obtained above enable us to express the minimal annulling polynomial explicitly in terms of basic singularity parameters of a matrix B.

Consider a singular $n \times n$-matrix B of rank r and its eigenvalues μ_i with algebraic multiplicities $s_i = n - r'_i$ $(i = 1, \ldots, q)$, $\mu_1 = 0$ (in the sequel, we omit the index $i = 1$ of a singular matrix parameters), for example, any *eigenmatrix* $B_i = B - \mu_i I$. From (27), the Hamilton-Cayley Theorem, with use of prime factorization and with replacement of the scalar coefficients by the Viète sums, as in (5), we have

$$K_1(B,n) = \sum_{t=0}^{n} (-B)^{n-t} k(B,t) = (-B)^{s'} \sum_{t=0}^{r'} (-B)^{r'-t} k(B,t) =$$

$$= (-B)^{s'} K_1(B,r') = (-B)^{s'} \prod_{i=2}^{q} (\mu_i I - B)^{s'_i} = Z \qquad (32)$$

This is the *annulling characteristic polynomial* in B.

From the other hand, each characteristic coefficient of order r' is nonzero, that is why

$$K_1(B,r') = \prod_{i=2}^{q} (\mu_i I - B)^{s'_i} \neq Z, \quad k(B,r') = \prod_{i=2}^{q} \mu_i^{s'_i} \neq 0. \qquad (33)$$

The recurrent Souriau formula (25) in the interval $r' < t \leq r''$ gives us the *nilpotent matrix coefficients*

$$K_1(B,t) = (-B)^{t-r'} K_1(B,r') = -K_2(B,t) \neq Z. \tag{34}$$

Further, if $t = r'' + 1$, then

$$K_1(B,r''+1) = (-B)^{r''-r'+1} K_1(B,r') = (-B)^{r''-r'+1} \prod_{i=2}^{q} (\mu_i I - B)^{s'_i} =$$

$$= Z = (-B)^{r''-r'+1} \prod_{i=2}^{q} (\mu_i I - B)^{s_i^0} = (-B)^{s^0} \prod_{i=2}^{q} (-B_i)^{s_i^0}, \tag{35}$$

where each s_i^0 is the exponent of the eigenmatrix B_i in a minimal annulling polynomial, it is called the *annulling multiplicity* of μ_i.

From (35) and condition in (34)

$$(-B)^{r''-r'} \prod_{i=2}^{q} (\mu_i I - B)^{s_i^0} \neq Z.$$

We obtain the **main result** – *formulae for the annulling multiplicities* of $\mu_1 = 0$ and consequently of all μ_i of the eigenmatrices B_i in the minimal annulling polynomial:

$$\boxed{s^0 = r'' - r' + 1, \qquad s_i^0 = r''_i - r'_i + 1.} \tag{36}$$

The annulling multiplicities satisfy the classic inequalities $1 \leq s_i^0 \leq s'_i$ [4, p. 24] due to $r'_i \leq r''_i$ and (32). Replace s_i^0 in the classic inequalities by their values (36), obtain the weak inequality $r''_i \leq n-1$. Therefore the classic inequalities may be strengthened, the upper bound is more precise:

$$1 \leq s_i^0 \leq r_i - r'_i + 1 \leq s'_i. \tag{37}$$

Now we can see that expressing the unknown 2-nd rock in terms of the given s_i^0 from (36) can not lead to restriction $r'' \leq r$. That is why the 1-st and the 2-nd rocks are the primary parameters of a singular matrix, while the annulling multiplicity is the secondary notion.

The upper bound in (37) is attained when $r''_i = r_i$, in that number if $r''_i = n - 1 = r_i \geq r'_i$.

Find condition for attaining the lower bound in (37), i. e. for equality $r'_i = r''_i$. Take advantage of the classic Sylvester Inequality [21, p. 394]:

$$\min(r_1, r_2) \geq rank\ (C_1 C_2) \geq r_1 + r_2 - n.$$

If $k \geq 2$ matrices are multiplied (or a power of a matrix is analysed), their singularities are more suitable than the ranks. Then the following *two inequalities in general forms* are expressed in terms of its factors singularities briefly and do not depend on n:

$$\max(sing\ C_i) \leq sing \prod_{i=1}^{k} C_i \leq n, \quad sing \prod_{i=1}^{k} C_i \leq \sum_{i=1}^{k} sing\ C_i, \tag{38}$$

$$sing\ C \leq sing\ C^h \leq n, \quad sing\ C^h \leq h \cdot sing\ C, \tag{39}$$

where h is an arbitrary positive integer.

The upper bounds in right inequalities of (38) are attained if the following two conditions do hold together:
(*i*) $\langle ker\ C_i\rangle \oplus \langle im\ C_i\rangle \equiv \langle \mathcal{A}^n\rangle$,
(*ii*) $\langle ker\ C_i\rangle \subset \langle\ im\ \prod_{j=i+1}^{k} C_j\rangle,\ i = 1, \ldots, k-1.$

They seem sufficiently clear and are useful in further considerations. In particular, if C_i are the eigenmatrices, then their powers pairly commute and conditions above are transformed into

$$\langle ker\ B_i^{h_i}\rangle \cap \langle ker\ B_j^{h_j}\rangle = \langle \mathbf{0}\rangle,\ i \neq j.$$

Then, due to (38) and conditions $(i), (ii)$, for all $h_i \geq s_i^0$ there holds

$$n = sing\left(\prod_{1\leq i\leq q,\ h_i\geq s_i^0} B_i^{h_i}\right) = sing\ Z = \sum_{1\leq i\leq q,\ h_i\geq s_i^0} sing\ B_i^{h_i}.$$

From the other hand, $rank\ B_i^{h_i} \geq r_i'$ (and this is equivalent to $sing\ B_i^{h_i} \leq s_i'$) as the algebraic multiplicity and the 1-st rock are invariant under powering a matrix. Consequently, due to $\sum_{i=1}^{q} s_i' = n$, we obtain

$$\left.\begin{array}{l} sing\ B_i^{h_i} < s_i' \text{ iff } h_i < s_i^0, \\ sing\ B_i^{h_i} = s_i' \text{ iff } h_i \geq s_i^0. \end{array}\right\} \tag{40}$$

The value $s = n - r$ is the geometric multiplicity. In particular, $sing\ B^{s^0} = s'$, $sing\ B_i^{s_i^0} = s_i'$. This fact and (39) lead to the following special inequalities:

$$\left.\begin{array}{l} s_i^0 s_i \geq s_i' \ \ (s_i^0 \leq s_i' \text{ and } s_i \leq s_i'), \\ s^0 s \geq s' \ \ (s^0 \leq s' \text{ and } s \leq s'). \end{array}\right\} \tag{41}$$

The set $\langle sing\ B_i^{h_i}\rangle$ as well as the set $\langle rank\ B_i^{h_i}\rangle$ determines [4, p. 143] the set of the Jordan subcells in the ultrainvariant $s_i' \times s_i'$-cell, and the critical exponent of the matrix in (40) determines the maximal size of the Jordan $s_i^0 \times s_i^0$-subcell.

If $s_i^0 = 1$ (it is equivalent to $r_i' = r_i''$), then, due to (41), $s_i = s_i'$. Conversely, if $s_i = s_i'$, then $r_i' = r_i'' = r_i$. Thus, for lower boundaries of s_i^0 there holds:

$$\left.\begin{array}{l} s_i^0 = 1 \ \Leftrightarrow\ r_i' = r_i'' \ \Leftrightarrow\ r_i' = r_i, \\ s^0 = 1 \ \Leftrightarrow\ r' = r'' \ \Leftrightarrow\ r' = r. \end{array}\right\} \tag{42}$$

For example, the following fact is well known:

$$s_i^0 = 1,\ i = 1, \ldots, q, \ \Leftrightarrow\ s_i = s_i',\ i = 1, \ldots, q \ \Leftrightarrow\ B \in \langle P\rangle.$$

The Jordan form J_B is used for inferring them [4, p. 143], however it immediately follows from (42), if to let $i = 1, \ldots, q$.

On the other hand, for upper boundaries of s_i^0 there holds:

$$s_i^0 = s_i' \ \Leftrightarrow\ s_i = 1 \ \Leftrightarrow\ r_i'' = n - 1 = r_i. \tag{43}$$

They are determined by (41).

So, the theory of minimal annuling polynomial is exposed more completely, and this polynomial is expressed in explicit form due to results obtained in the previous section.

1.6 Null-prime and null-defective singular matrices

A singular matrix is called *null-prime* if its 1-st rock is equal to its rank. We shall use notation Bp for null-prime matrices if necessary.

Of the fact above follows: if B is null-prime, then B' is null-prime. Obviously, for the eigenspace corresponding to its eigenvalue zero holds $\langle ker\ Bp\rangle \equiv \langle ker\ (Bp)^h\rangle$. In this subspace, the matrix Bp behaves as a prime one. Indicate more widely the properties and definitions of Bp.

The following assertions are equivalent:

(i) *a square matrix B of rank r is null-prime,*

(ii) $r' = r''$,

(iii) $r' = r$,

(iv) $rank(B^2) = r$,

(v) $\langle ker\ B\rangle \cap \langle im\ B\rangle \equiv \langle \mathbf{0}\rangle$,

(vi) $\langle ker\ B\rangle \cup \langle im\ B\rangle \equiv \langle ker\ B\rangle \oplus \langle im\ B\rangle \equiv \langle \mathcal{A}^n\rangle$.

Due to (vi), any null-prime matrix possesses the characteristic affine projectors in the linear spaces.

A square matrix B is called *null-defective* if $r' < r$ (its 1-st rock $r' = rank\ B^{s^0}$ also is the minimal value of $rank\ B^h$). According to (35), for a null-defective matrix B, there exists the characteristic nilpotent matrix

$$O_1 = \{K_1(B^{s^0}, r'_B)/k(B^{s^0}, r'_B)\}B,\ O_1^{s^0} = Z,\ [(I \pm O_1)^{s^0} - I]^{s^0} = Z, \tag{44}$$

where all the matrices commute with each other as polynomials in B (see in details in the next sect. 2.2).

The nilpotent matrix O_B in (21) is, in its turn, the sum of all the eigenmatrices $O_1, \dots, O_q$.

The parameters of the nilpotent matrix for a null-defective matrix B are the following:

$$r' = 0, \qquad r'' = s^0 - 1,$$

where s^0 is the nilpotency degree, and

$$s^0 - 1 = r'' \le rank\ O_1 \le n[r''/(r''+1)] = n[(s^0-1)/s^0] \le n - 1, \tag{45}$$

$$0 \le rank\ O_1 \le r, \qquad n - s^0(n - r) \le rank\ O_1. \tag{46}$$

Inequalities (45), (46) follow from (39). More precise bounds for the parameters

$$(n-1) - (s'_i - s^0_i) \le r_i \le n - 1, \tag{47}$$

$$s_i \le s'_i - (s^0_i - 1), \quad s^0_i \le s'_i - (s_i - 1) \tag{48}$$

follow from (37).

In matrix Jordan form (see [10, part 2]), the value $s^0_i - 1 = r''_i - r'_i$ is the maximal quantity of nonseparated units in the adjacent diagonal of the i-th ultrainvariant $s'_i \times s'_i$-cell. The total number of units in the cell is $s'_i - s_i = r_i - r'_i$. This gives the sense to estimations (47) and (48), and the notions of the 1-st and 2-nd rock.

Inequality (41) may be interpreted in terms of the Jordan form too, namely, by the following arguments. The adjacent diagonal of the matrix Jordan form contains, as well known, only units and zeros; moreover, k nonseparated units in it correspond to the Jordan subcell of size $k+1$. Among them there exists a subcell (may be not unique) of the maximal size s_i^0. Consider this $s_i^0 \times s_i^0$-subcell and add to the end of its array of units one zero element (outside the subcell). When s_i^0 is fixed, the total number of units in the adjacent diagonal takes the maximal value if its partition into segments is almost uniform: all the segments (but may be one) are of length s_i^0, and the last segment may be shorter, its length is equal to the nonzero remainder of division s_i' by s_i^0. Each segment ends with a zero, all other its elements are units. Therefore,

$$\min s_i = \lfloor s_i'/s_i^0 \rfloor$$

and the equality in (41) holds iff s_i'/s_i^0 is an integer. Inequalities (41) are equivalent to each of the following:

$$(n - r_i)(r_i'' - r_i') \geq r_i - r_i', \tag{49}$$

$$r_i' + [(s_i' - s_i)/s_i] \leq r_i'' \leq (n-1) - (s_i' - s_i^0)/s_i^0]. \tag{50}$$

Hence estimations (41), (49), (50) for r'' and s_i^0 are effective only under condition $r_i'' < r$. In this case, we obtain

$$s_i < s_i', \quad s_i^0 < s_i', \quad s_i' > 3, \quad s_i > 2, \quad s_i^0 > 1, \quad n > 3.$$

The parameter $r_i - r_i''$ is called *the i-th different* of a matrix. A defective matrix is called *null-different* if $r'' < r$. The maximal value of the different (particular and total) is $(\sqrt{n}-1)^2$, it is less than $n-3$. This follows from (49). The different is maximal if the integer n is a square. In this case,

$$r = n - \sqrt{n}, \quad r'' = \sqrt{n} - 1, \quad r' = 0, \quad q = 1.$$

Due to (49), the matrix B is *null-indifferent* in the following special cases:

$$\left.\begin{array}{ll} (i) & r_i = 1 \ (\Leftrightarrow r_i'' = 1); \\ (ii) & r_i = 2 \ (\Leftrightarrow r_i'' = 2); \\ (iii) & n \leq 3; \\ (iv) & s_i' \leq 3. \end{array}\right\} \tag{51}$$

Therefore, the different is zero if the dimension of the whole space or the dimension of the ultrainvariant space does not exceed 3. This may be useful for constructing the minimal annulling polynomial in terms of the ranks. Note the sense of condition (ii): units in the adjacent diagonals of the Jordan cells cannot be separated by zeros.

A singular square matrix B is null-indifferent iff

$$rank\ B^h = rank\ B^{h-1} - 1, \qquad h = 2, 3, \ldots, s^0$$

($rank\ B^{s^0} = r'$ is minimal).

Null-prime and null-defective matrices as well as prime and defective ones according to their definitions are pure affine notions. But they relate only to the eigenvalue zero of singular matrices, in particular, of the eigenmatrices $B_i = B - \mu_i I$. For the definition, it is not meaning, the matrix is real-valued or complex-valued one.

These notions are important especially in theory of eigenprojectors connected with given singular matrix B, and in its numerous applications. One of them is spectral decomposition of a matrix B up to its invariant and ultrainvariant subspaces for each eigenvalue μ_i, with reducing original matrix into the basic canonical form or only into the null-cell form (see in the next sect. 2.3).

Further, we shall often deal with matrices-multiplications of the types $B = A_1 A_2'$ and $B' = A_2 A_1'$, where A_1 and A_2 are $n \times m$-matrices set certain geometric objects in a n-dimensional affine or metric space. In the case, angular geometric relations between these objects in the space are determine the matrix-multiplication B as a null-normal one or a null-defective one.

It is clear that in the minimal polynomial of a prime matrix P, all the eigenmatrices $P_i = P - \mu_i I$ are null-normal ones, and all they have powers 1 in it. However in the minimal polynomial of a defective matrix B, some of its eigenmatrices $B_i = B - \mu_i I$ are null-defective ones, and they have powers $s_i^0 > 1$ in it. Then $B_i^{s_i^0}$ became by null-normal matrix with this minimal power.

1.7 The reduced form of characteristic coefficients

We conclude the chapter with evaluating all the characteristic coefficients of a given matrix B in so called *reduced form*, where the fraction numerator and denominator in (1) are polynomials in $\epsilon = -\mu$ of the least degree. This reduced form is obtained through dividing by the greatest common divisor the numerator and the denominator. The similar method for computing the minimal annulling polynomial of a matrix is well known – see, for example, in [4, p. 123].

Dividing the numerator and the denominator of fraction (1) by their greatest common divisor leads to reducing the Hamilton–Cayley zero polynomial as well as all the characteristic coefficients, their connection formulae, and the Souriau algorithm [27] (see above). Reducing in (24) yields the reduced analogues of the scalar and matrix characteristic polynomials $k_B(\epsilon)$ and $K_B(\epsilon)$ from (1):

$$q_B(\epsilon)I = (B + \epsilon I)Q_B(\epsilon). \tag{52}$$

These reduced polynomials have also the reduced scalar and matrix characteristic coefficients $q(B,t)$ and $Q_1(B,t)$, where t is the order of these coefficients:

$$q_B(\epsilon) = \sum_{t=0}^{n^0} q(B,t)\epsilon^{n^0-t},$$

$$Q_B(\epsilon) = \sum_{t=0}^{n^0-1} Q_1(B,t)\epsilon^{n^0-t-1}.$$

As well as (24), formula (52) is valid also for the matrix parameter E, and in special case $E = -B$ it gives the matrix minimal annulling polynomial of $E = -B$ (where scalar one depends on $\epsilon = -\mu$).

From here, the reduced Hamilton–Cayley Theorem and the reduced secular equation:

$$q_B(-B) = Q_1(B, n^0) =$$

$$= \sum_{t=0}^{n^0} q(B,t)(-B)^{n^0-t} = \prod_{i=1}^{q} (\mu_i I - B)^{s_i^0} = Z, \tag{53}$$

$$q_B(-\mu) = \sum_{t=0}^{n^0} q(B,t)(-\mu)^{n^0-t} = \prod_{i=1}^{q} (\mu_i - \mu)^{s_i^0} = 0. \tag{54}$$

Thus n^0 is the order of the minimal annulling polynomial (53). Reducing results in only those parts of (53), (54) that do not contain μ_i and s_i^0. The values μ_i and s_i^0 are determined by solving the secular equation in (54).

When these values are known, the reduced Viète theorem

$$q(B,t) = \sum_{(C^t_{n^0} \text{ terms })} \prod_{(t \text{ values})} \mu_i \qquad (q \le n^0 = \sum_{i=1}^{q} s_i^0 \le n) \tag{55}$$

follows from (53). This leads to reducing (25)–(29). In the reduced Souriau algorithm [27] (see above), the initial values are as usually, but further computations use the reduced trace, etc.:

$$Q_1(B,t) = I, \quad Q_2(B,0) = Z, \quad Q_2(B,1) = B,$$

and $q(B,1) = \sum_{i=1}^{q} s_i^0 \mu_i$ is the matrix B reduced trace. The reduced determinant is

$$q(B, n^0) = \prod_{i=1}^{q} \mu_i^{s_i^0}.$$

The inverse nonsingular matrix is

$$B^{-1} = Q_1(B, n^0 - 1)/q(B, n^0).$$

Note that quantity of the eigenvalues decreases up to n^0.

The highest coefficients of the eigenmatrices $B_i = B - \mu_i I$ as functions of μ_i have the following reduced form:

$$Q_1(B_i, r_i^0) = \prod_{j=1}^{q} (\mu_j I - B)^{s_j^0}, \quad q(B_i, r_i^0) = \prod_{j=1}^{q} (\mu_j - \mu_i)^{s_j^0}, \ j \ne i, \tag{56}$$

where $r_i^0 = n^0 - s_i^0$ is the reduced 1-st rock. The second rock is equal to $n^0 - 1$ after reducing. Particular reducing (of the fixed eigenvalue μ_i quantity) is equal to $s_i' - s_i^0 = (n-1) - r_i''$, the total reducing (for all μ_i) is $n - n^0$.

The sum of the basic particular parameters satisfies inequalities

$$nq - 1 = \sum_{i=1}^{q} r_i' \le \sum_{i=1}^{q} r_i'' \le \sum_{i=1}^{q} r_i \le nq - q.$$

If the matrix is prime ($B \in \langle P \rangle$), then

$$n^0 = q, \quad s_i^0 = 1, \quad q(P^h, 1) = \sum_{i=1}^{q} \mu_i^h, \quad q(P^h, n^0) = q^n(P, n^0) = \left(\prod_{i=1}^{q} \mu_i \right)^n.$$

And the coefficients for its eigenmatrices are

$$Q_1(P_i, n^0 - 1) = \prod_{j=1}^{q}(\mu_j I - P), \quad q(P_i, n^0 - 1) = \prod_{j=1}^{q}(\mu_j - \mu_i), j \neq i. \tag{57}$$

Note, the general spectral representation of a matrix (see in sect. 2.2) may apply the minimal annulling polynomial and, perhaps, other types of annulling polynomials, for example, these:

$$\prod_{j=1}^{q}(\mu_j I - B)^{\max s_j^0} = \prod_{j=1}^{q}(-B_j)^{\max s_j^0} = Z, \tag{58}$$

$$\prod_{j=1}^{q}(\mu_j^I - B)^{\max s_j'} = \prod_{j=1}^{q}(-B_j)^{\max s_j'} = Z. \tag{59}$$

Here the matrix $(-B_j)$ powers are null-prime matrices too.

These reduced forms of exact matrices scalar and matrix characteristic coefficients are important, of course, from the theoretical point of view. They demonstrate in some extent similarity between the Number theory and the Matrix algebra. In both cases, we deals with cancellation of greatest common divisor, but here it is as scalar or as matrix polynomial. Progenitor of such procedure with this divisor is the most ancient algorithm of Euclid in the Number theory!

In this first chapter, we have dealt with a lot of theoretical aspects, which were needing in more detailed consideration and studying. Contrary, from the practical point of view, its most valuable results are the general inequality of all means, the serial limit method and limit formulae of solving algebraic equations on it's basis, the found structure and properties of the matrix characteristic coefficients, the explicit form of the minimal annuling polynomial of a square matrix, the fundamental inequality for basic singularity parameters of a square singular matrix with their dependence on structure of the scalar and matrix characteristic coefficients, and an introduction of the very useful null-prime matrices with their unical properties. Particular attention was paid to higher order coefficients of a singular matrix (note that all the eigenmatrices of an arbitrary square matrix $B_i = B - \mu_i I$ are always singular ones).

All these new relationships and notions for a square matrix will be used in subsequent theoretical and practical considerations. So, the results of the chapter give new opportunities for inferring explicit formulae expressing eigenprojectors and modal matrices in terms of the scalar and matrix characteristic coefficients. This advantage is used widely in development of tensor trigonometry in further divisions of the book.

Chapter 2

Affine and orthogonal eigenprojectors

2.1 Affine (oblique) projectors and quasi-inverse matrix

Let $\langle\mathcal{A}^n\rangle$ be an affine n-dimensional space. Suppose Bp is a null-prime matrix of rank r, then $k(Bp,r) \neq 0$. Formula (26) is transforming into

$$K_1(Bp,r)/k(Bp,r) + K_2(Bp,r)/k(Bp,r) = \overrightarrow{Bp} + \overleftarrow{Bp} = I. \quad (60)$$

Further $\overrightarrow{Bp}$ and $\overleftarrow{Bp}$ stand for the so-called *affine eigenprojectors* of Bp. These projectors are also idempotent matrices (in general case, they are non-symmetric). In the Euclidean space they are also the *oblique eigenprojectors* in the metric sense. We claim that in the affine space $\overleftarrow{Bp}$ is a projector into the image $\langle im\ Bp\rangle$ parallel to the kernel $\langle ker\ Bp\rangle$, and $\overrightarrow{Bp}$ is a projector into $\langle ker\ Bp\rangle$ parallel to $\langle im\ Bp\rangle$. Indeed,

$$K_2(Bp,r) = BpK_1(Bp,r-1) = K_1(Bp,r-1)Bp;$$

$$\overrightarrow{Bp} + \overleftarrow{Bp} = I, \quad \overrightarrow{Bp}\cdot\overleftarrow{Bp} = \overleftarrow{Bp}\cdot\overrightarrow{Bp} = Z;$$

$$(\overrightarrow{Bp})^2 = \overrightarrow{Bp}(I - \overleftarrow{Bp}) = \overrightarrow{Bp}, \quad (\overleftarrow{Bp})^2 = \overleftarrow{Bp}(I - \overrightarrow{Bp}) = \overleftarrow{Bp};$$

$$\langle ker\ Bp\rangle \oplus \langle im\ Bp\rangle = \langle\mathcal{A}^n\rangle, \quad \mathbf{x} = \overrightarrow{Bp}\mathbf{x} + \overleftarrow{Bp}\mathbf{x} = \overrightarrow{\mathbf{x}} + \overleftarrow{\mathbf{x}}.$$

Any element $\mathbf{x}$ is uniquely decomposed into the sum of its projections in $\langle\mathcal{A}^n\rangle$ as above. Therefore,

$$\overrightarrow{Bp} = K_1(Bp,r)/k(Bp,r), \quad (61)$$

$$\overleftarrow{Bp} = K_2(Bp,r)/k(Bp,r) =$$

$$= BpK_1(Bp,r-1)/k(Bp,r) = K_1(Bp,r-1)Bp/k(Bp,r). \quad (62)$$

The matrix Bp and both its eigenprojectors commute with each another as polynomials in Bp (compare with formula (27)). In particular, for a scalar we get:

$$\overrightarrow{a} = 0, \quad \overleftarrow{a} = 1$$

and in some other trivial cases,

$$\overrightarrow{Z} = I, \quad \overrightarrow{I} = Z;$$

$$\left.\begin{aligned} \langle im\ K_1(Bp,r)\rangle \equiv \langle ker\ K_2(Bp,r)\rangle \equiv \langle ker\ Bp\rangle, \\ \langle ker\ K_1(Bp,r)\rangle \equiv \langle im\ K_2(Bp,r)\rangle \equiv \langle im\ Bp\rangle; \end{aligned}\right\} \quad (63)$$

$$rank\ K_1(Bp,r) = sing\ Bp, \quad rank\ K_2(Bp,r) = rank\ Bp; \quad (64)$$

$$\left.\begin{aligned} \overrightarrow{(Bp')} = (\overrightarrow{Bp})', \ \overleftarrow{(Bp')} = (\overleftarrow{Bp})', \\ \overrightarrow{\overleftarrow{Bp}} = \overleftarrow{\overrightarrow{Bp}} = \overleftarrow{Bp}, \ \overleftarrow{\overleftarrow{Bp}} = \overrightarrow{\overrightarrow{Bp}} = \overrightarrow{Bp}; \\ k(\overrightarrow{Bp},t) = C^t_{n-r}, \quad k(\overleftarrow{Bp},t) = C^t_r. \end{aligned}\right\} \quad (65)$$

For singular matrices Bp $(r = r')$, we have (as generalization of $\det B^h = \det^h B$):

$$k(Bp^h, r) = k^h(Bp, r) \neq 0; \tag{67}$$

$$K_j((Bp)^h, r) = K_j^h(Bp, r) = k^{h-1}(Bp, r)K_j(Bp, r), \; j = 1, 2. \tag{68}$$

In an affine space, the *affine quasi-inverse matrix* for a matrix Bp is the following:

$$Bp^- = \overleftarrow{Bp}[K_1(Bp, r-1)/k(Bp, r)] = [K_1(Bp, r-1)/k(Bp, r)]\overleftarrow{Bp}$$
$$= Bp[K_1(Bp, r-1)/k(Bp, r)]^2 = [K_1(Bp, r-1)/k(Bp, r)]^2 Bp. \tag{69}$$

It commutes with Bp and in the subspace $\langle im\ Bp\rangle$ it behaves as an usual inverse matrix, in $\langle ker\ Bp\rangle$ it plays the role of the zero matrix. It is uniquely determined by equations

$$Bp^- Bp = BpBp^- = \overleftarrow{Bp}, \quad Bp^- = \overleftarrow{Bp}Bp^- = Bp^-\overleftarrow{Bp}. \tag{70}$$

The following formulae hold:

$$rank\ Bp^- = rank\ Bp;$$
$$\langle im\ Bp^-\rangle \equiv \langle im\ Bp\rangle, \quad \langle ker\ Bp^-\rangle \equiv \langle ker\ Bp\rangle;$$
$$BpBp^- Bp = Bp; \quad Bp^- BpBp^- = Bp^-;$$
$$(Bp^-)^- = Bp; \quad (Bp^h)^- = (Bp^-)^h; \quad (Bp')^- = (Bp^-)'.$$

Moreover,

$$B^- = B^{-1} \Leftrightarrow det\ B \neq 0.$$

Due to (1), (61), (62), and (69), the affine eigenprojectors and the quasi-inverse matrix are represented as limits

$$\overrightarrow{Bp} = \lim_{\epsilon \to 0} [\epsilon(Bp + \epsilon I)^{-1}] = \lim_{N \to \infty} (NBp + I)^{-1}, \tag{71}$$

$$\overleftarrow{Bp} = \lim_{\epsilon \to 0} [Bp(Bp + \epsilon I)^{-1}] = \lim_{N \to \infty} [NBp(NBp + I)^{-1}], \tag{72}$$

$$Bp^- = \lim_{\epsilon \to 0} [Bp(Bp + \epsilon I)^{-2}] = \lim_{N \to \infty} [(N^2 Bp(NBp + I)^{-2}] \tag{73}$$

$$(\overrightarrow{Bp} + \overleftarrow{Bp} = I, \quad Bp^- Bp = BpBp^- = \overleftarrow{Bp}, \quad N = 1/\epsilon).$$

These limit formulae have most common *affine form*. They are gotten here by the algebraic manner using a resolvent of Bp (see also in sect. 3.4).

2.2 Spectral presentation of $n \times n$-matrix with basic canonical form

In all ultrainvariant subspaces (their sums are direct), the affine eigenprojectors (61) of a prime matrix P may be represented, due to (57), by two manners as follows:

$$\overrightarrow{P_i} = \frac{K_1(P_i, r_i)}{k(P_i, r_i)} = \frac{Q_1(P_i, r^0)}{q(P_i, r^0)} = \prod_{1 \le j \le q, \; j \neq i} \frac{\mu_j I - P}{\mu_j - \mu_i}, \tag{74}$$

where $r^0 = n^0 - 1 = q - 1$ (see the last manner also, for example, in [4, p. 156]).

The affine projectors of a non-prime (defective) matrix B are represented according to (61), (33), (56), and (58)–(60), by two different manners as follows:

$$\overrightarrow{Bp_{(i)}} = \frac{K_1(B_i, r_i')}{k(B_i, r_i')} = \frac{Q_1(B_i, r_i^0)}{q(B_i, r_i^0)} =$$

$$= \prod_{1 \le j \le q,\ j \ne i} \frac{(\mu_j I - B)^{s_j^0}}{(\mu_j - \mu_i)^{s_j^0}} = \prod_{1 \le j \le q,\ j \ne i} \frac{(\mu_j I - B)^h}{(\mu_j - \mu_i)^h} = \overrightarrow{(B_i^h)}, \tag{75}$$

where $Bp_{(i)} = B_i^{s_i^0}$, $h \ge \max s_i^0$ (see the last manner, for example, in [11, p. 128–143]).

Note, that eigenmatrices $P_i = P - \mu_i I$ and the power matrices

$$B^h, h \ge s^0, \quad B_i^{h_i},\ h_i \ge s_i^0,$$

are trivial special cases of null-prime singular matrices Bp.

Spectral representation of a non-prime matrix B up to its *ultrainvariant eigen subspaces* corresponding to each μ_i determines decomposition of the matrix B into the unique sum of two its characteristic matrices – prime one and nilpotent one (see before (21) and (44)):

$$B = B\sum_{i=1}^{q} \overrightarrow{Bp_{(i)}} = \sum_{i=1}^{q} \mu_i \overrightarrow{Bp_{(i)}} + \sum_{i=1}^{q} B_i \overrightarrow{Bp_{(i)}}$$

$$= \sum_{i=1}^{q} P_i + \sum_{i=1}^{q} O_i = P_B + O_B. \tag{76}$$

Note, $O_B^h = Z$ if $h \ge \max s_i^0$. This may be interpreted by the Jordan form.
In order to construct the canonical q-block-diagonal form of the matrix [4, p. 130], the modal matrix of transformation may be evaluated with use of the following coefficients (proportional to eigenprojectors), accordingly, due to (33) or theoretically to (56):
$K_1(B_i, r_i') = \prod_{1 \le j \le q,\ j \ne i} (\mu_j I - B)^{s_j'}$, $Q_1(B_i, r_i^0) = \prod_{1 \le j \le q,\ j \ne i} (\mu_j I - B)^{s_j^0}$.

Then

$$\left.\begin{array}{l} \langle im\ K_1(B_i, r_i')\rangle \equiv \langle im\ Q_1(B_i, r_i^0)\rangle \equiv \langle ker\ B_i^{s_i^0}\rangle, \\ \langle ker\ K_1(B_i, r_i')\rangle \equiv \langle ker\ Q_1(B_i, r_i^0)\rangle \equiv \langle im\ B_i^{s_i^0}\rangle. \end{array}\right\} \tag{77}$$

For a prime matrix P, the coefficients are simplified according to $r_i' = r_i$, or due to (57).

All the coefficients are null-prime matrices. However, such matrices have nonzero scalar coefficients of the highest order, that is why they contain a basis minor. This minor is the intersection of the basis $s_i' \times n$-submatrix of the rows and the basis $n \times s_i'$-submatrix of the columns. Therefore the total covariant and contravariant modal matrices consist of all the column submatrices and, respectively, of all the row ones ($i = 1, \ldots, q$):

$$V_{col}^{-1} B V_{col} = C_\mu(B), \qquad \tilde{E}_1 = V_{col}\tilde{E}, \tag{78}$$

$$V_{lig} B V_{lig}^{-1} = C_\mu(B), \qquad \tilde{E}_2 = V_{lig}^{-1}\tilde{E}, \tag{79}$$

$$(V_{lig}')^{-1} B' V_{lig}' = C_\mu'(B), \quad \tilde{E}_3 = V_{lig}'\tilde{E}, \tag{80}$$

$$(V_{lig}^*)^{-1} B^* V_{lig}^* = C_\mu^*(B), \quad \tilde{E}_4 = V_{lig}^*\tilde{E}, \tag{81}$$

where C_μ is the q-block-diagonal form of B with respect to its eigenvalues $\mu_1, \ldots, \mu_q$, $\tilde{E}$ and $\tilde{E}_k$, $k = 1, \ldots, 4$, are the original basis and one of these 4 canonical forms.

Each ultrainvariant space contains non invariant subspaces

$$\left.\begin{array}{c}\langle ker\ B_i^{s_i^0}\rangle \supset \langle im\ O_i^1\rangle \supset \cdots \supset \langle im\ O_i^{s_i^0-1}\rangle,\\ \langle ker\ B_i^{s_i^0}\rangle \supset \langle ker\ O_i^{s_i^0-1}\rangle \supset \cdots \supset \langle ker\ O_i^1\rangle,\end{array}\right\} \tag{82}$$

$$\left.\begin{array}{c}\langle im\ O_i^t\rangle \equiv \langle im\ K_1(B_i, r_i')B_i^t\rangle \equiv \langle im\ Q_1(B_i, r_i^0)B_i^t\rangle,\\ \langle ker\ O_i^t\rangle \equiv \langle im\ B_i^t\rangle, \quad t = 1, \ldots, s_i^0 - 1.\end{array}\right\} \tag{83}$$

Take a certain ultrainvariant cell of projection (76) and subtract its prime diagonal part. The result is its nilpotent cells. It may be transformed into subcells (82) till the final elementary subcells. After this the common process may be continued till the Jordan nilpotent form.

Formulae (78) and (79) determine here the various modal matrices for the prime matrix $P_B = \sum_{i=1}^{q} P_i$ in (76). The general formula of the covariant modal matrix is

$$\langle V_{col}\rangle \equiv V_{col}\langle C_q\rangle, \qquad V_{lig}^{-1} \in \langle V_{col}\rangle. \tag{84}$$

Here C_q is an arbitrary nonsingular cell matrix consisting of nonsingular blocks $c_1, \ldots, c_q$. The quantity of nilpotent Jordan $t \times t$-subcells in the i-th cell of the basic canonical form for the matrix B are

$$(rank\ O_i^t - rank\ O_i^{t+1}) - (rank\ O_i^{t+1} - rank\ O_i^{t+2}),$$

see, for example, [10, part 2, p. 95]. General spectral representation of the matrix B analytical functions may be computed with use of the Lagrange and Hermite interpolating polynomials with so called the component matrices [4, p. 155-159]:

$$B_{(ik)} = \frac{B_i^{k-1}}{(k-1)!}\overrightarrow{Bp_{(i)}}, \quad \langle im\ B_{(ik)}\rangle \equiv \langle im\ O_i^{k-1}\rangle, \quad k = 1, \ldots, s_i^0. \tag{85}$$

Substitute here $\overrightarrow{Bp_{(i)}}$ for (75), the result is the form depending only on the original matrix B.

2.3 Transforming a null-prime matrix in its null-cell form

Let Bp be a null-prime $n \times n$-matrix and $rank\ Bp = r$. Further, define the canonical *null-cell (two-block-diagonal) form* of the matrix Bp as the modal transformed $n \times n$-matrix Bc:

$$Bp \ \rightarrow \ Bc = \begin{bmatrix} Z_1 & Z \\ Z & B_1 \end{bmatrix}.$$

Here B_1 is a nonsingular $r \times r$-matrix ($det\ B_1 \neq 0$) and Z_1 is the zero $s \times s$-matrix, $s = n - r$ is the geometric and algebraic multiplicity of the eigenvalue 0 for Bp. Find the modal transformation of Bp into Bc. The high coefficients $K_1(Bp, r)$ and $K_2(Bp, r)$, where $r = rank\ Bp$, are proportional to the eigenprojectors (61) and (62), what are necessary here for the searched modal transformation. (But for their evaluating the eigenvalues of Bp are not necessary as for the full spectrum (76)). These coefficients are null-prime matrices, and thus they contain basis diagonal minors determining two basis $n \times s$- and $n \times r$-submatrices of columns. These submatrices generate the modal matrix of the base transformation:

$$V_{col}^{-1} Bp V_{col} = Bc, \quad \tilde{E}_1 = V_{col}\tilde{E}, \quad \langle V_{col}\rangle \equiv V_{col}\langle C_2\rangle. \tag{86}$$

Here C_2 is a two-cell analog of C_q from (84). So, *the transformation is found.*

Suppose there are two null-prime matrices Bp_1 and Bp_2 of the same order such that

$$\langle im\ Bp_1\rangle \equiv \langle im\ Bp_2\rangle,\ \langle im\ Bp_1'\rangle \equiv \langle im\ Bp_2'\rangle\ \ (\overrightarrow{Bp_1} = \overrightarrow{Bp_2},\ \overleftarrow{Bp_1} = \overleftarrow{Bp_2}).$$

Then, due to (86), we obtain

$$K_j(Bp_1Bp_2, r) = K_j(Bp_2Bp_1, r) = K_j(Bp_1, r)K_j(Bp_2, r),\ j = 1, 2,$$

$$k(Bp_1Bp_2, r) = k(Bp_2Bp_1, r) = k(Bp_1, r)k(Bp_2, r). \tag{87}$$

The last formula generalizes the well-known one for the determinant of square matrices multiplications

$$det\ (B_1B_2) = det\ (B_2B_1) = det\ B_1 \cdot det\ B_2.$$

One else simplest form for a null-prime matrix consists of zero $n \times (n - r)$-matrix and $n \times r$-matrix of the basis columns:

$$Bp = \left[\ Z_2 \mid A_2\ \right].$$

It may be also useful.

2.4 Null-normal singular matrices

There is an one-to-one correspondence between the pair of eigenprojectors $(\overrightarrow{Bp}, \overleftarrow{Bp})$ and the pair $(\langle im\ Bp\rangle, \langle ker\ Bp\rangle)$ of linear subspaces in an affine space $\langle \mathcal{A}^n\rangle$ with a certain base. Suppose this space is real. Consider the set of real so-called *null-normal* matrices $\langle Bm\rangle$ satisfying condition

$$\overrightarrow{Bm} = \overrightarrow{Bm'} = (\overrightarrow{Bm})' \Leftrightarrow \overleftarrow{Bm} = \overleftarrow{Bm'} = (\overleftarrow{Bm})'. \tag{88}$$

Geometrically, this means that

$$\langle ker\ Bm\rangle \equiv \langle ker\ Bm'\rangle \Leftrightarrow \langle im\ Bm\rangle \equiv \langle im\ Bm'\rangle. \tag{89}$$

The sum of $\langle im\ Bm\rangle$ and $\langle ker\ Bm\rangle$ in $\langle \mathcal{A}^n\rangle$ is direct as $k(Bm, r) \neq 0$. In the Euclidean space $\langle \mathcal{E}^n\rangle$ with an orthonormal base, we have

$$\left.\begin{array}{l}\langle ker\ Bm'\rangle \equiv \langle im\ Bm\rangle^{\perp} \equiv \langle ker\ Bm\rangle, \\ \langle im\ Bm'\rangle \equiv \langle ker\ Bm\rangle^{\perp} \equiv \langle im\ Bm\rangle;\end{array}\right\} \Leftrightarrow \tag{90}$$

$$\Leftrightarrow \langle im\ Bm\rangle \boxplus \langle ker\ Bm\rangle \equiv \langle \mathcal{E}^n\rangle.$$

This is the special geometric sense of matrices Bm: *In a real space $\langle \mathcal{E}^n\rangle$ with a fixed orthonormal base the characteristic eigenprojectors of a null-prime matrix Bp are symmetric iff its subspaces $\langle im\ Bp\rangle$ and $\langle ker\ Bp\rangle$ form the spherically orthogonal direct sum*, what is specially denoted in (90) (i. e., iff they are orthocomplements of each to another in $\langle \mathcal{E}^n\rangle$.)

In the eigenspace corresponding to the eigenvalue $\mu = 0$, the matrix Bm is similar to a normal matrix. That is why it is called *null-normal*. In the Euclidean space its eigenprojectors are orthogonal. Special cases of null-normal matrices are normal, symmetric, skew-symmetric, and nonsingular ones.

The following equivalences do hold:

$$\begin{array}{rcll}\overrightarrow{Bm} & = & \overrightarrow{Bm'} = K_1(Bm, r)/k(Bm, r)\ \Leftrightarrow K_1(Bm, r) & = & K_1'(Bm, r)\ (91) \\ \Updownarrow & & & \Updownarrow & \\ \overleftarrow{Bm} & = & \overleftarrow{Bm'} = K_2(Bm, r)/k(Bm, r)\ \Leftrightarrow K_2(Bm, r) & = & K_2'(Bm, r).\ (92)\end{array}$$

In $\langle\mathcal{E}^n\rangle$, $\overrightarrow{Bm}$ and $\overleftarrow{Bm}$ project into $\langle ker\ Bm\rangle$ and respectively $\langle im\ Bm\rangle$ by the orthogonal way, and $\overrightarrow{Bm} \perp \overleftarrow{Bm}$.

The following conditions are equivalent (see sect. 2.1):
(*i*) *all the eigenmatrices* B_t *are real and null-prime;*
(*ii*) *all these matrices have the real affine projectors* $\overrightarrow{B_t}$ *and* $\overleftarrow{B_t}$*;*
(*iii*) *the matrix* B *is real and prime, and all its eigenvalues are real numbers.*

A real normal matrix $B = M$ may be transformed into diagonal real one by a real orthogonal modal matrix iff the matrix M is symmetric ($M = S$).

For any symmetric matrix S, the kernel and the image of each its eigenmatrix S_t are the orthogonal complements of each other, and kernels form the direct orthogonal sum. Therefore, *all the eigenmatrices of a real matrix* B *are real and null-normal iff* B *is real and symmetric.* In particular, null-normal matrices B and B' of rank $n-1$ have the common eigenvector $\langle ker\ B\rangle \equiv \langle ker\ B'\rangle$ iff $B^V = (B^V)'$.

Take a null-normal matrix Bm and apply the Gram–Schmidt orthogonalization algorithm to columns of the two blocks of the matrix $V_{col} = V'_{lig}$ in (86) separately. The result is the orthogonal modal matrix for constructing the null-cell canonical form (86), i. e. congruent modal transformation:

$$Bc = R'_{col} Bm R_{col} \tag{93}$$

($\langle R_{col}\rangle \equiv R_{col}\langle R_2\rangle$, but $\langle V_{col}\rangle \equiv R_{col}\langle C_2\rangle$, see (86)). Structure of R_2 here is similar to C_2 in (86). If the original base is, for example, Cartesian, then the new orthonormal base is expressed in terms of the columns of the modal matrix $\{R_{col}\} = \{R'_{lig}\}$. And orientation of the base is changed under multiplying R_{col} by the alternating unity matrix on the right for its restoring. The modal orthogonal matrix R_{col} for constructing the diagonal form of a symmetric matrix S is computed by the way similar to (78). If all the eigenvalues of S are distinct, then n its unity length eigenvectors determined by $\langle ker\ S_t\rangle$ form the desired matrix R_{col}. If some of them are degenerative (under condition $s_t > 1$), then the Gram–Schmidt orthogonalization is applied.

The following *examples of null-normal matrices* are used in the sequel. These matrices are generated by the special $n \times m$-matrix A ($n \neq m$):

$$Bm_1 = A_1 A'_2, \quad Bm'_1 = A_2 A'_1 \tag{94}$$

$$(\langle im\ A_1\rangle \equiv \langle im\ A_2\rangle, \quad rank\ A_1 = rank\ A_2 = m < n),$$
$$Bm_2 = A'_1 A_2, \quad Bm'_2 = A'_2 A_1 \tag{95}$$

$$(\langle ker\ A_1\rangle \equiv \langle ker\ A_2\rangle, \quad rank\ A_1 = rank\ A_2 = n < m).$$

Note some other properties of all null-normal matrices.

$$\left.\begin{array}{c} \overrightarrow{Bm'Bm} = \overrightarrow{BmBm'} = \overrightarrow{Bm}, \quad \overleftarrow{Bm'Bm} = \overleftarrow{BmBm'} = \overleftarrow{Bm}, \\ \langle ker\ Bm'Bm\rangle \equiv \langle ker\ BmBm'\rangle \equiv \langle ker\ Bm\rangle, \\ \langle im\ Bm'Bm\rangle \equiv \langle im\ BmBm'\rangle \equiv \langle im\ Bm\rangle. \end{array}\right\} \tag{96}$$

Apply (87) to null-normal matrices Bm and Bm', we obtain

$$\left.\begin{array}{c} K_1(BmBm', r) = K_1(Bm'Bm, r) = K_1^2(Bm, r), \\ K_2(BmBm', r) = K_2(Bm'Bm, r) = K_2^2(Bm, r), \\ k(BmBm', r) = k(Bm'Bm, r) \quad = k^2(Bm, r). \end{array}\right\} \tag{97}$$

Formula (97) generalizes the well-known formula for determinants

$$det(BB') = det(B'B) = det^2 B.$$

(Singular matrices M and S are also the special cases of null-normal ones.)

2.5 Spherically orthogonal projectors and quasi-inverse matrices

In the previous section, we introduced the orthogonal eigenprojectors in addition to oblique ones. They were defined for null-normal matrices due to *spherical orthogonality* (90) of eigensubspaces in $\langle\mathcal{E}^n\rangle$. This property takes place only in $\langle\mathcal{E}^n\rangle$ and corresponds to to right tensor spherical angles (see in Ch. 5) between subspaces or lineors.

Let A be a real-valued $m \times n$-matrix of rank r which is less n and m. We have products $A'A, AA' \in \langle Bm\rangle$, and their rank is also equal to r. According to (91) and (92) we obtain

$$\overrightarrow{A'A} = K_1(A'A,r)/k(A'A,r), \quad \overrightarrow{AA'} = K_1(AA',r)/k(AA',r), \tag{98}$$

$$\left.\begin{array}{c} \overleftarrow{A'A} = K_2(A'A,r)/k(A'A,r) = A^+A = I_{n\times n} - \overrightarrow{A'A}, \\ \overleftarrow{AA'} = K_2(AA',r)/k(AA',r) = AA^+ = I_{m\times m} - \overrightarrow{AA'}, \\ \{k(AA',t)\} = \{k(A'A,t)\}; \end{array}\right\} \tag{99}$$

and in the two trivial cases, when either $rank\ A = r = m < n$, or $rank\ A = r = n < m$, they expressed as follow:

$$\left.\begin{array}{l} rank\ A = r = m < n \Rightarrow\ \overleftarrow{A'A} = A'\{AA'\}^{-1}\cdot A = A^+A, \\ rank\ A = r = n < m \Rightarrow\ \overleftarrow{AA'} = A\{A'A\}^{-1}\cdot A' = AA^+. \end{array}\right\} \tag{99'}$$

Here, for example, in $\langle\mathcal{E}^m\rangle$ with $m \times 1$-vectors $\mathbf{a}$:
$\overleftarrow{AA'}$ is the orthogonal projector onto $\langle im\ A\rangle \equiv \langle ker\ A'\rangle^\perp$; $\overleftarrow{\mathbf{aa}'} = \mathbf{aa}'/\mathbf{a}'\mathbf{a}, \{\mathbf{a}\}^+ = \mathbf{a}'/(\mathbf{a}'\mathbf{a})$;
$\overrightarrow{AA'}$ is the orthogonal projector onto $\langle ker\ A'\rangle \equiv \langle im\ A\rangle^\perp$; $\overrightarrow{\mathbf{aa}'} = I - \mathbf{aa}'/(\mathbf{a}'\mathbf{a})$;
A^+ is here the quasi-inverse Moor–Penrose $n \times m$-matrix [30–32], $rank\ A^+ = r$.

Contrary, for the two complementary subspaces of $\langle\mathcal{E}^n\rangle$ or of $\langle\mathcal{A}^n\rangle$, we obviously obtain

$$\overleftarrow{A'A} + \overrightarrow{A'A} = I_{n\times n}, \qquad \overleftarrow{B'B} + \overrightarrow{B'B} = I_{n\times n} = \overleftarrow{BB'} + \overrightarrow{BB'} \tag{100}$$

– equivalent to $\langle im\ A'\rangle \oplus \langle ker\ A\rangle \equiv \langle\mathcal{A}^n\rangle$, $\langle im\ B'\rangle \oplus \langle ker\ B\rangle \equiv \langle im\ B\rangle \oplus \langle ker\ B'\rangle \equiv \langle\mathcal{A}^n\rangle$.

Ii is in $\langle\mathcal{E}^n\rangle$ the subspaces are *orthogonally complementary*. According to (99) any matrix A^+ satisfies the two Penrose equations [32], which determine it independently:

$$AA^+A = A, \qquad A^+AA^+ = A^+.$$

From the latter and (62) we obtain exactly for any A, i. e., for $m \times n$ or $n \times m$:

$$A^+ = A' \cdot \frac{K_1(AA', r-1)}{k(AA',r)} = \frac{K_1(A'A, r-1)}{k(A'A,r)} \cdot A'. \tag{101}$$

First is the Decell's formula, inferred in [33] from the Souriau algorithm [27] (see in Ch. 1). Equality (101) can be checked by representing the matrix coefficients by polynomials (27).

The matrix A^+ behaves as the inverse matrix in $\langle im\ A\rangle$ and as the zero one in $\langle ker\ A'\rangle$ with respect to multiplication from the left:

$$A^+C = A^+[(\overleftarrow{AA'} + \overrightarrow{AA'})C] = A^+(\overleftarrow{AA'}C). \tag{102}$$

However with respect to multiplication from the right, the matrix A^+ plays the role of the inverse matrix in $\langle im\ A'\rangle$ and the zero matrix in $\langle ker\ A\rangle$:

$$CA^+ = [C(\overrightarrow{A'A} + \overleftarrow{A'A})]A^+ = (C\overleftarrow{A'A})A^+. \tag{103}$$

In particular, the matrix B commutes with B^+ exactly in $\langle im\ B\rangle \cap \langle im\ B'\rangle$. That is why the following equivalences hold for the matrix B^- from (69) (see sect. 2.1):

$$B^- = B^+ \Leftrightarrow B \in \langle Bm\rangle \Leftrightarrow B^+B = BB^+. \tag{104}$$

In the Euclidean space with a certain orthonormal base, a quasi-inverse orthogonal matrix has the following geometric sense: its Frobenius norm (the matrix norm of the 1-st order, see sect. 9.1) is minimal among all quasi-inverse matrices determined by 1-s Penrose equation $AXA = A$, i. e., this matrix is the normal solution of this equation from the left and from the right [30, 32] (see also below). Moreover, this matrix A^+ gives the *normal solutions* (i. e., with the minimal Frobenius norm) of the left, right, and mixed general linear equations

$$A_1(m\times n)\cdot X(n\times t) = A(m\times t) \ \Rightarrow \overset{\bullet}{X}(n\times t) = A_1^+A, \tag{105}$$

$$Y(t\times m)\cdot A_2(m\times n) = A(t\times n) \ \Rightarrow \overset{\bullet}{Y}(t\times m) = AA_2^+, \tag{106}$$

$$A_1(m_1\times n_1)\cdot X\cdot A_2(m_2\times n_2) = A(m_1\times n_2) \ \Rightarrow \overset{\bullet}{X}(n_1\times m_2) = A_1^+AA_2^+. \tag{107}$$

Equations *residuals* for full solutions have the minimal Frobenius norm too:

$$\left.\begin{array}{c} \overset{\bullet}{\Delta}_1 = -\overrightarrow{A_1A_1'}A, \\ \overset{\bullet}{\Delta}_1 = Z \ \Leftrightarrow \ A \in \langle \overleftarrow{A_1A_1'}\cdot \mathcal{E}^{m\times t}\rangle \equiv \langle KER_R\overrightarrow{A_1A_1'}\rangle, \end{array}\right\} \tag{108}$$

$$\left.\begin{array}{c} \overset{\bullet}{\Delta}_2 = -A\overrightarrow{A_2'A_2}, \\ \overset{\bullet}{\Delta}_2 = Z \ \Leftrightarrow \ A \in \langle \mathcal{E}^{t\times n}\cdot \overleftarrow{A_2'A_2}\rangle \equiv \langle KER_L\overrightarrow{A_2'A_2}\rangle, \end{array}\right\} \tag{109}$$

$$\left.\begin{array}{c} \overset{\bullet}{\Delta} = -\overrightarrow{A_1A_1'}A - A\overrightarrow{A_2'A_2} + \overrightarrow{A_1A_1'}A\overrightarrow{A_2'A_2}, \qquad \overset{\bullet}{\Delta} = Z \ \Leftrightarrow \\ A \in \langle \overleftarrow{A_1A_1'}\cdot \mathcal{E}^{m_1\times n_2}\cdot \overleftarrow{A_2'A_2}\rangle \equiv \langle KER_R\overrightarrow{A_1A_1'} \cap KER_L\overrightarrow{A_2'A_2}\rangle. \end{array}\right\} \tag{110}$$

Intersection of the set of all left quasi-inverse matrices and the set of all right ones determined by (99) consists of the unique element A^+ [30, 34]:

$$\langle A_R^-\rangle \equiv A^+ \oplus \langle \overrightarrow{A'A}\cdot \mathcal{E}^{n\times m}\cdot \overleftarrow{AA'}\rangle \tag{111}$$

(all these matrices produce orthoprojectors in (108), in particular, A^+),

$$\langle A_L^-\rangle \equiv A^+ \oplus \langle \overleftarrow{A'A}\cdot \mathcal{E}^{n\times m}\cdot \overrightarrow{AA'}\rangle \tag{112}$$

(all these matrices produce orthoprojectors (109), in particular, A^+),

$$A^+ = \langle A_R^-\rangle \cap \langle A_L^-\rangle. \tag{113}$$

In particular, from (108)–(110) we obtain

$$\left.\begin{array}{c} rank\ A_1 = m \ \Rightarrow \overset{\bullet}{\Delta}_1 = Z, \quad rank\ A_2 = n \ \Rightarrow \overset{\bullet}{\Delta}_2 = Z, \\ (rank\ A_1 = m,\ rank\ A_2 = n) \ \Rightarrow \overset{\bullet}{\Delta} = Z. \end{array}\right\} \tag{114}$$

Consider in details the exact normal solution of the classical linear equation $A\mathbf{x} = \mathbf{a}$ in the general form with the use of formula (101):

$$||A\mathbf{x} - \mathbf{a}|| \to \min, \quad \overset{\bullet}{\mathbf{x}} = A^{+}\mathbf{a} = [\overset{\bullet}{A}(r)/k(AA', r)]\mathbf{a}, \tag{115}$$

$$\overset{\bullet}{\mathbf{d}} = -\overrightarrow{AA'}\mathbf{a}. \tag{116}$$

We have

$$\overset{\bullet}{\mathbf{d}} = \mathbf{0} \ \Leftrightarrow \mathbf{a} \in \langle ker\overrightarrow{AA'}\rangle \equiv \langle ker\ K_1(AA', r)\rangle. \tag{117}$$

Here we get exact formulae (115) and (116) for the *normal solution and minimal residual* of the classic linear equation $A\mathbf{x} = \mathbf{a}$. The residual is antiprojection (116). Consequently, its Euclidean norm satisfies

$$||\overset{\bullet}{\mathbf{d}}||^2 = -\overset{\bullet}{\mathbf{d}}{}'\mathbf{a}, \tag{118}$$

$$||\overset{\bullet}{\mathbf{d}}|| = \sin\varphi\ ||\mathbf{a}||, \quad (\varphi \in [0; \pi/2]) \tag{119}$$

where φ is the introduced here scalar angle between the vector $\mathbf{a}$ and the subspace $\langle im\ A\rangle$.

We conclude the section with inferring from formula (101) the explicit expression for a (p, q)-element of the $n \times m$-matrix $\overset{\bullet}{A}(r)$ in (115). The most general Hermitean-like form of this element in the case of a complex initial linear equation is

$$(p, q) = \sum_{(C^r_{m-1}\text{ terms})} \ \sum_{(C^r_{n-1}\text{ terms})} \overline{\left\{\frac{(q,p) \in}{minor(r)}A\right\}} Ad_{q',p'} \left\{\frac{(q,p) \in}{minor(r)}A\right\},$$

where $p = 1, \ldots, m$, $q = 1, \ldots, n$, p' and q' are new indexes of a_{qp} in minors of A. Therefore, (115) generalizes here the Cramer formulae. *In special case* $r = n = m$, (115) represents the matrix solution of a nonsingular linear equation $A\mathbf{x} = \mathbf{a}$, because $\overset{\bullet}{A}(n) = \overline{det\ A} \cdot A^V$, $k(AA^*, n) = \overline{det\ A} \cdot det\ A$ and, consequently, the solution is $\mathbf{x} = (A^V/det\ A)\mathbf{a} = A^{-1}\mathbf{a}$ (the special classic case see, for example, in [4, p. 38]). In the presentation as limit formula for the initial $m \times n$-matrix A, the quasi-inverse $n \times m$-matrix A^+ by Moor and Penrose is expressed according to initial (1) and to (101),as follows (see more in sect. 3.4):

$$A^+ = \lim_{\epsilon\to 0}[A'(AA' + \epsilon I)^{-1}] = \lim_{\epsilon\to 0}[(A'A + \epsilon I)^{-1}A']$$

The exact normal solution of the linear equation $A\mathbf{x} = \mathbf{a}$ together with these general limit and exact real and complex formulae for the matrix of Moor and Penrose were established by the author with very detailed derivations yet in early of 1981, with introduction of both rocks for a matrix B, and along with all the new concepts introduced in Chapters 1 and 2. The draft of the author's 1st math atricle, with these formulae and the structures of the matrix characterisic coefficients, was submitted to the main mathematical journal of the USSR, and it lay with its 1st Soviet reviewer for 2 years, after which it was rejected by him with the wording that it was not suitable for this respectable journal. However, in the middle of this period, the new limit formula above mysteriously appeared in this reviewer's new book, by default, as his proper, and it is clear why: he probably liked it very muched.

The draft article continued to circulate in other Soviet math journals with the same result, and another "author" from the same math circle later published also very mysteriously my structure above of the matrix characteristic coefficients, but from him. How does the hand rise among "figures" in the field of exact sciences to pass off someone else's as their own?

"O tempora, o mores!" – Marcus Tullius Cicero (First-century BC).

(See more about these historical aspects on the author's web-site and in the end of Ch. 4.)

Chapter 3

Main scalar invariants of singular matrices

3.1 The minorant of a matrix and its applications

Let A_1 and A_2 be $n \times m$-matrices. Then $k(A_1A_2', t) = k(A_2A_1', t)$. The scalar coefficients of order t for $n \times n$-matrix A_1A_2' were shown to be the sums of all diagonal minors of order t. Represent each matrix of diagonal minors of A_1A_2' as the following multiplication of $t \times m$-matrices of rows:

$$\{D\text{-minor}(t)A_1A_2'\} = \{\text{lig}(t)A_1\}\{\text{lig}(t)A_2\}'.$$

By the Binet–Cauchy formula [4, p. 39], this minor (i. e., determinant of the left matrix) is the sum of all pair multiplications of all minors from the right submatrices of order t with the same set of columns. For the $m \times m$-matrix $A_1'A_2$, *in all these assertions, rows are changed for columns and columns are changed for rows.* Consider the two sets of $C_n^tC_m^t$ pair multiplications of order-t minors of A_1 and A_2. They form the two sums. The first sum is equal to the scalar coefficient $k(A_1A_2', t)$, the second sum is equal to the scalar coefficient $k(A_1'A_2, t)$. There exists a bijection between these two sets, it is described just above, thus for external and internal multiplications of these matrices we have

$$k(A_1A_2', t) = k(A_1'A_2., t) = k(A_2A_1', t) = k(A_2'A_1, t). \tag{120}$$

In the special case $A_1 = A_2 = A$, i. e., for these both *homomultiplications*, there holds

$$k(AA', t) = \sum_{(C_m^t \text{ terms})} \sum_{(C_n^t \text{ terms})} \text{minor}^2(t)A \; = k(A'A, t) = \mathcal{M}t^2(r)A \geq 0. \tag{121}$$

We introduced here the highest positive characteristic of an $n \times m$-matrix, its *minorant*:

$$\mathcal{M}t(r)A = \sqrt{k(AA', r)} = \sqrt{k(A'A, r)} = \mathcal{M}t(r)A' > 0.$$

It is the square root of the sum of all squared basic minors A, this follows from (121).

Note *the special cases.*

1. If $n > m = r$, then $\mathcal{M}t^2(m)A = det\ A'A$ (the Gram determinant for columns A).
2. If $m = 1$, then $\mathcal{M}t(1)\mathbf{a} = ||\mathbf{a}||_E$ (the Euclidean module $\mathbf{a}$).
3. If $n = m = r$, then $\mathcal{M}t(n)A = |det\ A|$ (the determinant A).

Formulae for the matrix poly-step homomultiplication minorant follow from (67):

$$\mathcal{M}t(r)\{\underbrace{AA'A\ldots}_{h}\} = \mathcal{M}t(r)\{\underbrace{A'AA'\ldots}_{h}\} =$$

$$= \sqrt{k[(AA')^h, r]} = \sqrt{k^h(AA', r)} = \mathcal{M}t^h(r)A.$$

Consider equation (115) and the matrix $\{A|\mathbf{a}\}$. If $n = m = r$, according (117), $\dot{\mathbf{d}} = \mathbf{0}$. When $n \geq m \geq r$, (116) and (119) give the general result:

$$\mathcal{M}t(r+1)\{A|\mathbf{a}\} = \sin\varphi \cdot ||\mathbf{a}|| \cdot \mathcal{M}t(r)A = ||\,\dot{\mathbf{d}}\,|| \cdot \mathcal{M}t(r)A. \tag{122}$$

Through graceful formula (122) and in term of *minorant* of the matrix $\{A|\mathbf{a}\}$ of order-$(r+1)$ with all squared minors, we prove clear **in one-line** the classic *Kronecker–Capelli Theorem*:

$$\mathcal{M}t^2(r+1)\{A|\mathbf{a}\} = \sum_{(C^{r+1}_{m+1})}\sum_{(C^{r+1}_{n})} \text{minor}^2(r+1)\{A|\mathbf{a}\} = 0 \Leftrightarrow \dot{\mathbf{d}} = \mathbf{0} \Leftrightarrow \sin\varphi = 0.$$

If $n > m = r$, then the Gram determinant may be also the analogous criterion as

$$\mathcal{M}t^2(r+1)\{A|\mathbf{a}\} = det[\{A|\mathbf{a}\}'\{A|\mathbf{a}\}] = ||\, \dot{\mathbf{d}}\, ||^2 \cdot \mathcal{M}t^2(r)A.$$

Formula (122) in the pure trigonometric form (where $\varphi \in (0;\pi/2]$) is

$$0 \leq \sin\varphi = \mathcal{M}t(r+1)\{A|\mathbf{a}\}/(\mathcal{M}t(r)A \cdot \mathcal{M}t(1)\mathbf{a}) \leq 1. \tag{123}$$

In particular, for the angle between two vectors ($\varphi_{12} \in (0;\pi/2]$) in $\langle\mathcal{E}^n\rangle$, we have:

$$0 \leq \sin\varphi_{12} = \mathcal{M}t(2)[\mathbf{a_1}|\mathbf{a_2}]\ /(\mathcal{M}t(1)\mathbf{a_1} \cdot \mathcal{M}t(1)\mathbf{a_2}) =$$

$$= \sqrt{det\{[\mathbf{a_1}|\mathbf{a_2}]' \cdot [\mathbf{a_1}|\mathbf{a_2}]\}}/(||\mathbf{a_1}|| \cdot ||\mathbf{a_2}||) = ||\mathbf{a_1} \times \mathbf{a_2}||/(||\mathbf{a_1}|| \cdot ||\mathbf{a_2}||) \leq 1. \tag{124}$$

Here on the left we gives a *scalar multiplication of sine type for two vectors* and on the right we gives identical to it a module of their vector multiplication. In the first variant, *for two vectors on a plane* ($n = 2$), *may be eigen*, i. e., in $\langle\mathcal{E}^2\rangle$!), the determinant in formula (124) disintegrates in two equal determinants. As result, there holds the simplified formula for the angle between two vectors on a plane with the angle sign:

$$-1 \leq \sin\varphi_{12} = det[\mathbf{a_1}|\mathbf{a_2}]/(||\mathbf{a_1}|| \cdot ||\mathbf{a_2}||) \leq +1, \quad (\varphi_{12} \in [-\pi/2;+\pi/2]).$$

Relation between the minorant of an $n \times r$-matrix A and the square root of the Gram determinant of its r columns enables one to clarify the geometric sense of the minorant as the volume of the parallelepiped, constructed on the vector-columns of the matrix A [5, p. 216]. In particular, put $m = r$. We often deal with such matrices in part II. They represent special linear geometric objects *lineors* of greater dimension ($r > 1$), then vectors. Consider the columns of a matrix A. Denote the submatrix formed by first j columns as A_j. Then $A_{j+1} = \{A_j|\mathbf{a_{j+1}}\}$ for each j. Apply formulae (119) and (122) to A_{j+1}, also the geometric interpretation of the Gram determinant square root may be used. Subsequent application of this operation gives the formula

$$\mathcal{M}t(r)A = v_r = ||\mathbf{a}_1|| \cdot \sin\varphi_{1,2} \cdot ||\mathbf{a}_2|| \cdot \sin\varphi_{1,2,3} \cdots ||\mathbf{a}_r|| \leq ||\mathbf{a}_1|| \cdot ||\mathbf{a}_2|| \cdots ||\mathbf{a}_r||, \tag{125}$$

where v_r is the volume of the r-dimensional parallelepiped with sides $\mathbf{a_1}, \ldots, \mathbf{a_r}$, and $\varphi_{1,2}, \varphi_{1,2,3}, \dots \in (0;\pi/2]$.

If $n = m = r$, then from (125) the sine Hadamard Inequality in its usual form [21, p. 35] is valid; and, if $r = 2$, it has particular form (124). Due to (74), the following does hold:

$$\mathcal{M}t(r)A = \sqrt{k(AA',r)} = \prod_{j=2}^{q}\sigma_j^{s_j} > 0, \quad \overrightarrow{AA'} = \prod_{j=2}^{q}(\sigma_j^2 I_{n\times n} - AA')/\sigma_j^2, \tag{126}$$

where $\sigma_j^2 > 0$ are the nonzero eigenvalues of AA' or $A'A$.

In general ($n \geq m \geq r \geq t$), the coefficients $k(AA', t) = k(A'A, t)$ can be expressed either geometrically as the sums of squared t-dimensional volumes (t-measures) or algebraically as the Viète sums of the eigenvalues of AA':

$$\left.\begin{aligned} k(AA',t) &= \sum_{(C_m^t \text{ terms})} v_{t(p)}^2 = s_t(\sigma_j^2) = v_t^2 > 0, \\ k(AA',1) &= \sum_{(m \text{ terms})} \ell_{(p)}^2 = s_1(\sigma_j^2) = \ell^2 = ||A||_F^2 > 0, \end{aligned}\right\} (\mathcal{M}t^2(r)A = v_r^2). \quad (127)$$

Here, in Cartesian coordinates, $v_{t(p)}$ is the volume v_t of the orthoprojection of the rank t. If $m = r$, then the ratio $v_{t(p)}/v_t = \cos\alpha_p$ is the p-th direction cosine.

Formulae (127) express the Pythagorean Theorem for the linear objects represented by $n \times r$-matrices. Further, they are called *lineors*. All the characteristics are always positive and invariant under orthogonal transformations of columns or rows of the matrix A and its Cartesian base. In particular, there holds

$$\mathcal{M}t(r)A = \mathcal{M}t(r)\{R_1AR_2\} = \mathcal{M}t(r)\sqrt{AA'} = \mathcal{M}t(r)\sqrt{A'A}. \quad (128)$$

Therefore, a minorant may be used as geometric characteristic for these lineors of different dimensions and ranks. In Ch. 9 this opportunity will be realized for introducing general norms of similar linear objects.

The arithmetic roots in (128) may be singular; in general, they are related with the matrix A by the *quasi-polar decompositions of A (i. e., QR-factorization)*:

$$A = S_1^{\oplus} \cdot Rq = \sqrt{AA'} \cdot \{(\sqrt{AA'})^+ \cdot A\}, \quad (129)$$

$$A = Rq \cdot S_2^{\oplus} = \{A \cdot (\sqrt{A'A})^+\} \cdot \sqrt{A'A}. \quad (130)$$

$$S_1^{\oplus} = Rq \cdot S_2^{\oplus} \cdot Rq' \;\Leftrightarrow\; AA' = Rq \cdot A'A \cdot Rq',$$

$$Rq = A \cdot (\sqrt{A'A})^+ = (\sqrt{AA'})^+ \cdot A \;\Rightarrow$$

$$Rq \cdot Rq' = \overleftarrow{AA'}, \quad Rq'Rq = \overleftarrow{A'A}, \quad Rq' = Rq^+.$$

The transformation $A \to Rq$ gives the same result as the Gram–Schmidt unity orthogonalization of m linearly independent vectors:

$$A = \{\mathbf{a_1}, \ldots, \mathbf{a_m}\} \to \{\mathbf{e_1}, \ldots, \mathbf{e_m}\} = Rq.$$

This algebraic transformation is the uniquely determined variant of the Gram–Schmidt orthogonalization (provided that the sequence of vectors is fixed).

In Euclidean space, this Gram–Schmidt orthogonalization can be expressed geometrically clearly with use of orthoprojectors:

$$\mathbf{v_1} = \mathbf{a_1}, \quad \mathbf{v_i} = \mathbf{a_i} - \sum_{\mathbf{k=1}}^{\mathbf{i-1}} [\mathbf{e_k} \cdot \mathbf{e_k'}] \cdot \mathbf{a_i} = \{I - \sum_{\mathbf{k=1}}^{\mathbf{i-1}} [\mathbf{e_k} \cdot \mathbf{e_k'}]\} \cdot \mathbf{a_i}, \quad (131)$$

where $\mathbf{e_k} \cdot \mathbf{e_k'} = \overleftarrow{\mathbf{e_k} \cdot \mathbf{e_k'}}$ – see sect. 2.1. The results of this procedure are the following $\mathbf{e_i} = \mathbf{v_i}/||\mathbf{v_i}||$, $i = 1, \ldots, m$, and additionally we have the matrix Rq for A.

For the *special* kind of $n \times m$-matrices, with $n > m = r$, prove the split formula for the minorant of their external multiplications:

$$\mathcal{M}t(r)A_1A_2' = \mathcal{M}t(r)A_1 \cdot \mathcal{M}t(r)A_2 = \sqrt{\det\,(A_1'A_1) \cdot \det\,(A_2'A_2)}. \quad (132)$$

With the definition of a minorant, the quasi-polar decompositions such as (129), (130), and also formula (128), we subsequently obtain

$$\mathcal{M}t^2(r)\{A_1A_2'\} = k[(A_1A_2'A_2A_1'),r] = k[(Rq_1 \cdot S_1^{\oplus} \cdot S_2^{\oplus} \cdot S_2^{\oplus} \cdot S_1^{\oplus} \cdot Rq_1'),r] =$$

$$= k[(S_1^{\oplus} \cdot S_2^{\oplus} \cdot S_2^{\oplus} \cdot S_1^{\oplus}),r] = det\ (A_1'A_1) \cdot det\ (A_2'A_2) = \mathcal{M}t^2(r)A_1 \cdot \mathcal{M}t^2(r)A_2.$$

Further, for such *external* and *internal multiplications* of $n \times m$-matrices we use notations:

$$B = A_1A_2', \quad B' = A_2A_1'; \qquad C = A_1'A_2, \quad C' = A_2'A_1.$$

For B, if $\langle im\ A_2'\rangle \cap \langle ker\ A_1\rangle = \mathbf{0}, \langle im\ A_1'\rangle \cap \langle ker\ A_2\rangle = \mathbf{0}$, there holds:
$\langle im\ B\rangle \equiv \langle im\ A_1\rangle \;\Leftrightarrow\; \langle ker\ B'\rangle \equiv \langle ker\ A_1'\rangle$ – see also (100),
$\langle im\ B'\rangle \equiv \langle im\ A_2\rangle \;\Leftrightarrow\; \langle ker\ B\rangle \equiv \langle ker\ A_2'\rangle$ – see also (100).
Due to additional condition $m = rank\ A_1 = rank\ A_2 = r$, the following does hold:

$$\left.\begin{array}{l} \overleftarrow{BB'} = \overleftarrow{A_1A_2'A_2A_1'} = \overleftarrow{A_1A_1'} = \overleftarrow{Rq_1Rq_1'}, \\ \overleftarrow{B'B} = \overleftarrow{A_2A_1'A_1A_2'} = \overleftarrow{A_2A_2'} = \overleftarrow{Rq_2Rq_2'}, \\ \overrightarrow{BB'} = \overrightarrow{A_1A_2'A_2A_1'} = \overrightarrow{A_1A_1'} = \overrightarrow{Rq_1Rq_1'}, \\ \overrightarrow{B'B} = \overrightarrow{A_2A_1'A_1A_2'} = \overrightarrow{A_2A_2'} = \overrightarrow{Rq_2Rq_2'}. \end{array}\right\} \tag{133}$$

Besides, $det\ C = det(A_1'A_2) \neq 0$. (See this in details in Part II, sect. 5.4.)
Then formulae

$$\left.\begin{array}{l} K_j[(A_1A_2'A_2A_1'),r] = det\ (A_2'A_2) \cdot K_j(A_1A_1',r), \\ K_j[(A_2A_1'A_1A_2'),r] = det\ (A_1'A_1) \cdot K_j(A_2A_2',r), \end{array}\right\} (j = 1, 2,) \tag{134}$$

follow from (61), (62), (132), (133).

3.2 Sine characteristics of matrices

Let $E = \{\mathbf{e}_i\}_{n\times n}$ be some $n \times n$-matrix, given as a linear unity geometric object in the 1-st quadrant of Cartesian base $\{I\}$ in a space $\langle \mathcal{E}^n\rangle$, where $||\mathbf{e}_i|| = 1$ for all i. Namely, the matrix $E = \{\mathbf{e}_i\}_{n\times n}$ determines an n-edges polyhedral tensor angle in the Euclidean space; $det\ E = \mathcal{M}t(n)E \leq 1$ is, due to the trigonometric value in Hadamard Inequality (125), its *sine characteristic*. This polyhedral angle corresponds one-to-one the unique mutual tensor angle, given by the matrix $\hat{E} = \{\hat{\mathbf{e}}_i\}_{n\times n} = \{\overrightarrow{E_iE_i'}\mathbf{e}_i \sec\beta_i\}$, where E_i is obtained from E by change of the column $\mathbf{e}_i$ on zero one, and for this tensor angle $\hat{E}$ unity its calibration by $\sec\beta_i$ is used. The orthoprojector of type $\overrightarrow{E_iE_i'}$ projects into the kernel $\langle ker\ E_i'\rangle$ orthogonally to the image $\langle im\ E_i\rangle$ (see sect. 2.5). There holds: $\cos\beta_i = \mathbf{e}_i'\hat{\mathbf{e}}_i = \hat{\mathbf{e}}_i'\mathbf{e}_i$ $(0 < \cos\beta_i \leq 1)$, $\mathbf{e}_i'\hat{\mathbf{e}}_j = 0$ or $E'\hat{E} = D_{\cos\beta} = \hat{E}'E, \to \cos^2\beta_i = \mathbf{e}_i'\overrightarrow{E_iE_i'}\mathbf{e}_i$, and the all values of $\cos\beta_i$ are finding. Then

$$det\ E \cdot det\ \hat{E}\ = det\ D_{\cos\beta} = \prod_{i=1}^{n}\cos\beta_i,\ \ |det\ E| \leq 1,\ \ |det\ \hat{E}| \leq 1;$$

$$E'E = D_{\cos\beta} \cdot (\hat{E}'\hat{E})^{-1} \cdot D_{\cos\beta}, \quad \hat{E}'\hat{E} = D_{\cos\beta} \cdot (E'E)^{-1} \cdot D_{\cos\beta},$$

$$G = \sqrt{D_{\sec\beta}} \cdot E'E \cdot \sqrt{D_{\sec\beta}} = \hat{G}^{-1} = [\sqrt{D_{\sec\beta}} \cdot \hat{E}'\hat{E} \cdot \sqrt{D_{\sec\beta}}\ ]^{-1}.$$

Here G and $\hat{G}$ are metric tensors in the stretched of these angles mutual affine bases, given in $\{I\}$ by modal matrices $\{E\sqrt{D_{\sec\beta}}\}$ and $\{\hat{E}\sqrt{D_{\sec\beta}}\}$.

However, in the book, we deal with tensor angles of the binary type, i. e., angles formed by pairs of linear subspaces (straight lines if $r = 1$) or linear objects A_1, A_2 (vectors if $r = 1$) in spaces with quadratic metrics.

At first, consider the *sine characteristic* of binary angles. For this we suppose that $r_1 = rank\ A_1$ and $r_2 = rank\ A_2$, but $r_1 + r_2 \leq n$. The block matrix $\{A_1|A_2\}$ is called the *external summation* of A_1 and A_2. Introduce for the rectangular matrices (or lineors) A_1 and A_2 the scalar characteristic *sine ratio* (see more in sect. 8.4):

$$|\{A_1|A_2\}|_{\sin} = \mathcal{M}t(r_1+r_2)\{A_1|A_2\}/(\mathcal{M}t(r_1)A_1 \cdot \mathcal{M}t(r_2)A_2) = \qquad (135)$$

$$= \sqrt{det\ \left[\begin{array}{c|c} A_1'A_1 & A_1'A_2 \\ \hline A_2'A_1 & A_2'A_2 \end{array}\right]} / \sqrt{det\ (A_1'A_1) \cdot det\ (A_2'A_2)} = detG_{1,2}/\mathcal{M}t(r)A_1A_2' \geq 0.$$

It generalizes (123) and ratio (124) for the sine of the angle between two vectors. The matrix in numerator generalizes *the internal multiplication of two vectors of sine type* used in (124). This ratio is the *sine positively definite norm* for a pair of A_1 and A_2.

The Kronecker–Capelli Theorem may be generalized to matrix linear equations such as (105)–(107). The generalization is expressed also in terms of the minorant:

$$\mathcal{M}t^2(r_1+r_2+1)\left[\begin{array}{c|c} A_1 & A \\ \hline Z & A_2 \end{array}\right] = 0 \ \Leftrightarrow\ \overset{\bullet}{\Delta} = Z. \qquad (136)$$

3.3 Cosine characteristics of matrices

Denote the highest scalar characteristic of a square singular matrix, its *dianal* :

$$\mathcal{D}l(r)B = k(B,r) = \mathcal{D}l(r)B' \quad (det\ B = 0),$$

So, $\mathcal{D}l(r)\{AA'\} = \mathcal{D}l(r)\{A'A\} = k(AA',r) = \mathcal{M}t^2(r)A$ – see sect. 3.1. And from formula (122) we have: $\mathcal{M}t^2(r+1)\{A|\mathbf{a}\} = \mathcal{D}l(r+1)\{[A|\mathbf{a}][A|\mathbf{a}]'\} = 0 \Leftrightarrow \overset{\bullet}{\mathbf{d}} = \mathbf{0} \ \Leftrightarrow\ \sin\varphi = 0!$

Then the new scalar characteristic for a singular square matrix B, its sign-indefinite *cosine ratio* (see more in Ch. 8), is expressed in terms of the minorant and the dianal:

$$\{B\}_{\cos} = \mathcal{D}l(r)B/\sqrt{\mathcal{D}l(r)BB'} = \mathcal{D}l(r)B/\mathcal{M}t(r)B = \prod_{i=2}^{q_1}\mu_i^{s'_{1,i}} \Big/ \prod_{j=2}^{q_2}\sigma_j^{s_{2,j}}. \qquad (137)$$

We may preliminary introduce the *cosine norm* for B as follows (see more in sect. 8.1):

$$1 \geq |\{B\}|_{\cos} = |\mathcal{D}l(r)B|/\mathcal{M}t(r)B = \prod_{i=2}^{q_1}|\mu_i|^{s'_{1,i}} \Big/ \prod_{j=2}^{q_2}\sigma_j^{s_{2,j}} \geq 0. \qquad (138)$$

The cosine ratio of null-defective B is 0 ($r' < r$), and it is $+1$ or -1 for null-normal B. Formula (137) to the right contains the eigenvalues μ_i with their algebraic multiplicities $s'_{1,i}$ for the matrix B and its singular numbers $\sigma_j > 0$ (for the square root of the matrix BB' or $B'B$) with their algebraic (geometric) multiplicities $s_{2,j}$ in $\mathcal{M}t(r)B$ as in (126).

Let A_1 and A_2 be $n \times m$-matrices with their *external and internal multiplications of cosine type* $B = A_1A_2'$ and $B' = A_2A_1'$, $C = A_1'A_2$ and $C' = A_2'A_1$. Then the cosine ratio for a pair of matrices (or lineors) A_1 and A_2 may be expressed as

$$\{A_1 \cdot A_2'\}_{\cos} = \{A_2 \cdot A_1'\}_{\cos} = \mathcal{D}l(r)\{A_1 \cdot A_2'\}/\mathcal{M}t(r)\{A_1 \cdot A_2'\}. \qquad (139)$$

If A_1 and A_2 are equirank $n \times r$-matrices, then, due to (120) and (132),

$$\{A_1 \cdot A_2'\}_{\cos} = \mathcal{D}l(r)\{A_1 \cdot A_2'\}/(\mathcal{M}t(r)A_1 \cdot \mathcal{M}t(r)A_2) =$$

$$= det\ \{A_1'A_2\}/[\sqrt{det\ \{A_1'A_1\}} \cdot \sqrt{det\ \{A_2'A_2\}}]. \qquad (140)$$

In particular, for the angle between two vectors in the Euclidean space $\langle\mathcal{E}^n\rangle$ we have

$$-1 \le \cos\varphi_{12} = \mathbf{a}_1'\mathbf{a}_2/||\mathbf{a}_1|| \cdot ||\mathbf{a}_2|| = \mathbf{a}_2'\mathbf{a}_1/||\mathbf{a}_2|| \cdot ||\mathbf{a}_1|| \le +1, \quad (\varphi_{12} \in (0;\pi]). \qquad (141)$$

We note here especially, that both left and right sides in formulae (135) or (140) may be considered as some identical algebraic expressions of trigonometric (sine or cosine) nature for coordinates of geometric objects (lineors) represented by $n \times r$-matrices A_1 and A_2. The angle sign is defined only for two vectors on a plane, *may be eigen*, i. e., in $\langle\mathcal{E}^2\rangle$.

For two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ (i. e. if $r = 1$), the expressions in (135), (140) at $n \ge 2$ separately or as the sum of their squared forms give a number of algebraic inequalities or identities of trigonometric (sine and cosine) nature. Their examples are well-known as sine Hadamard Inequality, for example in form (125) at $r = 2$; cosine Cauchy Inequality, for example in form (141). The scalar multiplications of two vectors of sine type in (124) and cosine type in (141) give these Summary identity for their coordinates (here in Euclidean space), which equivalent to Lagrange Identity also for two vectors:

$$[\mathcal{M}t(2)[\mathbf{a}_1|\mathbf{a}_2]/(\mathcal{M}t(1)\mathbf{a}_1 \cdot \mathcal{M}t(1)\mathbf{a}_2)]^2 + [tr\ \mathbf{a}_1\mathbf{a}_2'/(\mathcal{M}t(1)\mathbf{a}_1 \cdot \mathcal{M}t(1)\mathbf{a}_2)]^2 =$$

$$= [det([\mathbf{a}_1|\mathbf{a}_2]'[\mathbf{a}_1|\mathbf{a}_2])]/[\mathbf{a}_1'\mathbf{a}_1 \cdot \mathbf{a}_2'\mathbf{a}_2] + [(\mathbf{a}_1'\mathbf{a}_2)^2]/[\mathbf{a}_1'\mathbf{a}_1 \cdot \mathbf{a}_2'\mathbf{a}_2] = \qquad (142)$$

$$= \sin^2\varphi_{1,2} + \cos^2\varphi_{1,2} = 1 = (\mathbf{a}_1 \times \mathbf{a}_2)^2/||\mathbf{a}_1||^2 \cdot ||\mathbf{a}_2||^2 + (\mathbf{a}_1 \cdot \mathbf{a}_2)^2/||\mathbf{a}_1||^2 \cdot ||\mathbf{a}_2||^2,$$

(where the last variant is a classical sine-cosine Identity of Lagrange for two vectors). Note, that formula (142) enables one to normalize the angles between vectors in Euclidean spaces. In part II of the book, similar constructions for more general linear objects as *lineors*, represented by $n \times m$-matrices A_1 and A_2, will be analyzed.

3.4 Limit evaluation of eigenprojectors and quasi-inverse matrices

According to (1) and (101), the following limit formulae do hold:

$$A^+ = \lim_{\epsilon\to 0}[A'(AA' + \epsilon I)^{-1}] = \lim_{\epsilon\to 0}[(A'A + \epsilon I)^{-1}A'] = \qquad (143)$$

$$= \lim_{N\to\infty}[NA'(NAA' + I)^{-1}] = \lim_{N\to\infty}[(NA'A + I)^{-1}NA'], \qquad (144)$$

($\overrightarrow{A'A}A' = Z = A'\overrightarrow{AA'} \quad \Rightarrow \quad K_1(A'A, r)A' = Z = A'K_1(AA', r)$).

As well as general formulae (71)–(73), the special limit formulae (143), (144) are inferred by pure algebraic way, with use of the resolvent (1).

A. N. Tikhonov [26] was the first who expressed the normal solution of the linear equation $A\mathbf{x} = \mathbf{a}$ as a limit. He used his regularization method in the special case of a conditional extremum problem: find the value of the argument with the minimal Euclidean norm on a given set corresponding to the minimal residual of equation

$$U(\mathbf{x}, \epsilon) = \epsilon F_1(\mathbf{x}) + F_2(\mathbf{x}) = \min, \quad dU/d\mathbf{x} = \mathbf{0} \ \ (\epsilon \to 0). \qquad (145)$$

Here: $(F_1(\mathbf{x}) = \mathbf{x}'\mathbf{x},\ F_2(\mathbf{x}) = \mathbf{d}'(\mathbf{x}) \cdot \mathbf{d}(\mathbf{x})$, and where the residual is $\mathbf{d}(\mathbf{x}) = A\mathbf{x} - \mathbf{a})$.

Note, that similar results, but in limit form (144), might be obtained long before the publication of A. N. Tikhonov by Courant's penalty functions method [18]:

$$W(\mathbf{x}, N) = F_1(\mathbf{x}) + N \cdot F_2(\mathbf{x}) = \min, \quad dW/d\mathbf{x} = \mathbf{0} \ \ (N \to \infty). \tag{146}$$

In this task, both the methods are in one-to-one correspondence consisting in multiplying or dividing by a scalar limit parameter.

Courant's penalty functions method finds the conditional extremum of $F_1(\mathbf{x})$ with the gradient $1 \times n$-vector function in the constraint equation $\mathbf{h}'(\mathbf{x}) = dF_2/d\mathbf{x} = \mathbf{0}$. Integration converts the usual vector form into the equivalent scalar form:

$$h(\mathbf{x}) = \int_{\mathbf{x}_s}^{\mathbf{x}} \mathbf{h}'(\mathbf{x}) d\mathbf{x} = 0 = const. \tag{147}$$

Then in (146) we obtain the Lagrange function $W(\mathbf{x}, N)$ and the *unique* scalar Lagrange multiplier $N \to \infty$, as

$$(dh/d\mathbf{x}) \cdot N = \mathbf{h}'(\mathbf{x}) \cdot N = \mathbf{0} \cdot N = -dF_1/d\mathbf{x} \neq \mathbf{0}$$

follows from the differential equation (146), and consequently $N \to \infty$.

In particular, these limit methods are applicable for finding conditional extremum of $F_1(\mathbf{x})$ on the stationary set of $F_2(\mathbf{x})$. Chains in equations (145) and (146) may be continued by polynomials in ϵ or N. The sufficient condition for applicability these two limit methods in the *differential form* (with the small or large parameter) is, due to (147), integrability of the $1 \times n$-vector function $\mathbf{h}'(\mathbf{x})$ from the constraint equation and consequently symmetry of its Jacobi matrix: $(d\mathbf{h}/d\mathbf{x})' = d\mathbf{h}/d\mathbf{x}$. If the normal solution of equation $A\mathbf{x} = \mathbf{a}$ is searched for, this symmetric Jacobi matrix is $A'A$.

Due to General optimization limit method *differential equation* $\epsilon dF_1/d\mathbf{x} + \mathbf{h}'(\mathbf{x}) = \mathbf{0}$, $\epsilon \to 0$ *or* $dF_1/d\mathbf{x} + N\mathbf{h}'(\mathbf{x}) = \mathbf{0}$, $N \to \infty$, *determines a complete solution according to conditional stationarity of* $F_1(\mathbf{x})$ *under constraint* $\mathbf{h}'(\mathbf{x}) = \mathbf{0}$ *iff the Jacobi matrix of the constraint vector function* $\mathbf{h}'(\mathbf{x})$ *is null-normal*, i. e., $\langle ker\ d\mathbf{h}/d\mathbf{x}\rangle \equiv \langle ker\ (d\mathbf{h}/d\mathbf{x})'\rangle$. (And at the stationarity point of $F_1(\mathbf{x})$ for the $1 \times n$-vector of the conditional gradient, obviously, there holds: $dF_1/d\mathbf{x} \cdot \overrightarrow{d\mathbf{h}/d\mathbf{x}} \in \langle ker\ d\mathbf{h}/d\mathbf{x}\rangle$.)

The conditional stationarity nature of $F_1(\mathbf{x})$ *(i. e., either a conditional minimum or a conditional maximum, or a conditional saddle without extremum) is determined by the limit conditional Hesse matrix of* $F_1(\mathbf{x})$ *up to scalar parameter* ϵ *or* N.

See detailed exposition of this General optimization limit method and its applications in other our monograph [17, p. 97–112]. In particular, this method gives, by such simple way, the exact solutions for a conditional extremum of the second-order scalar function $Q(\mathbf{x})$ under the linear constraint equation $Bm \cdot \mathbf{x} = \mathbf{a}$, including $Bm = S$.

Moreover, the constant singular Jacoby null-prime matrix Bp for the linear constraint equation $Bp \cdot \mathbf{x} = \mathbf{a}$ may be transformed into the null-normal matrix Bm by a suitable modal transformation of the initial base (further, this limit method may by applied). As example, for a null-prime matrix Bp, its affine quasi-inverse matrix Bp^-, see (69), may be computed by the same limit way with preliminary use of linear base transformation for converting Bp into Bm. Then one calculates Bm^- by the limit method due to its value in (104), i. e., factually as the Moor-Penrose quasi-inverse matrix. Having finished these operations, one returns to the initial base by the reverse modal transformation, and get the matrix Bp^-.

Further, in Ch. 8, the trigonometric sense of the sine and cosine ratios from sections. 3.2, 3.3 will be explained on the basis of the trigonometric spectrum of a null normal B.

Chapter 4

Main alternative variants of complexification

4.1 Comparing alternative variants of complexification

Until this chapter, we have not particularly touched on the question: what arithmetic content is permissible and can appear instead of letter designations in formulae, inequalities, and in various statements using them. From the preliminary section "Notations", together with its complete at first Matrix Alphabet, it is clear that these abstract letter notions will appear in the further presented Tensor Trigonometry and in its numerous mathematical and physical applications else in a very large number using all Latin and Greek alphabet. Of course, the notations of logical operations linking these literal notions have nothing to do with what was said above. For scalar notions, instead of their letter designations, we can mean, under the conditions of their admissibility and expediency, specific kinds of numbers, including those with zero. Here we consider the main variants associated with the use of complex numbers, what is usually defined as complexification of the original real concepts.

Nature of complex numbers gives rise to main two and quite different approaches for implementing operations over initially given complex algebraic or numerical elements. The complex elements may have due to these operations the corresponding form of presentations.

By the *adequate* approach, operations over complex-number elements are formally the same as over real-number ones. This allows one to use results previously obtained for real-number analogous objects. However, there are some exceptions: inequalities (unless parameters are only real), module notions. The special case is *pseudoization*, when real and imaginary parts of complex elements form direct sums of the same type.

The *symbiotic* approach supposes the use of standard operations applied to real numbers as well as the additional operation of complex conjugation independent on usual ones. In particular, it takes place in the *Hermitean* approach for vectors and matrices with complex entries: their transposition is always accompanied by complex conjugation. The Hermite's variant of complexification allows one to use in the self-conjugate form notions of the real positive module or norm as well as similar self-conjugate form for a lot of inequality relations.

These different variants of complexification point out the two independent directions for further development of theories and their applications in complex spaces.

So, identities of types $\langle im\ B\rangle \equiv \langle im\ B'\rangle$ and $\langle im\ B\rangle \equiv \langle imB^{*}\rangle$ determine accordingly adequately and Hermitean null-normal matrices. But adequately and Hermitean orthogonal eigenprojectors and quasi-inverse matrices are defined by different ways using (98)–(101). Adequate complex characteristics no always exist in such determined form in what Hermitean ones exist. As example, $\mathcal{M}t^2(r)A = k(AA', r) = k(A'A, r)$ for a complex matrix, where $r = rangA$, may have any complex values including zero.

But for pseudoized vectors and matrices their squared minorant may have only real values — positive, negative and zero. From the other hand, in the Hermitean variant there holds $k(AA^{*}, t) = k(A^{*}A, t) > 0, \quad t \leq r$.

In any case, all eigenprojectors of a null-prime matrix Bp exist and are spectrally nonnegative semi-definite matrices, because their eigenvalues are equal to $+1$ and 0. Moreover, for matrices Bp affine eigenprojectors and quasi-inverse matrices do not depend on the complexification variant. If a matrix B is complex and nonsingular, then $\langle im\ B\rangle \equiv \langle imB'\rangle \equiv \langle imB^{*}\rangle \equiv \langle \mathcal{A}^n\rangle$, that is why the complex inverse matrix B^{-1} for such quadratic matrix B is uniquely determined.

Forms of representing any complex number "a" with the imaginary unit "i" are well-known and various. They are simplest arithmetic form, trigonometric Moivre's and polar forms, exponential Euler's form, pseudoized vectorial form, stereographic Riemann's form. For further aims, we use a *normal* 2×2-matrix form – without using the imaginary unit "i":

$$\left.\begin{array}{l} W(a) \equiv F(\rho,\varphi),\ \ (\varphi \in [-\pi;+\pi]): \\ \left[\begin{array}{c|c} p & -q \\ \hline +q & p \end{array}\right] = \rho \left[\begin{array}{cc} \cos\varphi & -\sin\varphi \\ +\sin\varphi & \cos\varphi \end{array}\right] = S + K \\ (a = p + iq); \\ \\ W'(a) \equiv F'(\rho,\varphi): \\ \left[\begin{array}{c|c} p & +q \\ \hline -q & p \end{array}\right] = \rho \left[\begin{array}{cc} \cos\varphi & +\sin\varphi \\ -\sin\varphi & \cos\varphi \end{array}\right] = S - K \\ (\bar{a} = p - iq), \end{array}\right\} \qquad (148)$$

Then, we have the properties:
$W(a)\cdot W'(a) = W'(a)\cdot W(a) = \rho^2 \cdot I_{2\times 2}$, $S = S'$, $K = -K'$, $SK = KS$.

Note especially, that this real form $W(a)$ is also single-valued as usual one. In particular, such form may be used in its simplest 2×2-matrix normal form for representation of paired solutions of a real-valued algebraic equation formally with conjugate roots, and with possible following generalization.

For instance, with such unusual approach one may proved simply that a *real-valued* algebraic equation of power n has always a complete *real-valued* similar simplest matrix solution unique up to admitted permutations of its 2×2-cells!

There holds $W(a_1)\cdot W(a_2) = W(a_1\cdot a_2) \equiv F(\rho_1,\varphi_1)\cdot F(\rho_2,\varphi_2) = F[\rho_1\cdot\rho_2,(\varphi_1+\varphi_2)]$. The form $W(a)$ executes summation and multiplication so as the arithmetic form a.

Besides, the real forms $W(a_j)$ of complex numbers a_j as well as the scalar complex form a_j are commutative in their summations and multiplications, and satisfy all formulae and identities for complex numbers. They compile the pairs of mutually transposed matrices in (148) similarly to the pairs of conjugate complex numbers.

Formally $W(a)$ represents a given complex number a in the arithmetical affine of the *real normal matrices space* $\langle\mathcal{A}^{2\times 2}\rangle$ of the binary type.

The *trigonometric form* $F(\rho,\varphi)$ in (148) represents the complex number a in the arithmetical Euclidean of the *real normal matrices space* $\langle\mathcal{E}^{2\times 2}\rangle$ of the binary type.

From this point of view, a *real-valued normal* $n\times n$-matrix M represents in a certain affine or Cartesian base $2k \le \lfloor n \rfloor$ complex conjugate numbers and $n-2k$ real-valued ones, i. e., $M = RWR'$. A *real-valued prime* matrix $P = VWV^{-1}$ represents in a certain affine base these numbers. Generally, here W is a *canonical normal monobinary* cell form of the matrices M and P. Their decompositions, as a direct sums, contain only real 1×1- and 2×2-cells.

In general, the matrix W, up to permutations of its cells, is the simplest *real solution* of secular equation $c(\mu) = 0$. Applying the Cayley-Hamilton Theorem to the prime matrix P gives $V^{-1}\{c(P)\}V = c(W) = Z$. Similar W-forms of such simplest matrices will be use in Part II of the book for clear inferring of the tensor trigonometry some formulae.

In its turn, real matrix form (148) may be complexified too, either in the adequate or Hermitian variant. In the first case, there holds

$$\left.\begin{array}{l} W(z_1): \\ \left[\begin{array}{c|c} u & -v \\ \hline v & u \end{array}\right] = \rho \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} = S + K \\ (z_1 = u + iv), \\ W'(z_1) = W(z_2): \\ \left[\begin{array}{c|c} u & v \\ \hline -v & u \end{array}\right] = \rho \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} = S - K \\ (z_2 = u - iv). \end{array}\right\} \qquad (149)$$

Then, we have the properties:

$[W(z) \cdot W'(z) = W'(z) \cdot W(z) = \rho^2 \cdot I_{2\times 2},\ S = S',\ K = -K',\ SK = KS.]$

Complex adequately normal W-form (149) is implemented in a some adequately Cartesian base of the *complex-valued* Euclidean space $\langle \mathcal{E}^{2\times 2}\rangle$ over $\mathbb{C}$. A complex adequately normal $n \times n$-matrix $M = R\,W R'$ may represent double quantity of non-conjugate complex numbers (i. e., as z_1 and z_2) in the similar bases. All the elements of its W-form are complex numbers, including the module ρ and the angle ψ. The complex normal matrix M may be simplified with some adequately orthogonal transformation R (also complex) and represented in complex canonical W-form (149).

In the second case, in the Hermitean variant, there holds

$$\begin{array}{ll} W(z): & W^*(z) = W'(\bar z): \\ \left[\begin{array}{c|c} u & -\bar v \\ \hline v & \bar u \end{array}\right] = H + Q & \left[\begin{array}{c|c} \bar u & \bar v \\ \hline -v & u \end{array}\right] = H - Q, \\ (z = u + iv), & (\bar z = \bar u - i\bar v), \end{array} \qquad (150)$$

$$[W(z) \cdot W^*(z) = W^*(z) \cdot W(z),\ H = H^*,\ Q = -Q^*,\ HQ = QH].$$

Complex Hermitean normal W-form (150) is implemented in a certain Cartesian base of the unitary space $\langle \mathcal{U}^{2\times 2}\rangle$. Its two eigenvalues are the complex conjugate numbers so as in (148). Hence, this complex normal form is simplified with some Hermitean orthogonal transformation U till converting into real W-form of type (148). The full set $\langle UWU^*\rangle$ is the *specified set of complex normal matrices*, that may by reduced by some modal transformations till canonical forms (150) and (148).

These normal matrices are interesting in Hermitean tensor trigonometry. Their conjugate eigenvalues are $d_t = \rho_t \exp(\pm i\beta_t)$, $\rho_t \in (-\infty + \infty)$, $\beta_t \in [-\pi/2; +\pi/2]$; for *Hermitean orthogonal matrices:* $d_t = \exp(\pm i\beta_t)$. Moreover, a pair of conjugate elements in their diagonal forms correspond to a trigonometric 2×2-cell of some Hermitean rotation for the geometric transformation of elements in a basic unitary space. (But general complex $n \times n$-normal matrices are simplified with some unitary transformations till their diagonal forms with n entries of the type $d_t = \rho_t \exp(i\beta_t)$!)

These questions are discussed more in details in Part II, Ch. 10.

4.2 Examples of adequate and pseudoized complexifications

Typical examples of *adequate complexification* are the following:

• formulae for roots of algebraic equations with complex coefficients,

• algebraic identities including ones of trigonometric nature – see in Chs. 3 and 8,

• trigonometric formulae for complex angles and their functions,

• analytical (holomorphic) functions, their expansions into power series,

• formulae for derivatives, differentials and integrals for functions of scalar and vectorial complex arguments.

(Everywhere real-number elements are substituted by complex ones.)

In a space over $\mathbb{C}$ with an adequate type of metric, the measures of length and angles are necessary complex. However, in a pseudo-Euclidean space, these measures may be real, zero or imaginary. Give below the following main examples for the pseudo-Euclidean space of index $q = 1$ (see more in Part II, Chs. 6, 11, 12, and in the large Appendix):

• Minkowski Geometry and pseudo-Euclidean tensor trigonometry in *elementary form* as the additional new important part of this Geometry,

• external pseudo-spherical non-Euclidean geometries on the spheres of the imaginary and real radius (i. e., of two types), embedded into pseudo-Euclidean space. (These two geometries with tensor hyperbolic and orthospherical functions in elementary forms are isometric to Lobachevsky–Bolyai and Beltrami geometries).

Consider examples of applications of the *adequate complexification in theory of analytical functions of scalar and vectorial complex variable* and in theory of matrices.

Let $\mathbf{x}, \mathbf{y} \in \langle \mathcal{E}^n \rangle$, and $\mathbf{z} = \mathbf{x} + \mathbf{iy}$ be an $(n\times 1)$-vector argument in a n-dimensional complex Euclidean space, $F(\mathbf{z}) = F_1(\mathbf{x}, \mathbf{y}) + iF_2(\mathbf{x}, \mathbf{y})$ be a certain scalar complex analytical function of $\mathbf{z}$. Differentiation and integration with respect to an $(n \times 1)$-vector-argument in the Euclidean space are expressed in Cartesian coordinates. Total derivatives, differentials, and integrals have adequate analogues from which partial characteristics and their relations are clear and obviously inferred:

$$dF = \mathbf{h}(\mathbf{z})d\mathbf{z} \;\Leftrightarrow\; dF = dF_1 + idF_2 = (\mathbf{h_1}(\mathbf{x}, \mathbf{y}) + i\mathbf{h_2}(\mathbf{x}, \mathbf{y}))(d\mathbf{x} + id\mathbf{y}) =$$

$$= [\mathbf{h_1}(\mathbf{x}, \mathbf{y})d\mathbf{x} - \mathbf{h_2}(\mathbf{x}, \mathbf{y})d\mathbf{y}] + i[\mathbf{h_1}(\mathbf{x}, \mathbf{y})d\mathbf{y} + \mathbf{h_2}(\mathbf{x}, \mathbf{y})d\mathbf{x}].$$

Here the $1 \times n$-vector partial derivatives (gradients) form pairs:

$$\left.\begin{aligned} \mathbf{h_1}(\mathbf{x}, \mathbf{y}) &= \frac{\partial F_1}{\partial \mathbf{x}} = \frac{\partial F_2}{\partial \mathbf{y}}, \\ \mathbf{h_2}(\mathbf{x}, \mathbf{y}) &= -\frac{\partial F_1}{\partial \mathbf{y}} = \frac{\partial F_2}{\partial \mathbf{x}}. \end{aligned}\right\} \qquad (a)$$

This is the vector-form of classical d'Alembert–Euler Equations for the scalar functions F_1, F_2 totally differentiable with respect to arguments $\mathbf{x}$, $\mathbf{y}$ (or for totality of two differential expressions above in square brackets).

Apply the same scheme of reasoning to the $1 \times n$-vector function

$\dfrac{dF}{d\mathbf{z}} = \mathbf{h}(\mathbf{z}) = \mathbf{h_1}(\mathbf{x}, \mathbf{y}) + i\mathbf{h_2}(\mathbf{x}, \mathbf{y})$:

$$\left.\begin{aligned} \frac{\partial \mathbf{h_1}}{\partial \mathbf{x}} &= \frac{\partial \mathbf{h_2}}{\partial \mathbf{y}} = \frac{\partial^2 F_1}{\partial \mathbf{x}^2} = -\frac{\partial^2 F_1}{\partial \mathbf{y}^2} = \frac{\partial^2 F_2}{\partial \mathbf{x}\partial \mathbf{y}} = \frac{\partial^2 F_2}{\partial \mathbf{y}\partial \mathbf{x}} = \left(\frac{\partial \mathbf{h_1}}{\partial \mathbf{x}}\right)', \\ \frac{\partial \mathbf{h_1}}{\partial \mathbf{y}} &= -\frac{\partial \mathbf{h_2}}{\partial \mathbf{x}} = \frac{\partial^2 F_2}{\partial \mathbf{y}^2} - \frac{\partial^2 F_2}{\partial \mathbf{x}^2} = \frac{\partial^2 F_1}{\partial \mathbf{y}\partial \mathbf{x}} = \frac{\partial^2 F_1}{\partial \mathbf{x}\partial \mathbf{y}} = \left(\frac{\partial \mathbf{h_1}}{\partial \mathbf{y}}\right)'. \end{aligned}\right\} \qquad (b)$$

The first equalities in chains (b) are the matrix-form d'Alembert–Euler equations for the vector functions $\mathbf{h_1}$ and $\mathbf{h_2}$ totally differentiable in terms of $\mathbf{x}$, $\mathbf{y}$. Together they express, as well as symmetry of Jacobi matrices due to symmetry of Hesse matrices, necessary and sufficient conditions for totality of the second differential F also in terms of $\mathbf{x}$, $\mathbf{y}$. The matrix-forms Laplace Equations for the harmonic functions F_1, F_2 of the real variables $\mathbf{x}$, $\mathbf{y}$ follow from the additional matrix equations in (b).

In a pseudo-Euclidean space $\langle \mathcal{E}^{n+q} \rangle$ (in the binary complex form), due to its special structure, the characteristics described above are changed:

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ i\mathbf{y} \end{bmatrix}; \quad dF = \mathbf{h}(\mathbf{z})d\mathbf{z} \Leftrightarrow dF = dF_1 + idF_2 =$$

$$= ([\ \mathbf{h_1} \mid \mathbf{t_1}\ ] + i[\ \mathbf{h_2} \mid \mathbf{t_2}\ ]) \left[\frac{d\mathbf{x}}{id\mathbf{y}} \right] =$$

$$= [\mathbf{h_1}(\mathbf{x},\mathbf{y})d\mathbf{x} - \mathbf{t_2}(\mathbf{x},\mathbf{y})d\mathbf{y}] + i[\mathbf{t_1}(\mathbf{x},\mathbf{y})d\mathbf{y} + \mathbf{h_2}(\mathbf{x},\mathbf{y})d\mathbf{x}].$$

Here

$$\left. \begin{array}{l} \mathbf{h_1}(\mathbf{x},\mathbf{y}) = \dfrac{\partial F_1}{\partial \mathbf{x}}, \quad \mathbf{h_2}(\mathbf{x},\mathbf{y}) = \dfrac{\partial F_2}{\partial \mathbf{x}}, \\ \mathbf{t_1}(\mathbf{x},\mathbf{y}) = \dfrac{\partial F_2}{\partial \mathbf{y}}, \quad \mathbf{t_2}(\mathbf{x},\mathbf{y}) = -\dfrac{\partial F_1}{\partial \mathbf{y}}; \end{array} \right\} \qquad (a')$$

$$\left. \begin{array}{l} \dfrac{\partial \mathbf{h_1}}{\partial \mathbf{x}} = \dfrac{\partial^2 F_1}{\partial \mathbf{x}^2} = \left(\dfrac{\partial \mathbf{h_1}}{\partial \mathbf{x}}\right)', \quad \dfrac{\partial \mathbf{t_1}}{\partial \mathbf{y}} = \dfrac{\partial^2 F_2}{\partial \mathbf{y}^2} = \left(\dfrac{\partial \mathbf{t_1}}{\partial \mathbf{y}}\right)', \\ \dfrac{\partial \mathbf{h_2}}{\partial \mathbf{x}} = \dfrac{\partial^2 F_2}{\partial \mathbf{x}^2} = \left(\dfrac{\partial \mathbf{h_2}}{\partial \mathbf{x}}\right)', \quad \dfrac{\partial \mathbf{t_2}}{\partial \mathbf{y}} = -\dfrac{\partial^2 F_1}{\partial \mathbf{y}^2} = \left(\dfrac{\partial \mathbf{t_2}}{\partial \mathbf{y}}\right)', \\ \dfrac{\partial \mathbf{h_1}}{\partial \mathbf{y}} = \dfrac{\partial^2 F_1}{\partial \mathbf{x}\partial \mathbf{y}} = \left(\dfrac{\partial^2 F_1}{\partial \mathbf{y}\partial \mathbf{x}}\right)' = -\left(\dfrac{\partial \mathbf{t_2}}{\partial \mathbf{x}}\right)', \\ \dfrac{\partial \mathbf{t_1}}{\partial \mathbf{x}} = \dfrac{\partial^2 F_2}{\partial \mathbf{y}\partial \mathbf{x}} = \left(\dfrac{\partial^2 F_2}{\partial \mathbf{x}\partial \mathbf{y}}\right)' = \left(\dfrac{\partial \mathbf{h_2}}{\partial \mathbf{y}}\right)'. \end{array} \right\} \qquad (b')$$

In this case, $F_1(\mathbf{x},\mathbf{y})$, $F_2(\mathbf{x},\mathbf{y})$ are not harmonic in the Sense of Laplace.

The real analogues exist for *purely real* parameters used previously. In particular, for matrices they are the rank, the 1-st and 2-nd rock. Parallelism of linear objects is an affine property, that is why it does not depend on the complexification variant. However, optimal procedures for parallelism checking in a real space and complex one may differ.

Suppose that $n \times m$-matrices A_1 and A_2 determine linear subspaces (or linear objects) in the affine space $\langle \mathcal{A}^n \rangle$. The procedure for parallelism recognizing uses here characteristic symmetric projectors. If ranks of A_1 and A_2 are equal, then process (94) may be run in the simplest variant.

In more general case, consider an $n \times n$-matrix with the same image, i. e., $\langle im\ AC \rangle \equiv \langle im\ A \rangle$, where C is an $m \times n$-matrix such that:

1) $\langle im\ C \rangle \cap \langle ker\ A \rangle = \mathbf{0} \Leftrightarrow rank\ AC = rank\ A$,

2) $k(AC, r) \neq 0$.

In a space over $\mathbb{R}$ one may put $C = A'$, in a space over $\mathbb{C}$ put $C = A^*$. In general, the following holds:

1. $\langle im\ A_2\rangle \subseteq \langle im\ A_1\rangle \ \Leftrightarrow\ \overleftarrow{A_1C_1} \cdot A_2 = A_2 \ \Leftrightarrow\ \overrightarrow{A_1C_1} \cdot A_2 = Z,$

   $\langle im\ A_1\rangle \subseteq \langle im\ A_2\rangle \ \Leftrightarrow\ \overleftarrow{A_2C_2} \cdot A_1 = A_1 \ \Leftrightarrow\ \overrightarrow{A_2C_2} \cdot A_1 = Z.$

2. $\langle im\ A_2\rangle \equiv \langle im\ A_1\rangle \ \Leftrightarrow\ \overrightarrow{A_1C_1} \cdot A_2 = Z = \overrightarrow{A_2C_2} \cdot A_1.$

On the other hand, orthogonality of linear objects is the notion depending on a metric in a given space.

In a real Euclidean space or in a complex Euclidean space with the adequate metric variant, orthogonality is recognized by the condition:

$$\langle im\ A_1\rangle \perp \langle im\ A_2\rangle \ \Leftrightarrow\ A_1'A_2 = Z = A_2'A_1.$$

But in a complex Euclidean space with the Hermitean metric variant, it is recognized by the condition:

$$\langle im\ A_1\rangle \perp \langle im\ A_2\rangle \ \Leftrightarrow\ A_1^*A_2 = Z = A_2^*A_1.$$

Here the both (left and right) conditions equations are equivalent.

4.3 Examples of Hermitean and symbiotic complexification

Hermite's complexification may be used almost in any case when it is necessary to decide problems in a complex space with vectorial objects. Hence we indicate only some examples, most close to our theme:

• positive norms for lengths, surfaces, volumes etc. of a different geometric objects in the Hermitean space;

• positive norms for the angle and its functions in an Hermitean plane;

• previous results expressed in the self-conjugate form, in particular, formulae and a lot of inequalities (98)–(103), (115)–(130), (132)–(144) with Lagrange Identity (142), especially:

• • minorant positivity for the linear objects in an Hermitean space,

• • formulae (122) and (136) expressing the Kronecker–Capelli Theorem,

• • Hadamard and Cauchy Inequalities of the sine and cosine types (Ch. 3), they are important for the trigonometry on an Hermitean plane with definition of Hermitian spherical trigonometric functions of angles between vectors using Hermiteized them as normalizing;

• • Sine general and Cosine general Inequalities (Chs. 3 and 8), they are important for the tensor trigonometry in an Hermitean space (see further in sect. 10.1 as the basis for definition of Hermitean spherical tensor trigonometric functions of angles between lineors using the Sine and Cosine Hermiteized normalizing inequalities);

• In particular, in the Quantum Mechanics, Hermiteanly orthogonal matrices are used to represent some observable paired physical values. This is based on the fact that the Heisenberg Uncertainty Principle is generated mathematically from the Hermiteized form of the Cosine Cauchy Inequality for a pair of complex vectors (see above in sect 2.3). And in addition, using the general Cosine Inequality for a pair of complex lineors, also in the Hermiteized form, it is possible to pass to more general quantume estimates;

• All the limit functional methods (sect. 3.4) act very well in the Hermiteized forms;

• *Maximum Modulus Principle*, in general form, it holds for scalar and vectorial complex functions of complex single and many variables – see this Principle's original inferring as a particular case in the our mathematical monograph [17, p. 127].

Most general is the *symbiotic approach*. Its application to the classical theory of analytical functions and basic operations of calculus (orthogonal differentiation and integration) gives the following *symbiotic analogues*:

• expansions into power series in conjugate variables $\mathbf{z}$ and $\overline{\mathbf{z}}$ for special analytical nonholomorphic functions of $\mathbf{z}$ and $\overline{\mathbf{z}}$, i. e., the not analytical functions in the Sense of Riemann,

• special rules of symbiotic (conjugate) differentiation and integration,

• special conditions for differentiability and analyticity for functions of the conjugate variables $\mathbf{z}$ and $\overline{\mathbf{z}}$,

• special conditions for integrability of a certain differential expression (i. e., of the differential totality),

• symbiotic methods for finding extrema of scalar real functions of conjugate variables (the preliminary necessary condition to such scalar function is its symmetry with respect to the conjugate arguments). This is further development of formal derivatives idea (see, for example, [19]) in analysis of nonholomorphic complex-variable functions. We illustrate the extremal problems by the following two examples, close to our theme,

• • extrema of the scalar real functions (from sect. 1.2) expressing the differences or ratios of corresponding means formed of all the algebraic equation roots, if the roots are positive and complex conjugate (see our methods of solving similar tasks in [17, p. 124–135]),

• • minimizing squared Hermitean module of complex equation residual (116), i. e., scalar real function $F = ||A\mathbf{x} - \mathbf{a}||^2_H$ with inferring complex limit formulae (143, 144).

* * *

Post scriptum to the Part I. In conclusion of this introductory part I as the initial basis for subsequent development of the tensor trigonometry in part II, the author considers it necessary to note the following. A lot of new provisions, characteristics and formulae of the part I were established by the author else at the beginning of 1981. However, they were not accepted then to publications in the leading Soviet mathematical journals – see more about this on the author's web-sites. These contents were published many later, in 2004, in his monograph [15]. In particular, this has place for the structure of matrix characteristic coefficients, for the new parameters of matrices singularity with fundamental relations and inequalities connecting them, for the explicit form of a minimal annulling polynomial, for the explicit formulae of all eigenprojectors and quasi-inverse matrices in terms of elements of an initial matrix, for the definition and applications of null-prime and null-normal matrices, for the exact and explicit normal solution of linear equations with formulae for pseudoinverse matrices – algebraic and limit ones, for the new algebraic notions as a minorant and a dianal of a matrix with their useful properties in the theory of linear algebraic equations and matrices, etc.. But some contents from this series began to appear later in publications from the same circle of mathematicians which did not accept all indicated above. For this reason, the author did not consider to make references to these publications with as if "their results". The same applies to plagiarist publications, in that number, in Wikipedia with borrowings from [15] of 2004 and later.

All of plagiarists were surpassed by the Ukrainian publishing house "Освіта України" ("Light of Ukraine") issued my "Tensor Trigonometry - 2004" after 10 years in 2015, without changes, but under other "author" name, with reviews to it from two Ukrainian Professors – Drs of sciences!!!

Probably, some, especially novice authors, have encountered with similar ethically unacceptable phenomena when, due to the lack of affiliations or connections for them to receive official reviews, their scientific works are not published, but then, due to the absence of these results in the literature, other "authors" with affiliations and connections quietly use them in their similar own publications. The author writes this here and sometimes further only for the purpose of additional information about generating a number of early suggested by him mathematical ideas, relations and results.

Therefore, in our overly politicized time, only the author himself should defend his scientific priorities and, along the way, the priorities of other authors appropriated by someones, when they have already left earthly life, and not hoping that others with increased a sense of justice will do it. Results presented in Part I are for 20th cent., results of Part II and Appendix are to September 2004.

Part II

Tensor Trigonometry: fundamental contents

This basic part of the book begins by large Chapter 5 in which Tensor Trigonometry is developing in spaces with Euclidean metric, and further in the way, with preliminary introducing the so-called *reflector tensor* of the so-called *quasi-Euclidean space* with their strong definitions. The reflector tensor is a symmetric matrix with eigenvalues -1 and $+1$, it is $\{I^{\pm}\}$ in the simplest case or $R\{I^{\pm}\}R'$ generally. It divides this binary quasi-Euclidean space into its direct orthogonal sum from two subspaces corresponding to these eigenvalues!

In the 1-st half of Chapter 5 (sect. 5.1–5.6), *projective and reflective* Tensor Trigonometry is constructed. It is developing with using eigenprojectors from the rectangular or square matrices. The projective spherical trigonometric functions and reflectors with tensor angles of also projective type between $n \times r$ lineors A_1 and A_2 or their images $\langle im\ A_1 \rangle$ and $\langle im\ A_2 \rangle$ (i. e., as planars of rank r_1 and r_2) are defined. In other interpretation, the tensor functions with their angles are defined by the same manner between two images of the singular null-prime $n \times n$-matrix $\langle im\ B \rangle$ and $\langle im\ B' \rangle$ (i. e., as planars of rank r). Then canonical structures of projective tensor trigonometric functions and reflectors are installed.

In the 2-nd half of Chapter 5 (sect. 5.7–5.12), we transit naturally into *rotational and deformational* Tensor Trigonometry constructed in Euclidean and quasi-Euclidean spaces. The motive tensor trigonometric functions in this version represent rotational (sine-cosine) and deformational (tangent-secant) matrix transformations). In the Chapter end, they are gotten in the so-called *elementary* forms, i. e., with one motive tensor spherical *eigen angle* of rotations–motions and corresponding to the given reflector tensor of the space of index $q = 1$. The reflector tensor determines the mono-binary canonical structure in some Cartesian base for all main concepts of the entire *quasi-Euclidean trigonometry* besides its Euclidean metric.

In Chapter 6 Tensor Trigonometry in the pseudo-Euclidean space with the identical *reflector metric tensor* $I^{\pm}$ and with corresponding to it sign-indefinite quadratic metrics is constructed with the wide use of the *abstract and specific spherical-hyperbolic analogies*. Scalar Trigonometry is exposed on the pseudoplane with the complete solution of the pseudo-Euclidean right triangles and with complete tensor trigonometric relations between principal and complementary hyperbolic angles. For geometries with principal hyperbolic angles, the *especial hyperbolic angle (number)* ω is introduced as the hyperbolic analog of $\pi/4$ (which corresponds to a hyperbola focus). The descriptive connections of spherical and hyperbolic principal angles, their functions, rotors and reflectors are given in the especial *Quart circle*. In the Chapter end, the motive hyperbolic functions are inferred in the *elementary* forms.

In Chapter 7 the trigonometric nature of matrices commutativity and anticommutativity is established as the separate important application for real-valued and Hermitian variants.

In Chapter 8 the trigonometric spectrums for a null-prime matrix and for a pair of lineor are established, which serve as a basis for inferring the general cosine and sine normalizing new matrix inequalities. They give opportunity for correct defining of trigonometric norms with cosine and sine relations for matrix objects. See preliminary about them in Chapter 3.

In Chapter 9 the correct quadratic norms of matrices and lineors as some geometric objects are defined with the use of the general inequality for average values from Chapter 1.

In Chapters 10, 11, 12 Tensor Trigonometry is developed in the complex adequate and Hermitian metric spaces, and in realifyed pseudoized spaces. Large attention is spared from the Tensor Trigonometry point of view to studying motions in the pseudo-Euclidean space of index q and separately in the Minkowskian space–time of index $q = 1$, with the embedded into them two concomitant hyperboloidal hyperspaces with hyperbolic geometry. So, various trigonometric models of two hyperbolic geometries in the large are inferred. In the end, the Special mathematical principle of relativity is formulated for its use in the large Appendix.

Chapter 5

Euclidean and Quasi-Euclidean tensor trigonometry

5.1 Objects of tensor trigonometry and their spatial relations

According to the Cantor–Dedekind Continuum Axiom [21, p. 99], affine and arithmetic spaces of the same dimension are isomorphic, therefore their metric forms are isomorphic too. Due to this, results, obtained by algebraic ways, may be geometrically interpreted; and vice versa. Primary elements of the n-dimensional affine space are points and free vectors, according to the axiomatic determination by Hermann Weyl. Their coordinates in a certain base are represented by n-tuples of numbers. Points and vectors form geometric objects. There are centralized and noncentralized geometric objects. Centralized geometric object has its application point in the center of a given coordinates system. There is the following correspondence between the equivalent algebraic and geometric forms of linear objects in these two spaces $\langle \mathcal{A}^n \rangle$:

a vector $\mathbf{a}$	–	a straight line segment,
an image $\langle im\ \mathbf{a} \rangle$	–	a straight line,
a kernel $\langle ker\ \mathbf{a}' \rangle$	–	a hyperplane,
$n \times r$-lineor A of rank r	–	an r-simplex,
an image $\langle im\ A \rangle$	–	a planar of rank r,
a kernel $\langle ker\ A' \rangle$	–	a planar of rank $n - r$.

Note, due to (100) there holds $\langle im\ A \rangle \oplus \langle ker\ A' \rangle \equiv \langle \mathcal{A}^n \rangle$ (direct and orthogonal in $\langle \mathcal{E}^n \rangle$ sum). These simplest linear geometric objects of developing tensor trigonometry have a valency 1. A valency for nonanalitic functions of objects may be other. For example, the internal and external multiplications of two vectors have the valency respectively 0 and 2:

$$\mathbf{a}_1'\mathbf{a}_2 = c = \mathbf{a}_2'\mathbf{a}_1, \quad \mathbf{a}_1\mathbf{a}_2' = B = \{\mathbf{a}_2\mathbf{a}_1'\}'. \qquad (151), (152)$$

Separate the class of *equirank* $n \times r$-lineors and planars. The planars may be determined also by any singular null-prime $n \times n$-matrices Bp (we shall denote the matrices briefly as B unless another sense is noted). Generally, for a pair of planars ($rank\ A_1 = r_1$, $rank\ A_2 = r_2$) relations of parallelism in $\langle \mathcal{A}^n \rangle$ and orthogonality in $\langle \mathcal{E}^n \rangle$ with the use of eigenprojectors from Ch. 2 in affine and Euclidean spaces are the following:

$$\left. \begin{array}{l} \langle im\ A_1 \rangle \equiv \langle im\ A_2 \rangle \ \Leftrightarrow\ \overleftarrow{A_1A_1'} = \overleftarrow{A_2A_2'} \ \Leftrightarrow \\ \Leftrightarrow\ \overrightarrow{A_1A_1'} = \overrightarrow{A_2A_2'} \ \Leftrightarrow\ \langle ker\ A_1' \rangle \equiv \langle ker\ A_2' \rangle, \end{array} \right\} \text{for equirank planars } (r_1 = r_2), \qquad (153)$$

$$\left. \begin{array}{l} \langle im\ A_2 \rangle \subseteq \langle im\ A_1 \rangle \ \Leftrightarrow\ \overleftarrow{A_1A_1'} \cdot \overleftarrow{A_2A_2'} = \overleftarrow{A_1A_1'} = \\ = \overleftarrow{A_2A_2'} \cdot \overleftarrow{A_1A_1'} \ \Leftrightarrow\ \overleftarrow{A_1A_1'} \cdot A_2 = A_2 \ \Leftrightarrow \\ \Leftrightarrow \overrightarrow{A_1A_1'} \cdot A_2 = Z = A_2' \cdot \overrightarrow{A_1A_1'} \Leftrightarrow \langle ker\ A_1' \rangle \subseteq \langle ker\ A_2' \rangle, \end{array} \right\} \quad (r_2 \leq r_1), \qquad (154)$$

$$\left. \begin{array}{l} \langle im\ A_2 \rangle \subseteq \langle ker\ A_1' \rangle \ \Leftrightarrow\ A_1'A_2 = Z_1,\ A_2'A_1 = Z_2 \ \Leftrightarrow \\ \Leftrightarrow\ \langle im\ A_1 \rangle \subseteq \langle ker\ A_2' \rangle \ \Rightarrow\ \langle im\ A_1 \rangle \cap \langle im\ A_2 \rangle = \mathbf{0}, \end{array} \right\} \Rightarrow (r_1 + r_2 \leq n), \qquad (155)$$

$$\left. \begin{array}{l} \langle ker\ A_1' \rangle \subseteq \langle im\ A_2 \rangle \ \Leftrightarrow\ \overleftarrow{A_2A_2'} \cdot \overrightarrow{A_1A_1'} = \overrightarrow{A_1A_1'} \ \Leftrightarrow \\ \Leftrightarrow\ \overrightarrow{A_2A_2'} \cdot \overrightarrow{A_1A_1'} = Z = \overrightarrow{A_1A_1'} \cdot \overrightarrow{A_2A_2'} \Leftrightarrow \\ \Leftrightarrow\ \langle ker\ A_2' \rangle \subseteq \langle im\ A_1 \rangle \ \Rightarrow\ \langle ker\ A_1' \rangle \cap \langle ker\ A_2' \rangle = \mathbf{0}, \end{array} \right\} \Rightarrow (r_1 + r_2 \geq n), \qquad (156)$$

These fundamental relations of parallelism in an affine space $\langle\mathcal{A}^n\rangle$ and orthogonality in an Euclidean space $\langle\mathcal{E}^n\rangle$ of lineors or planars, given by the lineors too, require further development and expansion so that on their basis we can derive tensor trigonometric functions and the basic trigonometric relations between them with their tensor angles – arguments.

If the linear subspaces are defined by null-prime $n \times n$-matrices Bp (Part I, sect. 1.6), then their affine eigenprojectors may be used also, for example,

$$\langle im\ Bp_1\rangle \equiv \langle im\ Bp_2\rangle, \langle ker\ Bp_1\rangle \equiv \langle ker\ Bp_2\rangle \ \Leftrightarrow\ \overleftrightarrow{Bp_1} = \overleftrightarrow{Bp_2}; \tag{157}$$

$$\left.\begin{array}{l} \langle im\ Bp_2\rangle \subseteq \langle im\ Bp_1\rangle \ \Leftrightarrow\ \overleftarrow{Bp_1}\cdot Bp_2 = Bp_2 \ \Leftrightarrow \\ \Leftrightarrow \overrightarrow{Bp_1}\cdot Bp_2 = Z = Bp_2'\cdot \overrightarrow{Bp_1'} \Leftrightarrow \langle ker\ Bp_1'\rangle \subseteq \langle ker\ Bp_2'\rangle. \end{array}\right\} \tag{158}$$

Affine relations (153)–(156) between planars determined by lineors A_1 and A_2 of their rank r_1 and r_2 may be naturally widen as follows. In the first extreme case, we have:

$$\left.\begin{array}{l} \langle im\ A_1\rangle \cap \langle im\ A_2\rangle = \mathbf{0} \ \Leftrightarrow\ rank\ (\overleftarrow{A_2A_2'} - \overleftarrow{A_1A_1'}) = \\ = r_1 + r_2 = rank\ (\overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'}) \le n. \end{array}\right\} \tag{159}$$

The image of this matrix $(\overleftarrow{A_2A_2'} - \overleftarrow{A_1A_1'})$ in any Cartesian base $\tilde{E}$ of an Euclidean space $\langle\mathcal{E}^n\rangle$ is the direct orthogonal sum $\langle im\ A_1\rangle \oplus \langle im\ A_2\rangle$ of dimension $(r_1 + r_2)$, and its kernel is the orthocomplement in the same $\langle\mathcal{E}^n\rangle$ to the image of dimension $n - (r_1 + r_2)$. In the second extreme case, we have:

$$\left.\begin{array}{l} \langle ker\ A_1'\rangle \cap \langle ker\ A_2'\rangle = \mathbf{0} \ \Leftrightarrow\ rank\ (\overleftarrow{A_2A_2'} - \overleftarrow{A_1A_1'}) = \\ = rank\ (\overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'}) = (n - r_1) + (n - r_2) \le n. \end{array}\right\} \tag{160}$$

Here the same matrix image, but in other interpretation $(\overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'})$, is the direct sum $\langle ker\ A_1'\rangle \oplus \langle ker\ A_2'\rangle$ of dimension $[(n - r_1) + (n - r_2)] = 2n - (r_1 + r_2)$, and its kernel is the orthocomplement in $\langle\mathcal{E}^n\rangle$ of dimension $(r_1 + r_2) - n$. Note, that (155) and (156) are only the special extreme cases of (159) and (160). Formulae (159) and (160) are compatible iff $n = r_1 + r_2$, i. e., in this especial case, there holds

$$\langle im\ A_1\rangle \oplus \langle im\ A_2\rangle \equiv \langle\mathcal{A}^n\rangle \equiv \langle ker\ A_1'\rangle \oplus \langle ker\ A_2'\rangle.$$

Under this condition, the matrix $(\overleftarrow{A_2A_2'} - \overleftarrow{A_1A_1'}) = (\overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'})$ is nonsingular. Similarly, in other cases, we have:

$$\langle im\ A_1\rangle \cap \langle im\ A_2\rangle \ne \mathbf{0} \ \Leftrightarrow\ rank\ (\overleftarrow{A_2A_2'} - \overleftarrow{A_1A_1'}) < r_1 + r_2, \tag{161}$$

$$\langle ker\ A_1'\rangle \cap \langle ker\ A_2'\rangle \ne \mathbf{0} \ \Leftrightarrow\ rank\ (\overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'}) < 2n - (r_1 + r_2). \tag{162}$$

Such in brackets and other similar wonderful matrices give us the way for defining further all the projective spherical tensor trigonometric functions of tensor angles as their arguments in terms of eigenprojectors corresponding for beginning to a pair of lineor A_1 and A_2, and then to a pair of matrix or linear matrix objects B and B'. Next, we turn to the construction of the tensor trigonometry, initially of projective type, in affine and Euclidean spaces.

5.2 Projective sine, cosine and spherically orthogonal reflectors

The following matrix characteristic

$$\sin\tilde{\Phi}_{12} = \overleftarrow{A_2A_2'} - \overleftarrow{A_1A_1'} = \overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'} = \sin'\tilde{\Phi}_{12} = -\sin\tilde{\Phi}_{21} \tag{163}$$

is called the *projective tensor sine* of the angle between two planars $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ (or between the lineors A_1 and A_2). The projective nature of the angle is pointed out by the tilde upper character. We have:

$$\tilde{\Phi}_{12} = (\tilde{\Phi}_{12})' = -\tilde{\Phi}_{21}. \tag{164}$$

The properties (164) of a projective tensor angle will be inferred further after converting with its tensor sine into the canonical monobinary and diagonal forms.

In tensor trigonometry, the concept of an angle with its orientation is defined mathematically very simply and correctly as the arcsine of the tensor sine. The introduction of the Euclidean quadratic metric in $\langle\mathcal{A}^n\rangle$ with transition into $\langle\mathcal{E}^n\rangle$ allows to rigorously define the concept of orthogonality with passing from the abstract affine value of angle to metric value. In scalar trigonometry, the definition without orientation is done through the relations in a right triangle, but for this it is necessary to strictly introduce the concept of a right angle.

According to (163), the angle between $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ is additively opposite to the angle between $\langle ker\ A_1'\rangle$ and $\langle ker\ A_2'\rangle$. These two angles together form the whole *binary structure* of $\tilde{\Phi}_{12}$. For example, the tensor sine of the angle between two non-oriented vectors or straight lines is

$$\sin\tilde{\Phi}_{12} = \overleftarrow{\mathbf{a}_2\mathbf{a}_2'} - \overleftarrow{\mathbf{a}_1\mathbf{a}_1'} = \frac{\mathbf{a}_2\mathbf{a}_2'}{\mathbf{a}_2'\mathbf{a}_2} - \frac{\mathbf{a}_1\mathbf{a}_1'}{\mathbf{a}_1'\mathbf{a}_1}. \tag{165 – I}$$

In addition, its algebraic structure on an *Euclidean plane* $\langle\mathcal{E}^2\rangle$ is

$$\sin\tilde{\Phi}_{12} = \sin\varphi_{12}\sqrt{I_{2\times2}}, \quad \sqrt{I_{2\times2}} = R\cdot\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot R',$$

where φ_{12} is the counter-clockwise angle in the right Cartesian base, $|\varphi_{12}| \le \pi$ for vectors or $|\varphi_{12}| \le \pi/2$ for straight lines, R is some orthogonal modal matrix.

Condition $\sin\tilde{\Phi}_{12} = \tilde{\Phi}_{12} = Z$ means parallelism (153) of the planars. In common these planars may be noncentralized as $<\mathbf{a}_1 + \langle im\ A_1\rangle>$ and $<\mathbf{a}_2 + \langle im\ A_2\rangle>$.

Relations similar to (154) have trigonometric analogues too:

$$\langle im\ A_1\rangle \subseteq \langle im\ A_2\rangle \ \Leftrightarrow\ \sin^2\tilde{\Phi}_{12} = +\sin\tilde{\Phi}_{12}, \tag{166}$$

$$\langle im\ A_2\rangle \subseteq \langle im\ A_1\rangle \ \Leftrightarrow\ \sin^2\tilde{\Phi}_{12} = -\sin\tilde{\Phi}_{12}. \tag{167}$$

Indeed,

$$\sin^2\tilde{\Phi}_{12} = \overleftarrow{A_1A_1'}\cdot\overrightarrow{A_2A_2'} + \overleftarrow{A_2A_2'}\cdot\overrightarrow{A_1A_1'} = \overrightarrow{A_1A_1'}\cdot\overleftarrow{A_2A_2'} + \overrightarrow{A_2A_2'}\cdot\overleftarrow{A_1A_1'}. \tag{168}$$

For example, in the case of formula (167), it may be inferred as:

$$\langle im\ A_2\rangle \subseteq \langle im\ A_1\rangle \ \Leftrightarrow\ \langle ker\ A_1'\rangle \subseteq \langle ker\ A_2'\rangle \ \Leftrightarrow$$

$$\Leftrightarrow\ \overleftarrow{A_1A_1'}\cdot\overleftarrow{A_2A_2'} = \overleftarrow{A_2A_2'}\ ,\ \overleftarrow{A_2A_2'}\cdot\overrightarrow{A_1A_1'} = Z \ \Leftrightarrow\ \sin^2\tilde{\Phi}_{12} = -\sin\tilde{\Phi}_{12}.$$

In special case (166), the tensor sine is a symmetric projector (its eigenvalues are 0 and +1); in special case (167) it is an antiprojector (the eigenvalues are 0 and −1).

The tensor angle between $\langle im\ B'\rangle$ and $\langle im\ B\rangle$ is additively opposite to the tensor angle between $\langle ker\ B\rangle$ and $\langle ker\ B'\rangle$. These two angles form entirely the whole binary structure of the projective tensor angle $\tilde{\Phi}_B$. Similarly to (163) and (164), there holds

$$\sin\tilde{\Phi}_B = \overleftarrow{B'B} - \overleftarrow{BB'} = \overrightarrow{BB'} - \overrightarrow{B'B} = \sin'\tilde{\Phi}_B = -\sin\tilde{\Phi}_{B'}; \tag{169}$$

$$\tilde{\Phi}_B = (\tilde{\Phi}_B)' = -\tilde{\Phi}_{B'}. \tag{170}$$

Condition $\sin\tilde{\Phi}_B = Z$ is equivalent to $\tilde{\Phi}_B = Z$ and $B \in \langle Bm\rangle$, it is the tensor trigonometric interpretation of null-normal matrices (Part I, sect. 2.4); $\sin\tilde{\Phi}_{12} = 0$ is equivalent to (153).

The trigonometric relations between two planars: image and kernel of matrices A_1 and A_2 or B and B' are characterized by the *projective tensor cosine* of tensor angle $\tilde{\Phi}_{12}$ or $\tilde{\Phi}_B$:

$$\left.\begin{array}{l}\cos\tilde{\Phi}_{12} = \overleftarrow{A_2A_2'} - \overrightarrow{A_1A_1'} = \overleftarrow{A_1A_1'} - \overrightarrow{A_2A_2'} = \\ = \overleftarrow{A_1A_1'} + \overleftarrow{A_2A_2'} - I = I - \overrightarrow{A_1A_1'} - \overrightarrow{A_2A_2'} = \\ = \cos'\tilde{\Phi}_{12} = \cos\tilde{\Phi}_{21} = \cos(-\tilde{\Phi}_{12}),\end{array}\right\} \tag{171}$$

$$\left.\begin{array}{l}\cos\tilde{\Phi}_B = \overleftarrow{BB'} - \overrightarrow{B'B} = \overleftarrow{B'B} - \overrightarrow{BB'} = \overleftarrow{BB'} + \overleftarrow{B'B} - I = \\ = I - \overrightarrow{BB'} - \overrightarrow{B'B} = \cos'\tilde{\Phi}_B = \cos\tilde{\Phi}_{B'} = \cos(-\tilde{\Phi}_{B'}).\end{array}\right\} \tag{172}$$

For two non-oriented vectors or straight lines on the Euclidean plane there holds:

$$\cos\tilde{\Phi}_{12} = \overleftarrow{\mathbf{a}_2\mathbf{a}_2'} + \overleftarrow{\mathbf{a}_1\mathbf{a}_1'} - I = \frac{\mathbf{a}_1\mathbf{a}_1'}{\mathbf{a}_1'\mathbf{a}_1} + \frac{\mathbf{a}_2\mathbf{a}_2'}{\mathbf{a}_2'\mathbf{a}_2} - I. \tag{165 – II}$$

$$\cos\tilde{\Phi}_{12} = \cos\varphi_{12}\sqrt{I_{2\times2}}, \quad \sqrt{I_{2\times2}} = R\cdot\begin{bmatrix} +1 & 0 \\ 0 & -1\end{bmatrix}\cdot R', \quad (\cos\varphi_{12} \geq 0).$$

The trigonometric analogues of conditions (155) and (156) follow from the formula

$$\cos^2\tilde{\Phi}_{12} = \overleftarrow{A_1A_1'}\cdot\overleftarrow{A_2A_2'} + \overrightarrow{A_2A_2'}\cdot\overrightarrow{A_1A_1'} = \overrightarrow{A_1A_1'}\cdot\overrightarrow{A_2A_2'} + \overleftarrow{A_2A_2'}\cdot\overleftarrow{A_1A_1'}. \tag{173}$$

Similarly to (168), equalities for the singular cosine (as projector and antiprojector)

$$\cos^2\tilde{\Phi}_{12} = +\cos\tilde{\Phi}_{12} \leftrightarrow (156),\ \cos^2\tilde{\Phi}_{12} = -\cos\tilde{\Phi}_{12} \leftrightarrow (155) \tag{174}$$

are equivalent to formulae (156) and (155), this follows from (173).

Tensor cosine of the main angle is equal also to tensor sine of the complementary angle with respect to the right angle compatible with it as $\tilde{\Xi} = (\tilde{\pi}/2 - \tilde{\Phi})$, and vice versa:

$$\cos\tilde{\Phi} = \sin\tilde{\Xi}, \quad \sin\tilde{\Phi} = \cos\tilde{\Xi}. \tag{175}$$

In an affine space $\langle\mathcal{A}^n\rangle$, tensor angle $\tilde{\Phi}$ has no quantitative sense unless this is zero or open. But in an Euclidean space $\langle\mathcal{E}^n\rangle$ the projective tensor angle as an argument expresses in its metric form the quantitative spatial angular relations between lineors or between planars.

In an Euclidean space, the right tensor angle is formed by pairs of planars $\langle im\ A\rangle$ and $\langle ker\ A'\rangle$. Hence, we obtain for the lineors A_1 and A_2 the pair of tensor mutual eigenreflectors, and this pair is bound one-to-one with the given projective tensor angle $\tilde{\Phi}_{12}$:

$$\overleftarrow{A_1A_1'} - \overrightarrow{A_1A_1'} = Ref\{A_1A_1'\} = \cos\tilde{\Phi}_{12} - \sin\tilde{\Phi}_{12} = Ref_{\boxplus}\{-\tilde{\Phi}_{12}\} = \cos\tilde{Z}_1, \tag{176}$$

$$\overleftarrow{A_2A_2'} - \overrightarrow{A_2A_2'} = Ref\{A_2A_2'\} = \cos\tilde{\Phi}_{12} + \sin\tilde{\Phi}_{12} = Ref_{\boxplus}\{+\tilde{\Phi}_{12}\} = \cos\tilde{Z}_2, \tag{177}$$

Due to the right tensor angle between $\langle im\ B\rangle$ and $\langle ker\ B'\rangle$ we get mutual eigenreflectors too

$$\overleftarrow{BB'} - \overrightarrow{BB'} = Ref\{BB'\} = \cos\tilde{\Phi}_B - \sin\tilde{\Phi}_B = Ref_{\boxplus}\{-\tilde{\Phi}_B\} = \cos\tilde{Z}_B, \tag{178}$$

$$\overleftarrow{B'B} - \overrightarrow{B'B} = Ref\{B'B\} = \cos\tilde{\Phi}_B + \sin\tilde{\Phi}_B = Ref_{\boxplus}\{+\tilde{\Phi}_B\} = \cos\tilde{Z}_{B'}. \tag{179}$$

They are tensor cosines of four zero tensor angles corresponding to planars $\langle im\ A_1\rangle$, $\langle im\ A_2\rangle$, or $\langle im\ B\rangle$, $\langle im\ B'\rangle$; and $\cos^2\tilde{Z} = I$. The symmetric square roots (176)–(179) such as $\sqrt{I} = (\sqrt{I})^{-1} = (\sqrt{I})'$ are *orthogonal spherical mutual reflectors* for $\tilde{\Phi}$ with eigenvalues ± 1. (Here $\tilde{\Phi}$ is variable *projective angle-argument for orthogonal function* $Ref_{\boxplus}$ *as were shown in definitions (176)-(179)!* For a pair of lineors or null-prime matrix Bp, we have 4 variants of the eigenreflectors as $\pm(\cos\tilde{\Phi} \mp \sin\tilde{\Phi})$. The symmetric tensor eigenreflectors carry out the *orthogonal reflections*: $+Ref\{AA'\}$ off the *mirror* $\langle ker\ A'\rangle$ parallel to $\langle im\ A\rangle$, $-Ref\{AA'\}$ off the *mirror* $\langle im\ A\rangle$ parallel to $\langle ker\ A'\rangle$; $+Ref\{BB'\}$ off the *mirror* $\langle ker\ B'\rangle$ parallel to $\langle im\ B\rangle$; $-Ref\{BB'\}$ off the *mirror* $\langle im\ B\rangle$ parallel to $\langle ker\ B'\rangle$. Some extreme cases are:

$$\sin\tilde{\Phi} = \tilde{Z} \ \Leftrightarrow\ \cos\tilde{\Phi} \subset \langle\sqrt{I_{n\times n}}\,\rangle_S,\ \cos\tilde{\Phi} = Z \ \Leftrightarrow\ \tilde{\Phi} = \tilde{\pi}/2 \ \Leftrightarrow\ \sin\tilde{\Phi} \subset \langle\sqrt{I_{n\times n}}\,\rangle_S,$$

$$\sin\tilde{\Phi}_{12} = +I \ \Leftrightarrow\ r_1 = 0,\ r_2 = n,\ \ \sin\tilde{\Phi}_{12} = -I \ \Leftrightarrow\ r_1 = n,\ r_2 = 0;\ (\sin\tilde{\Phi}_B \neq \pm I).$$

$$\cos\tilde{\Phi} = +I \ \Leftrightarrow\ rank\ A = rank\ B = n,\ \ \cos\tilde{\Phi} = -I \ \Leftrightarrow\ rank\ A = rank\ B = 0.$$

From one-to-one bond a pair of equirank reflectors with the tensor angle $\tilde{\Phi}$ we get:

$$\left.\begin{array}{l} \cos\{\tilde{\Phi}\} = \cos'\{\tilde{\Phi}\} = (F\{+\tilde{\Phi}\} + F\{-\tilde{\Phi}\})/2, \\ \sin\{\tilde{\Phi}\} = \sin'\{\tilde{\Phi}\} = (F\{+\tilde{\Phi}\} - F\{-\tilde{\Phi}\})/2. \end{array}\right\} \tag{180}$$

The following identities equivalent to $I \cdot I = I = I \cdot I$ are clearly valid:

$$\left.\begin{array}{l} (\overleftarrow{A_1A_1'} + \overrightarrow{A_1A_1'})(\overleftarrow{A_2A_2'} + \overrightarrow{A_2A_2'} = I = (\overleftarrow{A_2A_2'} + \overrightarrow{A_2A_2'})(\overleftarrow{A_1A_1'} + \overrightarrow{A_1A_1'}), \\ (\overleftarrow{BB'} + \overrightarrow{BB'})(\overleftarrow{B'B} + \overrightarrow{B'B} = I = (\overleftarrow{B'B} + \overrightarrow{B'B})(\overleftarrow{BB'} + \overrightarrow{BB'}). \end{array}\right\} \tag{181}$$

They give trigonometric formulae for a sine-cosine pair in the projective version:

$$\sin^2\tilde{\Phi} + \cos^2\tilde{\Phi} = I = \cos^2\tilde{\Xi} + \sin^2\tilde{\Xi} \ \ \text{(Ptolemy Tensor ortho-Projective Invariant)}, \tag{182}$$

$$\sin\tilde{\Phi}\cdot\cos\tilde{\Phi} = -\cos\tilde{\Phi}\cdot\sin\tilde{\Phi}, \tag{183}$$

$$\sin^2\tilde{\Phi}\cdot\cos^2\tilde{\Phi} = \cos^2\tilde{\Phi}\cdot\sin^2\tilde{\Phi}, \tag{184}$$

Note, that *the projective sine-cosine tensor pair is anticommutative.*

The Table of multiplication for differgenesis eigenprojectors

$$\begin{array}{|ll|} \overleftarrow{B}\cdot\overleftarrow{BB'} = \overleftarrow{BB'} = \overleftarrow{BB'}\cdot\overleftarrow{B'}, & \overrightarrow{B}\cdot\overrightarrow{B'B} = \overrightarrow{B'B} = \overrightarrow{B'B}\cdot\overrightarrow{B'}, \\ \overleftarrow{B'}\cdot\overleftarrow{B'B} = \overleftarrow{B'B} = \overleftarrow{B'B}\cdot\overleftarrow{B}, & \overrightarrow{B'}\cdot\overrightarrow{BB'} = \overrightarrow{BB'} = \overrightarrow{BB'}\cdot\overrightarrow{B}, \\ \overleftarrow{B}\cdot\overleftarrow{B'B} = \overleftarrow{B} = \overleftarrow{BB'}\cdot\overleftarrow{B}, & \overrightarrow{B}\cdot\overrightarrow{BB'} = \overrightarrow{B} = \overrightarrow{B'B}\cdot\overrightarrow{B}, \\ \overleftarrow{B'}\cdot\overleftarrow{BB'} = \overleftarrow{B'} = \overleftarrow{B'B}\cdot\overleftarrow{B'}, & \overrightarrow{B'}\cdot\overrightarrow{B'B} = \overrightarrow{B'} = \overrightarrow{BB'}\cdot\overrightarrow{B'}. \end{array} \tag{185}$$

This Table of multiplication may be inferred easy with the use of transposition operations!

Projective nature of introduced above tensor trigonometric functions is illustrated by the cosine formulae, associated with solving a flat right triangle:

$$\overleftarrow{BB'} = +\overleftarrow{B}\cdot\cos\tilde{\Phi} = +\cos\tilde{\Phi}\cdot\overleftarrow{B'}, \tag{186}$$

$$\overleftarrow{B'B} = +\overleftarrow{B'}\cdot\cos\tilde{\Phi} = +\cos\tilde{\Phi}\cdot\overleftarrow{B}, \tag{187}$$

$$\overrightarrow{B'B} = -\overrightarrow{B}\cdot\cos\tilde{\Phi} = -\cos\tilde{\Phi}\cdot\overrightarrow{B'}, \tag{188}$$

$$\overrightarrow{BB'} = -\overrightarrow{B'}\cdot\cos\tilde{\Phi} = -\cos\tilde{\Phi}\cdot\overrightarrow{B}, \tag{189}$$

In the Euclidean space $\overleftarrow{B}$ and $\overrightarrow{B}$ are the oblique eigenprojectors for the null-prime matrix B (see in sect. 2.1). Here they play a role of the hypotenuse in such tensor right triangles.

But the sine formulae give us the surprising four nilpotent legs:

$$\overleftarrow{B} - \overleftarrow{BB'} = +(\sqrt{Z})_1 = +\overleftarrow{B}\cdot\sin\tilde{\Phi} = +\overleftarrow{B}\cdot\overrightarrow{BB'} = -\overleftarrow{BB'}\cdot\overrightarrow{B}, \tag{190}$$

$$\overrightarrow{B} - \overrightarrow{B'B} = +(\sqrt{Z})_2 = +\overrightarrow{B}\cdot\sin\tilde{\Phi} = -\overrightarrow{B'B}\cdot\overleftarrow{B} = +\overrightarrow{B}\cdot\overleftarrow{B'B}, \tag{191}$$

$$\overleftarrow{B'} - \overleftarrow{B'B} = -(\sqrt{Z})_2' = -\overleftarrow{B'}\cdot\sin\tilde{\Phi} = -\overleftarrow{B'B}\cdot\overrightarrow{B'} = +\overleftarrow{B'}\cdot\overrightarrow{B'B}, \tag{192}$$

$$\overrightarrow{B'} - \overrightarrow{BB'} = -(\sqrt{Z})_1' = -\overrightarrow{B'}\cdot\sin\tilde{\Phi} = +\overrightarrow{B}\cdot\overleftarrow{BB'} = -\overrightarrow{BB'}\cdot\overleftarrow{B'}, \tag{193}$$

(When these formulae are transposed, then the sine sign changes.) The indicated differences of oblique and orthogonal projectors of the same type are nilpotent matrices of order 2.

Quadrating and multiplying of simple formulae (186)–(189) give the cosine formulae for the multiplications of oblique as well as orthogonal projectors of the same type:

$$\overleftarrow{BB'} = (\overleftarrow{B}\cdot\cos\tilde{\Phi})^2 = \overleftarrow{B}\cdot\overleftarrow{B'}\cdot\cos^2\tilde{\Phi} = \overleftarrow{B}\cdot\cos^2\tilde{\Phi}\cdot\overleftarrow{B'} = \cos^2\tilde{\Phi}\cdot\overleftarrow{B}\overleftarrow{B'}, \tag{194}$$

$$\overrightarrow{BB'} = (-\cos\tilde{\Phi}\cdot\overrightarrow{B})^2 = \overrightarrow{B'}\cdot\overrightarrow{B}\cdot\cos^2\tilde{\Phi} = \overrightarrow{B'}\cdot\cos^2\tilde{\Phi}\cdot\overrightarrow{B} = \cos^2\tilde{\Phi}\cdot\overrightarrow{B'}\overrightarrow{B}, \tag{195}$$

$$\overleftarrow{BB'}\cdot\overleftarrow{B'B} = (\overleftarrow{B}\cdot\cos\tilde{\Phi})\cdot(\overleftarrow{B'}\cdot\cos\tilde{\Phi}) = \cos^2\tilde{\Phi}\cdot\overleftarrow{B} = \overleftarrow{B}\cdot\cos^2\tilde{\Phi}, \tag{196}$$

$$\overrightarrow{B'B}\cdot\overrightarrow{BB'} = (-\cos\tilde{\Phi}\cdot\overrightarrow{B'})\cdot(-\cos\tilde{\Phi}\cdot\overrightarrow{B}) = \cos^2\tilde{\Phi}\cdot\overrightarrow{B} = \overrightarrow{B}\cdot\cos^2\tilde{\Phi}. \tag{197}$$

Projective trigonometric nature of the tensor angles is illustrated with the *symbolic tensor octahedron* formed by eight eigenprojectors of null-prime B in 2-valent $\langle\mathcal{E}^{n\times n}\rangle$ (Figure 1). For null-normal B, this octahedron is reduced to the tensor right triangle with hypotenuse I.

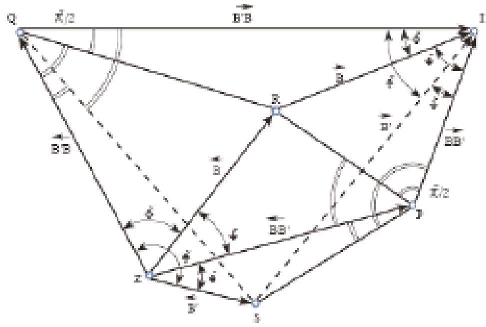

Figure 1. Symbolic tensor octahedron from 8 eigenprojectors for illustration of the projective tensor angles.

5.3 Projective secant, tangent and affine (oblique) reflectors

The tensor secant (and further tangent) of a projective angle is defined in terms of oblique eigenprojectors. The matrix trigonometric function

$$\left.\begin{aligned} &\sec\tilde{\Phi}_B = \overleftarrow{B'} - \overrightarrow{B} = \overleftarrow{B} - \overrightarrow{B'} = \overleftarrow{B} + \overleftarrow{B'} - I = \\ &= I - \overrightarrow{B} - \overrightarrow{B'} = \sec'\ \tilde{\Phi}_B = \sec\tilde{\Phi}_{B'} = \sec\,(-\tilde{\Phi}_B) = \\ &= (\overleftarrow{B'})' - \overrightarrow{B} = \overleftarrow{B} - (\overrightarrow{B})' \end{aligned}\right\} \qquad (198)$$

is called *the projective tensor secant* of the tensor angle $\tilde{\Phi}_B$.

These formulae are easily inferred by the following way. Summation of (186) and (189), (187) and (188) gives

$$(\overleftarrow{B'} - \overrightarrow{B})\cdot\cos\tilde{\Phi} = \cos\tilde{\Phi}\cdot(\overleftarrow{B'} - \overrightarrow{B}) = I = \cos\tilde{\Phi}\cdot(\overleftarrow{B} - \overrightarrow{B'}) = (\overleftarrow{B} - \overrightarrow{B'})\cdot\cos\tilde{\Phi}.$$

These equalities determine the tensor secant.

According to (172), $\cos\tilde{\Phi}_B$ is nonsingular iff $\langle im\ B\rangle \cap \langle ker\ B\rangle = \mathbf{0}$, i. e., $B \in \langle Bp\rangle$ is a null-prime matrix (see Part I, sect. 1.6), therefore,

$$\sec\tilde{\Phi}_{Bp} = \cos^{-1}\tilde{\Phi}_{Bp}, \quad \sec\tilde{\Phi}_{Bp}\cdot\cos\tilde{\Phi}_{Bp} = I = \cos\tilde{\Phi}_{Bp}\cdot\sec\tilde{\Phi}_{Bp}; \qquad (199)$$

The matrix B may be null-defective, and there may exist no oblique eigenprojectors. Then the cosine of angle $\tilde{\Phi}_B$ is the zero matrix on the subspace $\langle im\ B\rangle \cap \langle ker\ B\rangle$ and

$$\sec\ \tilde{\Phi}_B = \cos^+\tilde{\Phi}_B, \quad \sec\tilde{\Phi}\cdot\cos\tilde{\Phi} = \overleftarrow{\cos\tilde{\Phi}} = \cos\tilde{\Phi}\cdot\sec\tilde{\Phi}. \qquad (200)$$

The formal definition of the tensor secant as *quasi-secant* takes advantage of the quasi-inverse Moor–Penrose matrix (see Part I, sect. 2.5) for the inversion of the singular tensor cosine. (Recall, that its matrix is symmetrical.) In this case, the multiplication of the tensor cosine and quasi-secant is the orthoprojector in formula (200). From the other hand, for a null-defective matrix B, the cosine of the angle between the subspaces $\langle im\ B^{s^0}\rangle$ and $\langle im\ (B')^{s^0}\rangle$ is a nonsingular matrix. Note, that for the null-normal matrix the tensor angle between $\langle im\ B\rangle$ and $\langle ker\ B\rangle$ is right. But for the main tensor angle and its functions, in the case, we have:

$$\sin\tilde{\Phi}_B = Z \;\Leftrightarrow\; \cos\tilde{\Phi}_B = \sqrt{I},\ \cos^2\tilde{\Phi}_B = I,\ \sec\tilde{\Phi}_B = \cos^{-1}\tilde{\Phi}_B.$$

For the tensor sine in the especial case, if $B \in \langle Bp\rangle$ and $r_B = n/2$, there holds

$$det\sin\tilde{\Phi}_B \neq 0 \;\Leftrightarrow\; \langle im\ B\rangle \cap \langle im\ B'\rangle = \mathbf{0},\ \langle ker\ B\rangle \cap \langle ker\ B'\rangle = \mathbf{0}. \qquad (201)$$

If the same tensor angle is defined by lineors A_1 and A_2, then conditions (159) and (160) should hold simultaneously. In other cases, the tensor sine is a singular matrix, and the *quasi-cosecant* is defined in terms of the quasi-inverse Moor–Penrose matrix:

$$\operatorname{cosec}\tilde{\Phi}_B = \sin^+\tilde{\Phi}_B = \operatorname{cosec}'\ \tilde{\Phi}_B = -\operatorname{cosec}\tilde{\Phi}_{B'} = -\operatorname{cosec}(-\tilde{\Phi}_B) = \sec\tilde{\Xi}. \qquad (202)$$

Further, subtracting (186) and (187) gives

$$\sin\tilde{\Phi}_B = -\cos\tilde{\Phi}_B\cdot(\overleftarrow{B'} - \overleftarrow{B}) = +(\overleftarrow{B'} - \overleftarrow{B})\cdot\cos\tilde{\Phi}_B.$$

These equalities determine the tensor function

$$\left.\begin{aligned} & i\tan\tilde{\Phi}_B = \overleftarrow{B}' - \overleftarrow{B} = \overrightarrow{B} - \overrightarrow{B}' = (\overleftarrow{B})\,' - \overleftarrow{B}) = \\ & = \overrightarrow{B} - (\overrightarrow{B})\,' = -(i\tan\tilde{\Phi}_B)' = -i\tan\Phi_{B'} = -i\tan(-\Phi_B), \end{aligned}\right\} \tag{203}$$

called the *projective realificated tensor tangent* of $\tilde{\Phi}_B$. In the realificated form it is a *real valued skewsymmetric matrix* with the eigenvalues $\mu_J = \pm i\tan\varphi_J$. Moreover (see also sect. 5.5 and 7), there hold the following anticommutative paired relations (!):

$$\left.\begin{aligned} & i\tan\tilde{\Phi} = +\sin\tilde{\Phi}\cdot\sec\tilde{\Phi} = -\sec\tilde{\Phi}\cdot\sin\tilde{\Phi} \leftrightarrow \\ & \leftrightarrow\ \sin\tilde{\Phi} = +i\tan\tilde{\Phi}\cdot\cos\tilde{\Phi} = -\cos\tilde{\Phi}\cdot i\tan\tilde{\Phi} \rightarrow \\ & \rightarrow +\sin\tilde{\Phi}\cdot i\tan\tilde{\Phi} = -i\tan\tilde{\Phi}\cdot\sin\tilde{\Phi}. \end{aligned}\right\} \tag{204}$$

For two vectors or two straight lines, due to (151) and (152), there holds

$$i\tan\tilde{\Phi}_B = \frac{B'}{trB'} - \frac{B}{trB} = \frac{B'-B}{trB} = \frac{\mathbf{a}_2\mathbf{a}_1'}{\mathbf{a}_1'\mathbf{a}_2} - \frac{\mathbf{a}_1\mathbf{a}_2'}{\mathbf{a}_2'\mathbf{a}_1} = \frac{\mathbf{a}_2\mathbf{a}_1' - \mathbf{a}_1\mathbf{a}_2'}{\mathbf{a}_1'\mathbf{a}_2} = i\tan\tilde{\Phi}_{12}. \tag{205}$$

Its structure is $[i\tan\tilde{\Phi}_{12} = \tan\varphi_{12}\sqrt{I_{2\times 2}}\,]$, $\quad\sqrt{I_{2\times 2}} = R\cdot\begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}\cdot R'.$

The *realificated quasi-cotangent* is defined, in the general case, as

$$i\cot\tilde{\Phi}_B = i\tan^+\tilde{\Phi}_B = -i\cot'\tilde{\Phi}_B = -i\cot\tilde{\Phi}_{B'} = -i\cot(-\tilde{\Phi}_B) = i\tan\tilde{\Xi}_B. \tag{206}$$

The following identities are affine (oblique) analogs of identities (181):

$$(\overleftarrow{B} + \overrightarrow{B})\cdot(\overleftarrow{B}' + \overrightarrow{B}') = I = (\overleftarrow{B}' + \overrightarrow{B}')\cdot(\overleftarrow{B} + \overrightarrow{B}) \tag{207}$$

They are clearly valid for the null-prime matrices. By the way, trigonometric formulae

$$\sec^2\tilde{\Phi} - \tan^2\tilde{\Phi} = I = \operatorname{cosec}^2\tilde{\Xi} - \cot^2\tilde{\Xi}\ \text{(Tensor oblique-Projective quasi-Invariants)}, \tag{208}$$

$$+i\tan\tilde{\Phi}\cdot\sec\tilde{\Phi} = -\sec\tilde{\Phi}\cdot i\tan\tilde{\Phi}, \tag{209}$$

$$\tan^2\tilde{\Phi}\cdot\sec^2\tilde{\Phi} = \sec^2\tilde{\Phi}\cdot\tan^2\tilde{\Phi} \tag{210}$$

complement formulae (182)–(184) for the tensor sine-cosine *anticommutative pair*. Note, $\tan\tilde{\Phi}$ is a *true projective tensor tangent* with the eigenvalues $\mu_J = \pm\tan\varphi_J$.

Note. We named (208) as the *quasi-invariants*, because, from the point of view of the kinds of transformations, secant-tangent (or cosecant-cotangent) reflections with these quasi-invariants do not have the property of two- and multistep applicability in the reflective transformations of coordinates or geometric objects. They are applicable only for one-step reflections. This distinction will be discussed in details in the Appendix, where tensor trigonometric invariants and quasi-invariants will play a large role in various non-Euclidean geometries and in the theory of relativity.

Rule 1. *Square and any even degrees of all the tensor trigonometric functions of the same angle (for the same pair of lineors or planars) commute with each other tensor trigonometric functions of the same angle, with all its eigenprojectors and all its eigenreflectors.*

If $B = Bp$ is null-prime matrix (not null-normal), then its two mutual *oblique spherical eigenreflectors* (reflecting with the trigonometric deformation – see in sect. 5.10) are defined similarly to formulae (176)–(179) in terms of the oblique eigenprojectors (see Part I, (60)):

$$\overleftarrow{B} - \overrightarrow{B} = I - 2\overrightarrow{B} = Ref\{B\} = \sec\tilde{\Phi}_B - i\tan\tilde{\Phi}_B = Ref_{\boxtimes}\{-\tilde{\Phi}_B\}, \qquad (211)$$

$$\overleftarrow{B'} - \overrightarrow{B'} = I - 2\overrightarrow{B'} = Ref\{B'\} = Ref'\{B\} = \sec\tilde{\Phi}_B + i\tan\tilde{\Phi}_B = Ref_{\boxtimes}\{+\tilde{\Phi}_B\}. \qquad (212)$$

(In notation $Ref_{\boxtimes}$ we used the oblique sub-sign.) From the algebraic point of view, they are asymmetric prime square roots of the unity matrix as $\sqrt{I} = (\sqrt{I})^{-1}$ with eigenvalues ± 1.

These mutual *asymmetric tensor eigenreflectors* carry out the *oblique reflection*, namely:
$+Ref\{B\}$ off the mirror $\langle ker\ B\rangle$ parallel to $\langle im\ B\rangle$ with the spherical deformation,
$-Ref\{Bp\}$ off mirror $\langle im\ B\rangle$ parallel to $\langle ker\ B\rangle$ with the spherical deformation.
They are inferred with use of (211) and (60). But iff Bp is a null-normal matrix Bm, then square roots (211) and (212) are symmetric, i. e., they transformed into (178), (179).

Each symmetric and asymmetric prime square roots of I geometrically are accordingly orthogonal and oblique reflectors. Moreover, each pair of the same kind roots corresponds to a unique pair of mutual eigenprojectors, and to a unique pair of mutual projective tensor trigonometric functions (sine-cosine tangent-secant) – see more about this in sect. 5.6.

Reflectors are nonsingular matrices, as in their defining formulae (176)–(179), (211), (212) we get that ranks of both matrices (left and right) are summated and their sum equal to n. (These questions will be consider in details in the following sect. 5.6., 5.7, 5.10.)

Thus we defined above with formulae all the trigonometric functions of tensor angles in the reflective version of tensor trigonometry. On the Euclidean and affine plane, these tensor formulae are applicable too! Here they determine completely the orientation of tensor angles, but their scalar invariants determine also classic flat trigonometry. In the Euclidean and affine linear space, classic flat trigonometry acts with scalar invariants too, but only on the eigenplane of any binary tensor angle without its specific nature.

5.4 Comparison of two ways for defining projective angles

These ways for the angles $\tilde{\Phi}_{12}$ and $\tilde{\Phi}_B$, are the following:

• in terms of $n \times m$-matrices of lineors A_1 and A_2, as geometric objects;

• in terms of $n \times m$-matrices B and B' (as multiplication of the lineors). Both these ways have already been used before (see Part I, sect. 3.3).

Find general conditions under which *tensor angle* $\tilde{\Phi}$ *and its trigonometric functions do not depend on a choice of the way from these two ways of the tensor angle defining*.

According to initial definitions in sect. 3.1, put external and internal multiplications:

$$B = A_1 A_2', \quad B' = A_2 A_1'; \qquad (213),(214)$$

$$C = A_1' A_2, \quad C' = A_2' A_1. \qquad (215),(216)$$

Then the matrices A_1 and A_2 should have the same sizes. Moreover, from the identity of the two tensor angles, i. e., $\tilde{\Phi}_{12} = \tilde{\Phi}_B$, the equalities of their projective sine-cosine trigonometric functions follow as well as the equalities of the corresponding orthogonal eigenprojectors (bound with the angles by exact formulae) follow too; and vice versa:

$$\tilde{\Phi}_{12} = \tilde{\Phi}_B \ \Leftrightarrow\ (\sin\tilde{\Phi}_{12} = \sin\tilde{\Phi}_B,\ \cos\tilde{\Phi}_{12} = \cos\tilde{\Phi}_B) \ \Leftrightarrow$$

$$\Leftrightarrow\ (\overleftrightarrow{A_1 A_1'} = \overleftrightarrow{BB'},\ \overleftrightarrow{A_2 A_2'} = \overleftrightarrow{B'B}).$$

However, the equalities of the corresponding affine (oblique) eigenprojectors $\overleftrightarrow{A_1A'_2} = \overleftrightarrow{B}$ (as bonded with the angle by other formulae) follow from definitions (213)–(214). What is more, these additional equalities are valid due to only existence of affine projectors for B (sect. 2.1). For their existence in the case, see below condition (230).

Equality of the orthoprojectors is equivalent to the following relations:

$$\langle im\ A_1\rangle \equiv \langle im\ B\rangle \quad \Leftrightarrow \quad \langle ker\ A'_1\rangle \equiv \langle ker\ B'\rangle, \qquad (217), (218)$$

$$\langle im\ A_2\rangle \equiv \langle im\ B'\rangle \quad \Leftrightarrow \quad \langle ker\ A'_2\rangle \equiv \langle ker\ B\rangle. \qquad (219), (220)$$

In their turn, the pairs of relations (217), (218) and (219), (220) are equivalent each to another due to the well-known fact, that the left and right sub-spaces in these pairs are complements each to another in $\langle \mathcal{A}^n\rangle$ and orthogonal ones in $\langle \mathcal{E}^n\rangle$ – see in Part I this well-known property (100).

At first, consider, when conditions (217) are valid. Obviously, that

$$\langle im\ B\rangle \equiv A_1\langle im\ A'_2\rangle \ \Leftarrow\ B = A_1A'_2,$$

$$\langle im\ A_1\rangle \equiv A_1\langle \mathcal{A}^{r_2}\rangle \equiv A_1(\langle im\ A'_2\rangle \oplus \langle ker\ A_2\rangle).$$

Therefore (217) is equivalent to the pair of obvious conditions in (213):

$$\langle im\ A'_2\rangle \cap \langle ker\ A_1\rangle = \mathbf{0}, \quad \langle ker\ A_2\rangle \subset \langle ker\ A_1\rangle. \qquad (221)$$

Similarly, (219) is equivalent to the pair of obvious conditions in (214):

$$\langle im\ A'_1\rangle \cap \langle ker\ A_2\rangle = \mathbf{0}, \quad \langle ker\ A_1\rangle \subset \langle ker\ A_2\rangle. \qquad (222)$$

It is seen that independent conditions (217), (219) hold simultaneously iff

$$\left.\begin{array}{l} \langle ker\ A_1\rangle \equiv \langle ker\ A_2\rangle \ \Leftrightarrow\ \langle im\ A'_1\rangle \equiv \langle im\ A'_2\rangle \Leftrightarrow \\ \Leftrightarrow\ \overrightarrow{A'_1A_1} = \overrightarrow{A'_2A_2} \ \Leftrightarrow\ \overleftarrow{A'_1A_1} = \overleftarrow{A'_2A_2} \end{array}\right\} \qquad (223)$$

and where it is necessary $r_1 = r_2 \leq m$.

Thus (223) is the necessary and sufficient condition answering the problem from beginning of the section. Obviously, (223) also implies the very simple and useful sufficient condition $r_1 = r_2 = r = m$. This condition, in its turn, has simple corollaries

$$\langle ker\ A_1\rangle \equiv \langle ker\ A_2\rangle = \mathbf{0}, \qquad \langle im\ A'_1\rangle \equiv \langle im\ A'_2\rangle \equiv \langle \mathcal{A}^r\rangle.$$

This special case is implied when one deals with external and internal multiplications such as (213)–(216) for these so called *equirank lineors* A_1 and A_2 under condition

$$r_1 = r_2 = r = m < n. \qquad (224)$$

(This holds always for two vectors.) From (120) and (213)–(216) we have

$$k(B, r) = k(B', r) = det\ C = det\ C'. \qquad (225)$$

If B is null-prime matrix, then $\langle im\ B\rangle \cap \langle ker\ B\rangle = \mathbf{0}$ and $k(B, r) = det\ C \neq 0$. In the case, if B is null-normal matrix, then $\langle im\ B\rangle \equiv \langle im\ B'\rangle$ and due to (97) (see Part I, sect. 2.4) we have $k(BB', r) = k(B'B, r) = k^2(B, r) = det^2 C > 0$. However if B is null-defective matrix, then $\langle im\ B\rangle \cap \langle ker\ B\rangle \neq \mathbf{0}$ and $k(B, r) = det\ C = 0$.

Under general condition (223) or particular condition (224), there holds

$$\overleftrightarrow{A_1A_1'} = \overleftrightarrow{BB'}, \qquad \overleftrightarrow{A_2A_2'} = \overleftrightarrow{B'B}. \tag{226}$$

In an affine space, the characteristic $det\ G = det[(A_1|A_2)'(A_1|A_2)]$ is the criterion for *at least* partial parallelism of these planars or partial *coplanarity* of these lineors – see this in sect. 8.4. In an Euclidean space, the characteristic $det\ C = det(A_1'A_2)$, under condition (224), is the criterion for *at least* their partial orthogonality.

$$det\ G = 0 \ \Leftrightarrow\ \langle im\ A_1\rangle \cap \langle im\ A_2\rangle \neq \mathbf{0}, \tag{227}$$

$$det\ G \neq 0 \ \Leftrightarrow\ \langle im\ A_1\rangle \cap \langle im\ A_2\rangle = \mathbf{0}, \tag{228}$$

$$det\ C = 0 \Leftrightarrow \langle im\ A_1\rangle \cap \langle ker\ A_2'\rangle \neq \mathbf{0} \Leftrightarrow \langle im\ A_2\rangle \cap \langle ker\ A_1'\rangle \neq \mathbf{0}, \tag{229}$$

$$det\ C \neq 0 \Leftrightarrow \langle im\ A_1\rangle \cap \langle ker\ A_2'\rangle = \mathbf{0} \Leftrightarrow \langle im\ A_2\rangle \cap \langle ker\ A_1'\rangle = \mathbf{0}. \tag{230}$$

In an Euclidean space there holds $\langle ker\ A'\rangle \equiv \langle im\ A\rangle^{\perp}$ – see, for example, in Part I, (100).

Total parallelism of planars (153) or *colplanarity* of equirank lineors – see this in sect. 8.4, under condition (224), means that the matrix $B = A_1A_2'$ is null-normal, i. e., $B \in \langle Bm\rangle$. Due to (97) and (132), this is equivalent to the relations:

$$\left.\begin{aligned} &|det\ C| = \sqrt{k(Bm \cdot Bm', r)} = |k(Bm, r)| = \\ &= \mathcal{M}t(r)(A_1 \cdot A_2') = \mathcal{M}t(r)A_1 \cdot \mathcal{M}t(r)A_2 = \\ &= \sqrt{det(A_1' \cdot A_1)} \cdot \sqrt{det(A_2' \cdot A_2)} \end{aligned}\right\} \tag{231}$$

and is also equivalent to parallelism (153) in an affine space. Formulae (227)–(231) may be interpreted trigonometrically, it will be done later.

Total orthogonality of planars or lineors, under condition (224), means that $B = A_1A_2'$ is a nilpotent matrix of order 2: $B^2 = Z$, or $C = Z$. The latter is also equivalent to orthogonality (155), if $r_1 = r_2$, in an Euclidean space. Their partial orthogonality means that B is a null-defective matrix.

The tensor angle $\tilde{\Phi}_{12}$ and its trigonometric functions are, of course, more general than the angle $\tilde{\Phi}_B$ and its functions, as matrices A_1 and A_2 may have distinct sizes $n \times r_1$ and $n \times r_2$ admissible only for $\tilde{\Phi}_{12}$. Moreover, if the lineors are partially or totally orthogonal, then only the angles $\tilde{\Phi}_{12}$ exist. Therefore the type of a tensor angle more convenient in the problem solving should be chosen.

5.5 Cell-forms of tensor trigonometric functions and reflectors

Parallelism and orthogonality correspond to extreme values of tensor angles between linear objects. In order to completely analyze all relations between objects, it is necessary to represent the trigonometric functions in canonic forms, to find their eigenvalues and to define informative scalar invariant characteristics for the tensor angle.

Consider differences of orthoprojectors similar to (163) and (171). They express the projective sine and cosine by two manners. According to (182)–(184) the sine and cosine eigenvalues are real paired ($\pm$) numbers belonging to $(-1; +1)$:

$$\mu_{i\sin}^2 + \mu_{i\cos}^2 = 1. \tag{232}$$

The paired sine and cosine eigenvalues in an *Euclidean space* correspond to values of *binary angles* on the trigonometric eigenplanes. Four eigen orthoprojectors in both variants of differences (163) and (171) are pairly orthogonal. The projectors correspond one-to-one to four pairly orthogonal subspaces: $\langle im\ A_1\rangle$, $\langle ker\ A_1'\rangle$ and $\langle im\ A_2\rangle$, $\langle ker\ A_2'\rangle$ – see Part I, (100):

$$\left.\begin{array}{c} \langle im\ A_1\rangle \perp \langle ker\ A_1'\rangle, \langle im\ A_1\rangle \oplus \langle ker\ A_1'\rangle \equiv \langle \mathcal{E}^n\rangle \Leftrightarrow \\ \Leftrightarrow\ \langle im\ A_2\rangle \perp \langle ker\ A_2'\rangle, \langle im\ A_2\rangle \oplus \langle ker\ A_2'\rangle \equiv \langle \mathcal{E}^n\rangle. \end{array}\right\} \qquad (233)$$

In the first variant of (163), i. e., as in (159), the sine is considered in the subspace $\langle im\ A_1 \cup im\ A_2\rangle$; in the second variant of (163), i. e., as in (160), the sine is considered in the subspace $\langle ker\ A_1' \cup ker\ A_2'\rangle$. Similarly, in the first variant of (171), the cosine is considered in $\langle im\ A_2 \cup ker\ A_1'\rangle$; in the second variant of (171), the cosine is considered in $\langle im\ A_1 \cup ker\ A_2'\rangle$.

The illustration is given in Figure 2. It is supposed without loss of generality that the first variant as in (154), i. e., $r_1 \le r_2$, $r_1 + r_2 \le n$ (or $2r \le n$), takes place. The space $\langle \mathcal{E}^n\rangle$ is partitioned due to this variant of differences (163) and (171) into four basic subspaces. Both they map the sine and cosine functions of the tensor angle $\tilde{\Phi}$ with its binary eigen angles $\pm\varphi_i$ – primaary and mutual to the first. (See more about such splitting in sect. 5.12.)

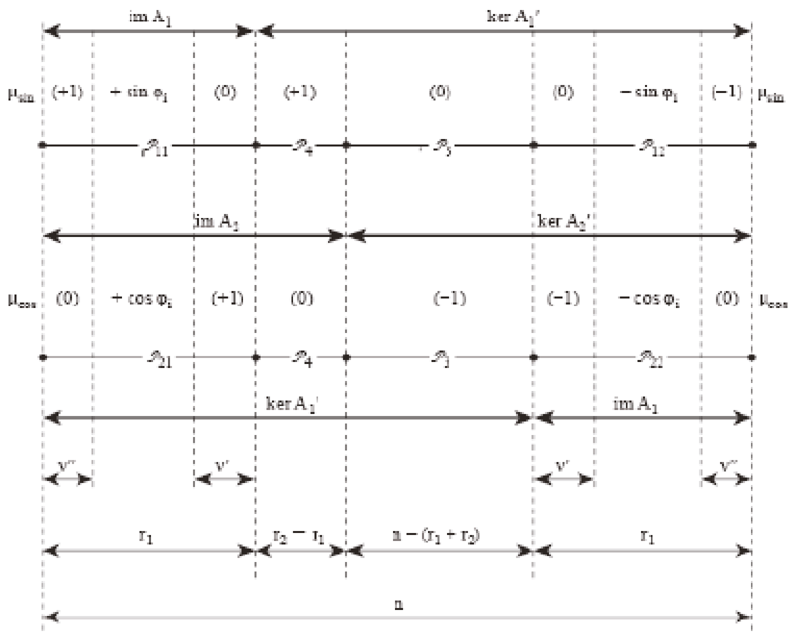

Figure 2. Distribution of projective sine and cosine eigenvalues in all the eigen subspaces of tensor angle between two lineors.

At Figure 2, as a Tensor Trigonometric Diagram for the sine-cosine pair of the projective tensor angle $\tilde{\Phi}_{12}$, we map abstractly the distribution in an Euclidean space in the logical sequence of paired $(\pm)$ eigenvalues of the tensor sine and cosine (with its *binary eigen angles* $\pm\varphi_i$ in D-form of tensor sine) and with corresponding to them eigen subspaces of this tensor angle orthogonal each to others. All these indicated subspaces are pairly orthogonal provided that in the *trigonometric subspace* of the tensor angle of dimension $2r$ there holds:

$$\sin\varphi_i \ne \pm 1, \quad \sin\varphi_i \ne\ 0, \quad (\cos\varphi_i \ne\ 0, \quad \cos\varphi_i \ne \pm 1). \qquad (234)$$

Otherwise orthogonalization may be used, so, by manner, suggested in (131), Ch. 3.

This whole binary trigonometric subspace is defined as the following direct sums of these four particular orthogonal subspaces (in the sine and cosine variants):

$$\langle \mathcal{P}_{11} \oplus \mathcal{P}_{12} \rangle \equiv \langle \mathcal{P}_{21} \oplus \mathcal{P}_{22} \rangle. \tag{235}$$

(These four subspaces are formed by eigenvectors of the tensor angle sine and cosine.)

Its even dimension 2τ is called further *the trigonometric rank of a tensor angle*, where $\tau = \min\{r_1, r_2, n - r_1, n - r_2\}$. Here we have $\tau = r_1$. The eigenvalues of the sine and cosine functions in (232) have the same absolute values in the two *mutual subspaces* (235), as the two sides of the binary angle in (163) are, due to (233), orthogonal; but their signs are opposite, as the projectors are ordered inversely in the two variants of differences (163) and (171) – see Figure 2.

If additional conditions (234) and $r_1 \leq r_2$ hold, the two intersections subspaces (the zero sine and the zero cosine) and their dimensions are expressed as follows:

$$\langle \mathcal{P}_3 \rangle \equiv \langle ker\ A_1' \cap ker\ A_2' \rangle, \quad dim \langle \mathcal{P}_3 \rangle = n - (r_1 + r_2), \quad (\sin\varphi = 0,\ \cos\varphi = -1);$$

$$\langle \mathcal{P}_4 \rangle \equiv \langle im\ A_2 \cap ker\ A_1' \rangle\ (\nu'' = 0), \quad dim \langle \mathcal{P}_4 \rangle = r_2 - r_1, \quad (\cos\varphi = 0,\ \sin\varphi = +1).$$

The projective tensor cosine and sine are symmetric (anticommutative) matrices, so they may be converted separately into their D-forms with certain modal orthogonal matrices R_1 and R_2 respectively in the bases $\tilde{E}_1 = R_1 \cdot \tilde{E}$ and $\tilde{E}_2 = R_2 \cdot \tilde{E}$. In order to give the trigonometric sense to the eigenvalues (232), we use an Euclidean space $\langle \mathcal{E}^n \rangle$ with the original base $\tilde{E}$ and then find an *local unity Cartesian base for the canonical W-forms* of the tensor trigonometric functions for the angle $\tilde{\Phi}$. We establish it below.

Each i-th trigonometric 2×2-cell with an unique pair of the cosine and sine ($\pm$) eigenvalues in the trigonometric subspace of a tensor angle $\tilde{\Phi}_{12}$ corresponds to its i-th eigenplane. It is determined here in $\tilde{E}$ by a pair of the *cosine orthogonal unity eigenvectors* u_i and v_i. They are two Cartesian axes of the *tensor cosine D-form base* (not yet oriented) and correspond to its eigenvalues $\pm\cos\varphi_i$, where $\varphi_i \in [-\pi/2; +\pi/2]$ are the eigenvalues of the tensor angle between planars or non-oriented lineors. In order to construct the canonical forms of the tensor trigonometric functions, dispose the trigonometric cells along the matrix diagonal with increasing the values $|\cos\varphi_i|$. Then along the diagonal dispose the 1×1-cells corresponding to the intersection sub-spaces $\langle \mathcal{P}_3 \rangle$ and $\langle \mathcal{P}_4 \rangle$. Denote the original base $\tilde{E}$ axes as $x_1, \ldots, x_n$, and the trigonometric part of the new axes as $u_1, \ldots, u_\tau$; $v_1, \ldots, v_\tau$ such that $x_1 \leftrightarrow u_1$, $x_2 \leftrightarrow v_1, \ldots,$ $x_{2i-1} \leftrightarrow u_i$, $x_{2i} \leftrightarrow v_i, \ldots,$ $x_{2\tau-1} \leftrightarrow u_\tau$, $x_{2\tau} \leftrightarrow v_\tau$. Direct the new axes in such way that each u_i and v_i form an acute angle. We found R_W for translating into new $\tilde{E}_1 = R_W\{\tilde{E}\} = \{I\}$.

In any trigonometric cell, $\sin^2\tilde{\Phi}_{12}$ and $\cos^2\tilde{\Phi}_{12}$ have the two positive (quadric) multiple eigenvalues from (232). As $\sin^2\tilde{\Phi}_{12}$ and $\cos^2\tilde{\Phi}_{12}$ commute in (184) and (185), then their and cosine D-forms are implemented together in the same *local base* $\tilde{E}_1$:

$\sin^2\tilde{\Phi}_{12}$

$\ddots$							
	$\sin^2\varphi_i$ 0 0 $\sin^2\varphi_i$						
		$\ddots$					
			0				$\mathcal{P}_3$
				$\ddots$			
					1		$\mathcal{P}_4$
						$\ddots$	

$\cos^2\tilde{\Phi}_{12}$

$\ddots$						
	$\cos^2\varphi_i$ 0 0 $\cos^2\varphi_i$					
		$\ddots$				
			1			
				$\ddots$		
					0	
						$\ddots$

.

Due to symmetry of $\sin\tilde{\Phi}_{12}$ and $\cos\tilde{\Phi}_{12}$, and anticommutativity condition (183) one represents these functions in the following *canonical W-forms* – see ((148) in sect. 1.4) in the *trigonometric base* $\tilde{E}_1 = \{I\}$ *of the diagonal cosine* (provided that $r_1 \le r_2$, $r_1 + r_2 \le n$):

$\sin\tilde{\Phi}_{12}$ $\qquad$ $\cos\tilde{\Phi}_{12}$ $\qquad$ (236), (237)

$\ddots$							
	$0 \quad +\sin\varphi_i$ $+\sin\varphi_i \quad 0$						
		$\ddots$					
			0				$\mathcal{P}_3$
				$\ddots$			
					$+1$		$\mathcal{P}_4$
						$\ddots$	

$\ddots$						
	$+\cos\varphi_i \quad 0$ $0 \quad -\cos\varphi_i$					
		$\ddots$				
			-1			
				$\ddots$		
					0	
						$\ddots$

,

In (236) and (237), the signs of the projective sine and cosine in the trigonometric cells are chosen out four possible variants, according to their definitions in (163), (171); but the signs of them in $\langle\mathcal{P}_3\rangle$, and $\langle\mathcal{P}_4\rangle$ are chosen, according to the additional conditions.

For the angle $\tilde{\Phi}_B$ ($B \in \langle Bp\rangle$, there holds $\langle\mathcal{P}_4\rangle = \mathbf{0}$, $dim\langle\mathcal{P}_3\rangle = n - 2r$. According to (199) we obtain the same base $\tilde{E}_1$ of the diagonal secant as for the cosine. From antisymmetry of $i\tan\tilde{\Phi}_{12}$ and anticommutativity condition in (203)-(204) one represents these functions in $\tilde{E}_1$ in the following *canonical W-forms* (provided that $2r < n$):

$\sec\tilde{\Phi}_B$ $\qquad$ $i\tan\ \tilde{\Phi}_B$ $\qquad$ (238), (239)

$\ddots$					
	$+\sec\ \varphi_i \quad 0$ $0 \quad -\sec\ \varphi_i$				
		$\ddots$			
			-1		$\mathcal{P}_3$
				$\ddots$	

$\ddots$				
	$0 \quad -\tan\varphi_i$ $+\tan\varphi_i \quad 0$			
		$\ddots$		
			0	
				$\ddots$

,

Formulae (236)–(239) are the canonical W-forms for all projective trigonometric functions in the directed base of the diagonal cosine $\tilde{E}_1$. This base is called *trigonometric* and used in W-forms representations. These forms also illustrate **Rule 1** (see above).

Under conditions (234) also there holds:

$$\left.\begin{array}{l} \vec{S_i} = \overrightarrow{\{\cos^2\tilde{\Phi} - \cos^2\varphi_i \cdot I\}} = \vec{S_{i1}} + \vec{S_{i2}}; \\ \vec{S_{i1}} = \overrightarrow{\{\cos\tilde{\Phi} - \cos\varphi_i \cdot I\}}, \quad \vec{S_{i2}} = \overrightarrow{\{\cos\tilde{\Phi} + \cos\varphi_i \cdot I\}}; \\ \vec{S_3} = \overrightarrow{\sin\tilde{\Phi}}, \quad \vec{S_4} = \overrightarrow{\cos\tilde{\Phi}}; \quad (\vec{S_1} + \vec{S_2} + \vec{S_3} + \vec{S_4} = I) \end{array}\right\}. \tag{240}$$

These are the orthoprojectors onto the following *characteristic eigen subspaces*: the i-th trigonometric cell, the axes $u_i \subset \langle\mathcal{P}_{21}\rangle$ and $v_i \subset \langle\mathcal{P}_{22}\rangle$, $\langle\mathcal{P}_3\rangle$, and $\langle\mathcal{P}_4\rangle$. Here their basis columns (as well as the basis rows) determine the subspaces indicated.

If some angle φ_i is multiple, then the i-th trigonometric cells are united, and orthogonalization of their homogeneous axes are necessary for preserving the binary trigonometric structure. Moreover, if simplest eigenvalues (0 and ± 1) of the projective cosine or sine are equal to the same ones in $\langle \mathcal{P}_3 \rangle$ and $\langle \mathcal{P}_4 \rangle$, one may also use orthogonalization for dividing the mixed trigonometric partial subspaces. (See sect. 3.1.)

Below we consider the extreme cases of the angles and the cases with the other primary additional assumptions (see Figure 2).

Return to conditions (234). They facilitate partitioning an Euclidean space $\langle \mathcal{E}^n \rangle$ into trigonometric subspaces due to the unary and binary parts of W-forms. At first, consider the additional case, when the eigenvalues $\sin\varphi_i = 0$ of the multiplicity $2\nu\,'$ are in $\langle \mathcal{P}_{11} \rangle$ and $\langle \mathcal{P}_{12} \rangle$. Besides they corresponds to the sine value 0 belonging to $\langle \mathcal{P}_3 \rangle$. Also they corresponds to the pair eigenvalues of the multiplicity ν' $\cos\varphi_i = +1$ in $\langle \mathcal{P}_{21} \rangle$ and $\cos\varphi_i = -1$ in $\langle \mathcal{P}_{22} \rangle$. The last value of the cosine corresponds to the cosine value -1 belonging to $\langle \mathcal{P}_3 \rangle$. The other additional case takes place, when the eigenvalues $\cos\varphi_i = 0$ of the multiplicity $2\nu''$ are in $\langle \mathcal{P}_{21} \rangle$ and $\langle \mathcal{P}_{22} \rangle$. Besides they corresponds to the cosine value 0 belonging to $\langle \mathcal{P}_4 \rangle$. Also they corresponds to the pair eigenvalues of the multiplicity ν'' $\sin\varphi_i = +1$ in $\langle \mathcal{P}_{11} \rangle$ and $\sin\varphi_i = -1$ in $\langle \mathcal{P}_{12} \rangle$. The first value of the sine corresponds to the sine value $+1$ belonging to $\langle \mathcal{P}_4 \rangle$. In order to separate all the characteristic eigenspaces, it is necessary to orthogonalize them. After that the partial subspaces $\langle \mathcal{P}_{11} \rangle$, $\langle \mathcal{P}_{21} \rangle$, $\langle \mathcal{P}_3 \rangle$, $\langle \mathcal{P}_4 \rangle$, $\langle \mathcal{P}_{12} \rangle$, $\langle \mathcal{P}_{22} \rangle$ become entirely orthogonal.

Now suppose that other assumptions, taken before, do not hold. If $r_1 + r_2 > n$, then $\langle \mathcal{P}_3 \rangle = \langle im\ A_1 \rangle \cap \langle im\ A_2 \rangle$. Besides, if $r_2 > r_1$, then $\langle \mathcal{P}_4 \rangle = \langle im\ A_2 \rangle \cap \langle ker\ A_1' \rangle$. In according with these new conditions, the signs of unitary sine and cosine eigenvalues in $\langle \mathcal{P}_3 \rangle$ and $\langle \mathcal{P}_4 \rangle$ should be changed. For equirank lineors the subspace $\langle \mathcal{P}_4 \rangle$ is absent!

All the bases used are right ($det\{R\} = +1$). Among them are the original Cartesian base $\tilde{E}$ and the new Cartesian bases in the planes $\langle u_i, v_i \rangle$, i. e., $\tilde{E}_1 = R_W\{\tilde{E}\} = \{I\}$ (they form the binary part of the *trigonometric base*). In the trigonometric base, one may find the contradiagonal values of the sine up to their signs according to (236), then the cosine signs are exactly determined by (237); and vice versa. Both determine completely the absolute value and the sign of the counter-clockwise scalar angle φ_i in $[-\pi/2; +\pi/2]$. This segment is the range of angles for planars or non-oriented lineors.

Analogous reasoning may be realized for distributions of the projective secant and tangent values in the four eigenspaces of the tensor spherical angle between two lineors, with correspondence to their mapping above in (238), (239).

Thus, with matrices of canonical forms (236)–(240), we have completed that fundamental part of tensor trigonometry, which relates to the definition, various properties, and primary application of tensor trigonometric functions and their angle-arguments of the projective type. And it remains for us in the same way to complete its fundamental part with similar canonical matrices, which relates to the definition, properties, and primary application of tensor trigonometric symmetric (orthogonal) and oblique (affine) eigenreflectors with the projective-type tensor angles figuring in them. But in eigenreflectors (176)–(179) and (211), (212), as mathematical operators, these angles are not arguments. Their arguments are singular square matrices (AA', BB', B'B, B and B') with images and kernels between which reflections occur. All these tensor projective trigonometric functions, as it is exposed above, in their sine-cosine (176)–(179) and tangent-secant (211)–(212) pairs forms the corresponding pairs of symmetric (orthogonal) and asymmetric (oblique) eigenprojectors with respect to the image or the kernel for each. Therefore the same trigonometric base (the directed base of the diagonal cosine) is used for canonical forms of both mutual orthogonal eigenreflectors (176), (177) and of both mutual affine eigenreflectors (211), (212). Their matrix tensor forms are the following:

$+Ref\{A_1A_1'\}$ $\qquad\qquad$ $+Ref\{A_2A_2'\}$ (241)

$\ddots$						
	$+\cos\varphi_i \quad -\sin\varphi_i$ $-\sin\varphi_i \quad -\cos\varphi_i$					
		$\ddots$				
			-1			
				$\ddots$		
					-1	
						$\ddots$

$\mathcal{P}_3$, $\mathcal{P}_4$

$\ddots$						
	$+\cos\varphi_i \quad +\sin\varphi_i$ $+\sin\varphi_i \quad -\cos\varphi_i$					
		$\ddots$				
			$+1$			
				$\ddots$		
					-1	
						$\ddots$

,

(they are the algebraic sum (237) and (236) provided that $r_1 \leq r_2,\ r_1 + r_2 \leq n$);

$+Ref\{B\}$ $\qquad\qquad$ $+Ref\{B'\}$ (242)

$\ddots$				
	$+\sec\ \varphi_i \quad +\tan\varphi_i$ $-\tan\varphi_i \quad -\sec\ \varphi_i$			
		$\ddots$		
			-1	
				$\ddots$

$\mathcal{P}_3$

$\ddots$				
	$+\sec\ \varphi_i \quad -\tan\varphi_i$ $+\tan\varphi_i \quad -\sec\ \varphi_i$			
		$\ddots$		
			$+1$	
				$\ddots$

,

(they are the algebraic sum (238) and (239) provided that $2r \leq n$).

5.6 The tensor trigonometric theory of prime roots $\sqrt{I}$

In this section, we describe briefly connection between the main notions of tensor trigonometry and the theory of prime roots $\sqrt{I}$ (i. e., without nilpotent matrix summand as in (21) or (76) – Part 1). Fix an original Cartesian base $\tilde{E}$ in $\langle\mathcal{E}^n\rangle$. In this base any *prime square root of* the matrix I is the reflector (sign-indefinite nonsingular matrix), either symmetric or nonsymmetric – see formulae (176)–(179) and (211), (212).

So, it is $(\sqrt{I})_s = Ref\{Bm\}$, in particular $(\sqrt{I})_s = Ref\{AA'\}$; or $\sqrt{I} = Ref\{Bp\}$.

They can be converted, with the certain modal transformation $T \cdot \{\tilde{E}\} = \tilde{E}_D$, into the dual block-unity D-form of $\sqrt{I} = Ref\{Bm\}$ or $\sqrt{I} = Ref\{Bp\}$:

$$R_W' \cdot \sqrt{I} \cdot R_W = I^{\pm} = \left[\begin{array}{c|c} +I & Z \\ \hline Z & -I \end{array}\right] \begin{matrix} q^+ \\ q^- \end{matrix} \qquad (q^+ + q^- = n,\ \ q^+ = rank\ B,\ \ q^- = sing\ B).$$

For any trigonometric matrix (i. e., matrix, bound with a tensor angle) its trigonometric rank is defined by the binary structure of the tensor angle (see, for example, in (235)). Here the trigonometric rank $2r$ also corresponds to an *index q of the reflector*:

$$2r = 2q = 2\min\{q^+, q^-\} = 2\min\{r, n-r\}.$$

Further, for W-forms we use also the trigonometric Cartesian base $\tilde{E}_1 = R_W\{\tilde{E}\} = \{I\}$.

Separate *symmetric* roots $(\sqrt{I})_s = (\sqrt{I})'_s$. For a null-normal matrix Bm, for example, $+Ref\{Bm\} = \overleftarrow{Bm} - \overrightarrow{Bm} = (\sqrt{I})_s$. Put, without loss of generality, $Bm = AA'$.

Let $(\sqrt{I})_1$ and $(\sqrt{I})_2$ be a pair of independent symmetric roots. Then, in $\langle \mathcal{E}^n \rangle$, these roots and the orthoreflectors are connected as follows:

$$\left\{ \begin{array}{ll} \overleftarrow{A_1A'_1} = \frac{(I+(\sqrt{I})_1)}{2}, & \overrightarrow{A_1A'_1} = \frac{(I-(\sqrt{I})_1)}{2}, \\ \overleftarrow{A_2A'_2} = \frac{(I+(\sqrt{I})_2)}{2}, & \overrightarrow{A_2A'_2} = \frac{(I-(\sqrt{I})_2)}{2}. \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} (\sqrt{I})_1 = \overleftarrow{A_1A'_1} - \overrightarrow{A_1A'_1}, \\ (\sqrt{I})_2 = \overleftarrow{A_2A'_2} - \overrightarrow{A_2A'_2}. \end{array} \right\} \tag{243}$$

From this, taking into account (163), (171), (176), and (177), we obtain

$$\left. \begin{array}{lll} \cos\tilde{\Phi}_{12} - \sin\tilde{\Phi}_{12} & = (\sqrt{I})_1 & = +Ref\{A_1A'_1\}, \\ \cos\tilde{\Phi}_{12} + \sin\tilde{\Phi}_{12} & = (\sqrt{I})_2 & = +Ref\{A_2A'_2\}, \end{array} \right\} \tag{244}$$

$$\cos\tilde{\Phi}_{12} = [(\sqrt{I})_1 + (\sqrt{I})_2]/2, \quad \sin\tilde{\Phi}_{12} = [(\sqrt{I})_2 - (\sqrt{I})_1]/2. \tag{245}$$

The homogeneous projectors are equirank, iff $(\sqrt{I})_1$ and $(\sqrt{I})_2$ have the same index, either q^-, or q^+ (as the trigonometric rank for a pair of lineors or null-prime matrix). Remember, that the orthoreflectors $+Ref\{AA'\}$ and $-Ref\{AA'\}$ have their mutually orthogonal mirrors $\langle ker\ A' \rangle$ and $\langle im\ A \rangle$ in the Euclidean space.

<u>Corollaries</u> (for $\langle \mathcal{E}^n \rangle$)

1. *A symmetric root* $\sqrt{I}$ *defines one-to-one a unique symmetric orthogonal reflector as well as a unique mutual pair of spherically orthogonal projectors and a unique right tensor angle of the same trigonometric rank.*

2. *Any pair of symmetric roots* $(\sqrt{I})_1$ *and* $(\sqrt{I})_2$ *defines a unique pair of spherically orthogonal projectors, a unique tensor angle* $\tilde{\Phi}_{12}$ *and its trigonometric functions.*

3. *If an original Cartesian base* $\tilde{E}$ *is fixed, then all the matrix notions, according to item* 2, *due to* (243) – (245), *may be converted into compatible monobinary W-forms in a trigonometric Cartesian base* $\tilde{E}_1$.

Separate *nonsymmetric* prime roots $\sqrt{I} \neq (\sqrt{I})'$. For a null-prime matrix Bp, for example, $+Ref\{Bp\} = \overleftarrow{Bp} - \overrightarrow{Bp} = \sqrt{I} \neq (\sqrt{I})'$. Denote the matrix Bp briefly as B. Then we have the following bond of these roots and oblique reflectors:

$$\left\{ \begin{array}{ll} \overleftarrow{B} = \frac{(I+\sqrt{I})}{2}, & \overrightarrow{B} = \frac{(I-\sqrt{I})}{2}, \\ \overleftarrow{B'} = \frac{(I+(\sqrt{I})')}{2}, & \overrightarrow{B'} = \frac{(I-(\sqrt{I})')}{2}. \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \sqrt{I} = \overleftarrow{B} - \overrightarrow{B}, \\ (\sqrt{I})' = \overleftarrow{B'} - \overrightarrow{B'}. \end{array} \right\} \tag{246}$$

From this, taking into account (198), (203), (211), (212), we obtain

$$\left. \begin{array}{lll} \sec\ \tilde{\Phi}_B - i\tan\tilde{\Phi}_B & = \sqrt{I} & = +Ref\{B\}, \\ \sec\ \tilde{\Phi}_B + i\tan\tilde{\Phi}_B & = (\sqrt{I})' & = +Ref\{B'\}, \end{array} \right\} \tag{247}$$

$$\sec\ \tilde{\Phi}_B = (\sqrt{I} + (\sqrt{I})')/2, \quad i\tan\tilde{\Phi}_B = ((\sqrt{I})' - \sqrt{I})/2. \tag{248}$$

The roots $\sqrt{I}$ and $(\sqrt{I})'$ always have the same trigonometric rank. Remember, that the oblique reflectors $+Ref\{Bp\}$ and $+Ref\{Bp\}$ have their mutually oblique mirrors $\langle ker\ B \rangle$ and $\langle im\ B \rangle$ in the Euclidean space – see for the non-transposed reflector.

<u>Corollaries</u> (for $\langle \mathcal{E}^n \rangle$)

1. *Any nonsymmetric prime root* $\sqrt{I}$ *defines a unique nonsymmetric oblique reflector as well as a unique mutual pair of spherically oblique projectors.*

2. *Any pair of nonsymmetric prime roots* $\sqrt{I}$ *and* $(\sqrt{I})'$ *define a unique pair of spherically oblique projectors, a unique tensor angle* $\tilde{\Phi}_B$ *and its trigonometric functions.*

3. *If a Cartesian base* $\tilde{E}$ *is fixed, then all the notions (item 2), due to* (246) – (248), *may be converted into compatible monobinary W-forms in a trigonometric Cartesian base* $\tilde{E}_1$.

* * *

In sections 5.1–5.6, we have laid out that fundamental content of tensor trigonometry which relates to its linear algebraic objects, to definitions of all the tensor trigonometric functions with their tensor angle–argument of projective type in the affine and Euclidean spaces, to all symmetric (orthogonal) and oblique (affine) eigenreflectors very logically produced from these trigonometric projective functions with their specific geometric reflective actions. We have established a one-to-one relationship between all these eigenreflectors and the symmetric or asymmetric prime roots from the unity matrix. When these roots–reflectors have also a specific origin, then they can have been linked to eigenprojectors and tensor trigonometric functions with projective angles–arguments.

Our next task is, using such eigenreflectors, firstly, to correctly introduce the quasi-Euclidean space (complementing the well-known pseudo-Euclidean space) and, secondly, to develop in natural way that part of tensor trigonometry which relates to representation of rotations as tensor trigonometric functions with their tensor angle-argument of motive type. In order to pass from tensor angles of projective type to ones of motive type in natural and correct way, we will resort to the known property of scalar trigonometry: the execution of two reflections for a vector on an Euclidean plane, where reflection and rotation are consistent trigonometrically, leads to its rotation (with transformation in tensor variant).

5.7 Rotational functions of motive tensor spherical angles

In the sequel, in order to infer some matrix formulae and connected with them equality and inequality we shall use so called *the principle of binarity*. It consists in the following.

The prime real matrices P_1 and P_2 are anticommutative iff they may be represented jointly in their real anticommutative monobinary cell forms W_1 and W_2 in a certain *real local base* (sect. 4.1). If the original affine base is $\tilde{E}$, then here the local base is $\tilde{E}_1 = V_W \cdot \tilde{E} = \{I\}$. The matrices P_1 and P_2 are anticommutative on their common real eigenspaces of dimensions 1 and 2 (see more in sect. 7.2). These forms W_1 and W_2 are a direct sum of their monobinary cells of the identical structure.

Moreover, any analytical function $F(P_1, P_2)$ in the base $\tilde{E}$ may be expressed in the base $\tilde{E}_1$ as $F(W_1, W_2)$. In particular, this realizes for elementary operations of summation and multiplication. The scalar invariants of $F(P_1, P_2)$ are the same invariants for $F(W_1, W_2)$. (In theory of matrices, the analogous *principle of unarity* is applied for analytical functions of several prime commutative matrices with their joint reducing to diagonal forms.) The principle of binarity is based on the fact that original and squares of anticommutative prime matrices P_1 and P_2 commute each with another. Both these principles enable one to generalize analytical operations over simplest cell structures and results onto original matrices and their analytical functions.

Suppose, in particular, in $\langle \mathcal{E}^n \rangle$: $P_1 = \cos \tilde{\Phi}_{12}$, $P_2 = \sin \tilde{\Phi}_{12}$ for the *equirank lineors* A_1 and A_2, according to formulae (163) and (171). Then $\langle \mathcal{P}_4 \rangle = \mathbf{0}$. But non-zero $\langle \mathcal{P}_3 \rangle$ exists iff it exists in canonical cosine form (237) (as positive or negative unity block).

By (176) and (177) for these anticommutative P_1 and P_2 we have the analytical function

$$F(P_1, P_2) = (P_1 + P_2) \cdot (P_1 - P_2) =$$

$$= [+Ref\{A_2 A_2'\}] \cdot [+Ref\{A_1 A_1'\}] = [-Ref\{A_2 A_2'\}] \cdot [-Ref\{A_1 A_1'\}].$$

Then there holds $T_W = R_W$.

So, the W-form of $F(P_1, P_2)$ in the *trigonometric base* $\tilde{E}_1 = R_W \cdot \tilde{E} = \{I\}$ is expressed by the *orthogonal rotational matrix* at the angle $2\Phi_{12}$ gotten with two mutual reflections:

$$\begin{array}{ccc} Ref\{A_2A_2'\} & Ref\{A_1A_1'\} & Rot(+2\Phi_{12}) \end{array} \qquad (249)$$

$$\left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & +\cos\varphi_i & +\sin\varphi_i & \\ & +\sin\varphi_i & -\cos\varphi_i & \\ \hline & & & \ddots \end{array}\right] \cdot \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & +\cos\varphi_i & -\sin\varphi_i & \\ & -\sin\varphi_i & -\cos\varphi_i & \\ \hline & & & \ddots \end{array}\right] = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \cos 2\varphi_i & -\sin 2\varphi_i & \\ & +\sin 2\varphi_i & \cos 2\varphi_i & \\ \hline & & & \ddots \end{array}\right],$$

where $\langle \mathcal{P}_3 \rangle$ is the *unity block* $+I$ as $(\pm 1)\cdot(\pm 1) = +1$ for unity cosine part in (237). This 2×2-cell implements rotation at the counter-clockwise angle $+2\varphi_i$ on trigonometric eigenplanes. In $\tilde{E}$, it implements *spherical rotation* at the *motive* tensor angle $+2\Phi_{12}$:

$$Ref\{A_2A_2'\} \cdot Ref\{A_1A_1'\} = (\cos\tilde{\Phi}_{12} + \sin\tilde{\Phi}_{12}) \cdot (\cos\tilde{\Phi}_{12} - \sin\tilde{\Phi}_{12}) =$$

$$= \cos^2\tilde{\Phi}_{12} - \sin^2\tilde{\Phi}_{12} + 2\sin\tilde{\Phi}_{12}\cos\tilde{\Phi}_{12} = \cos^2\Phi_{12} - \sin^2\Phi_{12} + 2i\sin\Phi_{12}\cos\Phi_{12} =$$

$$= \cos 2\Phi_{12} + i\sin 2\Phi_{12} = Rot\ 2\Phi_{12} = [-Ref\{A_2A_2'\}] \cdot [-Ref\{A_1A_1'\}], \qquad (249)$$

$$[\pm Ref\{A_1A_1'\}] \cdot [\pm Ref\{A_2A_2'\}] = Rot\ '2\Phi_{12} = Rot(-2\Phi_{12}) = Rot\ 2\Phi_{21}. \qquad (250)$$

Notation $Rot\ \Phi$ is used for tensor rotational functions of *binary motive type* tensor spherical angles Φ. Such tensor angles do not contain in their notation the tilde symbol necessary for projective tensor angles. The following *united properties* hold for the main sine-cosine pairs of projective and motive tensor angles (see more in sect 5.8): $\cos^2\tilde{\Phi} = \cos^2\Phi$, $\sin^2\tilde{\Phi} = \sin^2\Phi$; and $\sin\tilde{\Phi}\cdot\cos\tilde{\Phi} = i\sin\Phi\cdot\cos\Phi = \cos\Phi\cdot i\sin\Phi = -\cos\tilde{\Phi}\cdot\sin\tilde{\Phi}$. These formulae also illustrate **Rule 1** (sect. 5.3), but for the motive type trigonometric functions. Obviously, then in any rotational matrix subspace $\langle \mathcal{P}_3 \rangle$ has eigenvalues only $+1$, and $\langle \mathcal{P}_4 \rangle = \mathbf{0}$ (as in the case $r_1 = r_2$ at Figure 2). Most generally note, that $Rot\ \Phi_{12}$ as in (249) is a *trigonometric square root* ((i. e., as result of the original angle dimidiating in the each binary cell of W-form!):

$$Rot\ \Phi_{12} = \{[\pm Ref\{A_2A_2'\}] \cdot [\pm Ref\{A_1A_1'\}]\}^{1/2}. \qquad (251)$$

Formula (249) is interpreted as follows. Orthogonal reflection of $\langle im\ A_1 \rangle$ (or $\langle ker\ A_1' \rangle$) and then of $\langle im\ A_2 \rangle$ (or $\langle ker\ A_2' \rangle$) is equivalent to rotation at the doubled angle between $\langle im\ A_1 \rangle$ and $\langle im\ A_2 \rangle$. It is quite clear when we deal with two vectors or straight lines. The rotational matrix ($\tau = 1$), according to (249), (176), (177), is

$$Rot\ \Phi_{12} = [(I - 2\cdot\overleftarrow{\mathbf{a}_2\mathbf{a}_2'})\cdot(I - 2\cdot\overleftarrow{\mathbf{a}_1\mathbf{a}_1'})]^{1/2} =$$

$$= \left[I - 2\cdot\left(\frac{\mathbf{a}_1\mathbf{a}_1'}{\mathbf{a}_1'\mathbf{a}_1} + \frac{\mathbf{a}_2\mathbf{a}_2'}{\mathbf{a}_2'\mathbf{a}_2}\right) + 4\cos^2\varphi_{12}\cdot\frac{\mathbf{a}_2\mathbf{a}_1'}{\mathbf{a}_1'\mathbf{a}_2}\right]^{1/2}. \qquad (252)$$

Here the mirrors are either $\mathbf{a}$ or a hyperplane $\langle ker\ \mathbf{a}' \rangle$ – orthocomplement of $\langle im\ \mathbf{a} \rangle$,

$$Rot\ \Phi_{12}\cdot\overleftrightarrow{\mathbf{a}_1\mathbf{a}_1'}\cdot Rot\ (-\Phi_{12}) = \overleftrightarrow{\mathbf{a}_2\mathbf{a}_2'},\ \overleftarrow{\mathbf{aa}'} = \frac{\mathbf{aa}'}{\mathbf{a}'\mathbf{a}}, \overleftarrow{\mathbf{a}_2\mathbf{a}_1'} = \frac{\mathbf{a}_2\mathbf{a}_1'}{\mathbf{a}_1'\mathbf{a}_2}\ (\overleftarrow{\mathbf{ee}'} = \mathbf{ee}', \overleftarrow{\mathbf{e}_2\mathbf{e}_1'} = \frac{\mathbf{e}_2\mathbf{e}_1'}{\cos\varphi_{12}}).$$

If these $n\times 1$-vectors are oriented, then the angle φ_{12} in the trigonometric eigenplane of the matrix $Rot\ \Phi$ in the Euclidean space $\langle \mathcal{E}^n \rangle$ varies in $[-\pi;\pi]$. (For rotations of lineors we have $r_R = r_1 = r_2$ at Figure 2.) But rotation (249) is performed in the $2r_R$-dimensional subspace with respect to its orthocomplement of dimension $n - 2r_R$. The rotation in matrix (249) with the *eigen angles* $\pm i\varphi_k$ on *k-th trigonometric eigenplanes* is realized in $[-\pi/2;+\pi/2]$!

Real prime matrices are called *compatible* if their W-forms have the same structure in a common base. So, they may be commutative or anticommutative ones – see more in Ch. 7). Real normal matrices may be converted into W-forms with rotational transformations of the base, we denote them as R_W. For compatible normal matrices, R_W is same.

The most general variant of formulae (249) and (250) for compatible reflectors is

$$(\cos\tilde{\Phi}_{12} \pm \sin\tilde{\Phi}_{12})(\cos\tilde{\Phi}_{34} \pm \sin\tilde{\Phi}_{34}) = Rot\ \{\pm\Phi_{12} \pm \Phi_{34}\}.$$

In tensor trigonometry, besides the *mutual reflectors* in (249), so called the *mid-reflector* is very important. For a pair of the given lineors or planars their mid-reflector is single between $Ref\{A_1A'_1\}$ and $Ref\{A_2A'_2\}$ ($r_1 = r_2 = r$) or $Ref\{BB'\}$ and $Ref\{B'B\}$, i. e., for the middle subspace of tensor angle $\tilde{\Phi}_{12}$ or $\tilde{\Phi}_B$. But it is not attach only to this pair of objects. It is defined for the set of pairs of linear objects having such common mid-reflector. It has *the sign-alternating unity diagonal W-form* congruous to the *cosine diagonal form* (237) in the trigonometric base $\tilde{E}_1$ of the projective angle $\tilde{\Phi}$. The cosine axes in the zero sine subspace $\langle\mathcal{P}_3\rangle$ are the same with their eigenvalues $+1$ or -1; the zero cosine subspace $\langle\mathcal{P}_4\rangle = \mathbf{0}$ as $r_1 = r_2$. According to (171), (172) and to diagonal cosine (237), the projective cosine is the algebraic sum of two *orthogonal* terms with algebraically positive and negative eigenvalues:

$$\cos\tilde{\Phi}_{12} = \{\cos\tilde{\Phi}_{12}\}^{\oplus} + \{\cos\tilde{\Phi}_{12}\}^{\ominus}, \quad \{\cos\tilde{\Phi}_{12}\}^{\oplus}\cdot\{\cos\tilde{\Phi}_{12}\}^{\ominus} = Z.$$

These summands are singular matrices. Here the mid-reflector mirror is the subspace $\langle im\ \{\cos\tilde{\Phi}_{12}\}^{\ominus}\rangle$, given by axes v_i. According to (176) the mid-reflector is expressed as

$$Ref\{\cos\tilde{\Phi}_{12}\}^{\ominus}\} = \{\overrightarrow{\cos\tilde{\Phi}_{12}}\}^{\ominus} - \{\overleftarrow{\cos\tilde{\Phi}_{12}}\}^{\ominus} = \{\sqrt{I}\}_S = \{R_W\cdot I^{\pm}\cdot R'_W\}. \tag{253}$$

Define the **index** q of this mid-reflector of Φ_{12} or Φ_B as the quantity of its eigenvalues -1.

The mirror of this mid-reflector is situated in the middle between two mutual mirrors in (176) and (177) for the tensor angle $\tilde{\Phi}_{12}$ – see their structures (241). In order to prove this, we obtain this mid-reflector by four ways: by modal rotating the 1-st reflector at the angle $\{+\Phi_{12}/2\}$, by modal rotating the 2-nd reflector at the angle $\{-\Phi_{12}/2\}$ in the base $\tilde{E}_1$ and by only left and right rotating these reflectors at the angles $\{\pm\Phi_{12}\}$ as below:

$$\underset{Rot\ \Phi_{12}/2}{\begin{bmatrix} \ddots & & & \\ & \cos\varphi_i/2 & -\sin\varphi_i/2 & \\ & \sin\varphi_i/2 & \cos\varphi_i/2 & \\ & & & \ddots \end{bmatrix}} \cdot \underset{Ref\{A_1A'_1\}}{\begin{bmatrix} \ddots & & & \\ & +\cos\varphi_i & -\sin\varphi_i & \\ & -\sin\varphi_i & -\cos\varphi_i & \\ & & & \ddots \end{bmatrix}} \cdot \underset{Rot'\ \Phi_{12}/2}{\begin{bmatrix} \ddots & & & \\ & \cos\varphi_i/2 & \sin\varphi_i/2 & \\ & -\sin\varphi_i/2 & \cos\varphi_i/2 & \\ & & & \ddots \end{bmatrix}} =$$

$$= \underset{Rot'\ \Phi_{12}/2}{\begin{bmatrix} \ddots & & & \\ & \cos\varphi_i/2 & \sin\varphi_i/2 & \\ & -\sin\varphi_i/2 & \cos\varphi_i/2 & \\ & & & \ddots \end{bmatrix}} \cdot \underset{Ref\{A_2A'_2\}}{\begin{bmatrix} \ddots & & & \\ & \cos\varphi_i & +\sin\varphi_i & \\ & \sin\varphi_i & -\cos\varphi_i & \\ & & & \ddots \end{bmatrix}} \cdot \underset{Rot\ \Phi_{12}/2}{\begin{bmatrix} \ddots & & & \\ & \cos\varphi_i/2 & -\sin\varphi_i/2 & \\ & \sin\varphi_i/2 & \cos\varphi_i/2 & \\ & & & \ddots \end{bmatrix}} =$$

$$= Rot\ (+\Phi_{12})\cdot Ref\{A_1A'_1\} = Ref\{\cos\tilde{\Phi}_{12}\}^{\ominus} = Rot\ (-\Phi_{12})\cdot Ref\{A_2A'_2\} = \tag{254}$$

$$= (\sqrt{I})_S = \begin{bmatrix} \ddots & & & \\ & +1 & 0 & \\ & 0 & -1 & \\ & & & \ddots \end{bmatrix}.$$

The small reverse.

Now we may return to justification of projective trigonometric functions definition in the beginning of the chapter. Basically, it comes down to the choice of formulae for the cosine and secant, as well as for the orthogonal and oblique reflectors, in fact, from the two most optimal options. Factually, we have already fully outlined the first projective option. It remains to show below how the alternative option will differ. So, let us initially choose the opposite formulae for the cosine and secant:

$$\cos\bar{\Phi}_{12} = \overrightarrow{A_2A_2'} - \overleftarrow{A_1A_1'}, \quad \cos\bar{\Phi}_B = \overrightarrow{B'B} - \overleftarrow{BB'}; \quad \sec\bar{\Phi}_B = \overrightarrow{B'} - \overleftarrow{B}.$$

Then in all such formulae containing cosine and secant, they change sign to the opposite. However the essence of these formulae remains the same. Then, in order to preserve the having logical form for all useful relations with reflectors, we also choose opposite formulas for all these reflectors:

$$\left.\begin{array}{l} \overrightarrow{A_1A_1'} - \overleftarrow{A_1A_1'} = Ref\{A_1A_1'\} = \cos\bar{\Phi}_{12} + \sin\bar{\Phi}_{12} = F(+\bar{\Phi}_{12}), \\ \overrightarrow{A_2A_2'} - \overleftarrow{A_2A_2'} = Ref\{A_2A_2'\} = \cos\bar{\Phi}_{12} - \sin\bar{\Phi}_{12} = F(-\bar{\Phi}_{12}); \\ \overleftarrow{B} - \overrightarrow{B} = Ref\{B\} = \sec\bar{\Phi}_B + i\tan\bar{\Phi}_B = \Psi\{+\bar{\Phi}_B\}, \\ \overleftarrow{B'} - \overrightarrow{B'} = Ref\{B'\} = \sec\bar{\Phi}_B - i\tan\bar{\Phi}_B = \Psi\{-\bar{\Phi}_B\}. \end{array}\right\}$$

After that, in the notations of all reflectors, the mirror and its orthogonal or affine complement also change automatically places. Accordingly, under the given conditions at Figure 2, the sign of the unit eigenvalues of the cosine on the subspace of the zero sine becomes positive. In (249), the order of reflections and in (254), the order of rotations must be reversed for the positive *counter-clockwise* rotation angle as adopted! Of course, this is relative and does not introduce significant change in the projective part of tensor trigonometry! One can do as it seems more convenient – either according to the stated option, or according to this alternative option. Both options are equal in rights!

* * *

Further, when using the principle of binarity in 2×2-cells, we do not attach importance to the sign of unit eigenvalues corresponding to the subspace $\langle\mathcal{P}_3\rangle$ in the binary structure of the mid-reflector. (It can be either positive or negative, as usually specified.) For new matrices, gotten with the use of this principle, we give useful relations in addition to (249), (251), (254):

$$\left.\begin{array}{l} Ref\{A_1A_1'\} = Rot'\ \Phi_{12}\cdot Ref\{A_2A_2'\}\cdot Rot\ \Phi_{12} \leftrightarrow \\ \leftrightarrow Ref\{A_2A_2'\} = Rot\ \Phi_{12}\cdot Ref\{A_1A_1'\}\cdot Rot'\ \Phi_{12}. \end{array}\right\} \Rightarrow \tag{255}$$

$$\Rightarrow \left\{\begin{array}{l} Ref\{\cos\bar{\Phi}_{12}\}^{\ominus} = Rot\ (+\Phi_{12})\cdot Ref\{A_1A_1'\} = Ref\{A_1A_1'\}\cdot Rot\ (-\Phi_{12}) = \\ = Rot\ (-\Phi_{12})\cdot Ref\{A_2A_2'\} = Ref\{A_2A_2'\}\cdot Rot\ (+\Phi_{12}). \end{array}\right\}$$

$$\left.\begin{array}{l} Ref\{A_2A_2'\} = Ref\{\cos\bar{\Phi}_{12}\}^{\ominus}\cdot Ref\{A_1A_1'\}\cdot Ref\{\cos\bar{\Phi}_{12}\}^{\ominus}, \\ Ref\{A_1A_1'\} = Ref\{\cos\bar{\Phi}_{12}\}^{\ominus}\cdot Ref\{A_2A_2'\}\cdot Ref\{\cos\bar{\Phi}_{12}\}^{\ominus}. \end{array}\right\} \tag{256}$$

Rule 2. *Compatible spherical rotational matrices commute. In their multiplications the tensor argument angles of motive type form an algebraic sum.*

Rule 3. *In multiplications of a rotation and a symmetric reflector, if they are compatible, the rotation is transferred trough the reflector with the change of its tensor angle sign.*

Corollaries

1. *An orthogonal matrix* R *is a rotational function if* $\det R = +1$, *and* R *is a reflector if* $R = R'$. *What is more, when* $\det R = +1$, *these two properties may be compatible.*

2. *The types of tensor angle in eigenreflectors (i. e., when bound with the angle) and in rotational matrix functions are different in their tensor forms!!! In first case, it is projective. In second case, it is motive. In classic scalar forms of these angles, this difference is absent!*

3. *Compatible active rotational transformation of a reflector as a* 2*-valent tensor at an angle* Φ *is equivalent to its rotation as an* 1*-valent tensor at the angle* 2Φ *– see in (254).*

4. *So, due to (254),* $Ref\{\cos\bar{\Phi}_{12}\}^{\ominus} = Rot\ \Phi_{12}\cdot Ref\{A_1A_1'\} = Rot\ (-\Phi_{12})\cdot Ref\{A_2A_2'\}$. *is a mid-reflector (253) for the lineors* A_1 *and* A_2, *for their images and eigenorthoprojectors!*

Many other relations can be established with the binarity principle, but now we need these.

Further this Principle works great, as we use it for prime matrices associated with binary tensor angles between two objects. So, the elegant and useful formula immediately follows from (255):

$$\mathit{Rot}\ (\pm\Phi_{12}) \cdot \mathit{Ref}\{\cos\tilde{\Phi}_{12}\}^{\ominus} \cdot \mathit{Rot}\ (\pm\Phi_{12}) = \mathit{Ref}\{\cos\tilde{\Phi}_{12}\}^{\ominus} = \mathrm{Const}.$$

Of course, this relation works for a wider set of rotations than $\pm\Phi_{12}$. But only some rotations Φ pass with a change of angle's sign through the 2×2-cells of this mid-reflector from left and right. Such admissible rotations $\mathit{Rot}\ \Phi$ act *between* two Euclidean eigensubspaces of $\langle\mathcal{E}^n\rangle$ corresponding to eigenvalues $+1$ and -1 of this specific mid-reflector $\mathit{Ref}\{\cos\tilde{\Phi}_{12}\}^{\ominus}$. The remaining admissible rotations $\langle\mathit{Rot}\ \Theta\rangle$ from the complete set of admissible rotations with respect to this mid-reflector act only *within* these two Euclidean eigensubspaces. This complete set of rotations forms the group with respect to it and the conditions. In such broader sense, we can represent the complete formula:

$$\mathit{Rot}\ \Phi \cdot \mathit{Rot}'\ \Theta \cdot \mathit{Ref}\{\cos\tilde{\Phi}_{12}\}^{\ominus} \cdot \mathit{Rot}\ \Theta \cdot \mathit{Rot}\ \Phi = \mathit{Ref}\{\cos\tilde{\Phi}_{12}\}^{\ominus}.$$

We have obtained the type of formulae which act in Euclidean tensor trigonometry with the given tensor angles $\tilde{\Phi}_{12}$ of the projective kind and their mid-reflectors $\mathit{Ref}\{\cos\tilde{\Phi}_{12}\}^{\ominus}$ with the index q.

Let's introduce into mathematical arsenal the new *binary quasi-Euclidean space* [15], which naturally complements the well-known *binary pseudo-Euclidean space* [65] as Sputnik. The main idea is to extend the "mid-reflector", obtained in (253, 254), to the entire Euclidean space. Then such particular mid-reflector transforms into the *fundamental reflector tensor* of this *quasi-Euclidean space* of the index q and with an Euclidean metric! Define in our work this index q of this binary space as the quantity of eigenvalues -1 of its reflector tensor. As a consequence, the initial Euclidean space splits into two Euclidean parts that are spherically orthogonal to each other. Further Φ_{12} is a spherical angle of the *principal* rotations *between* two Euclidean parts, Θ_{12} is an *orthospherical angle* of the *secondary* rotations *inside* these Euclidean parts. Principal rotations often are called *boost*. $\langle\mathit{Rot}\ \Theta\rangle$ forms subgroup of the *quasi-Euclidean rotations group*. Transferring through the reflector tensor $\mathit{Ref}\{\cos\tilde{\Phi}\}^{\ominus}$, the principal rotation changes its angle sign annihilating; the secondary rotation is transferring through both unity parts of the reflector tensor without changes annihilating too! In quasi-Euclidean and non-Euclidean Geometries of spherical type, the principal angles Φ_{12} play a motive role, the orthospherical angles Θ_{12} give as a rule the rotations of bases or objects.

In the *motive version*, the compatible spherical rotations of two types satisfy relations:

$$\left.\begin{array}{l}\mathit{Rot}\ (\pm\Phi_{12}) \cdot \mathit{Ref}\{\cos\tilde{\Phi}\}^{\ominus} \cdot \mathit{Rot}\ (\pm\Phi_{12}) = \mathit{Ref}\{\cos\tilde{\Phi}\}^{\ominus}, \\ \mathit{Rot}'\ (\pm\Theta_{12}) \cdot \mathit{Ref}\{\cos\tilde{\Phi}\}^{\ominus} \cdot \mathit{Rot}\ (\pm\Theta_{12}) = \mathit{Ref}\{\cos\tilde{\Phi}\}^{\ominus}.\end{array}\right\} \qquad (257)$$

For the *projective version*, we use formula (256) with two mutual orthogonal reflectors. With its angular analogues in (175, 176) and adding to them one orthospherical reflector, we obtain the compatible spherical reflections of two types, which all satisfy relations:

$$\left.\begin{array}{l}\mathit{Ref}_{\boxplus}\{\mp\tilde{\Phi}_{12}\} \cdot \mathit{Ref}\{\cos\tilde{\Phi}\}^{\ominus} \cdot \mathit{Ref}_{\boxplus}\{\pm\tilde{\Phi}_{12}\} = \mathit{Ref}\{\cos\tilde{\Phi}\}^{\ominus}, \\ \mathit{Ref}_{\boxplus}\{\pm\tilde{\Theta}_{12}\} \cdot \mathit{Ref}\{\cos\tilde{\Phi}\}^{\ominus} \cdot \mathit{Ref}_{\boxplus}\{\pm\tilde{\Theta}_{12}\} = \mathit{Ref}\{\cos\tilde{\Phi}\}^{\ominus}.\end{array}\right\} \qquad (258)$$

And as simply follows, transferring through the reflector tensor $\mathit{Ref}\{\cos\tilde{\Phi}\}^{\ominus}$, the principal reflector is transformed into its mutual one annihilating; the secondary reflector is transferring through both unity parts of the reflector tensor without changes and annihilating too!

By (257), (258) we defined the new *Special group of transformations* of the also new *quasi-Euclidean space* $\langle\mathcal{Q}^{n+q}\rangle$ with admissible rotations and reflectors of two kinds! They complete the Lorentzian group of the pseudo-Euclidean space $\langle\mathcal{P}^{n+q}\rangle$. See their common definitions in sect. 6.3! The fundamental reflector tensor determines spherical tensor trigonometry of this space with Euclidean metric, internal and external multiplications as in sect. 5.4. Besides, in this space of the index $q=1$, relations (257) and (258) define the embedded Special *oriented hyperspheroid* of the constant positive radius R with its *external and internal non-Euclidean geometry of spherical type*. And we'll continue this new topic in Chs. 6, 6A, 8A and 10A with complete tensor and differential trigonometric descriptions of the internal motions and the equivalent external rotations in them with the laws of their summation.

5.8 Motive-type tensor sine, cosine, secant and tangent

The paired rotational matrices R and R' ($detR = +1$) – see (249), (250) consist of the *commutative* tensor sine and cosine of a *motive tensor angle* Φ_{12} or Φ_B with their paired *binary eigen angles – primary and mutual* $\pm i\varphi_k$ in their *eigen trigonometric planes*:

$$Rot\ \Phi = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \cos\varphi_j & -\sin\varphi_j & \\ & +\sin\varphi_j & \cos\varphi_j & \\ \hline & & & \ddots \end{array}\right], \quad Rot\ (-\Phi) = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \cos\varphi_j & +\sin\varphi_j & \\ & -\sin\varphi_j & \cos\varphi_j & \\ \hline & & & \ddots \end{array}\right], \tag{259}$$

$$\cos\Phi = \cos'\Phi = (Rot\ \Phi + Rot'\ \Phi)/2 = [Rot\ (+\Phi) + Rot\ (-\Phi)]/2, \tag{260}$$

$$i\sin\Phi = -(i\sin\Phi)' = (Rot\ \Phi - Rot'\ \Phi)/2 = [Rot\ (+\Phi) - Rot\ (-\Phi)]/2. \tag{261}$$

The *realificated motive sine* is a *real valued skewsymmetric matrix* with the eigenvalues $\mu_j = \pm i\sin\varphi_j$, but $\sin\Phi$ is a *true motive sine* – see below in (267). The motive secant and tangent will be define through $Def\ \Phi$ in sect. 5.10. Here we define them preliminary as:

$$\sec\ \Phi = \cos^{-1}\Phi = \sec'\ \Phi. \tag{262}$$

$$\tan\ \Phi = \sec\ \Phi\cdot\sin\ \Phi = \sin\ \Phi\cdot\sec\ \Phi = \tan'\Phi. \tag{263}$$

$$Rot\ \Phi\cdot Rot(-\Phi) = \sin^2\Phi + \cos^2\Phi = I = \cos^2\Xi + \sin^2\Xi\ \ \text{(Ptolemy Invariant)}. \tag{264}$$

$$Def\ \Phi\cdot Def(-\Phi) = \sec^2\Phi - \tan^2\Phi = I = \mathrm{cosec}^2\,\Xi - \cot^2\Xi\ \ \text{(Tensor quasi-Invariant)}. \tag{265}$$

$$\left.\begin{array}{l} \sin\Phi\cdot\cos\Phi = \cos\Phi\cdot\sin\Phi = \sin\tilde{\Phi}\cdot\cos\tilde{\Phi} = -\cos\tilde{\Phi}\cdot\sin\tilde{\Phi}, \\ \sec\Phi\cdot\tan\Phi = \tan\Phi\cdot\sec\Phi = \tan\tilde{\Phi}\cdot\sec\tilde{\Phi} = -\sec\tilde{\Phi}\cdot\tan\tilde{\Phi}. \end{array}\right\} \tag{266}$$

For the cosine and sine of a motive rotation angle in $A_2 = Rot\ \Phi\cdot A_1$, obviously, we have $\langle\mathcal{P}_4\rangle = \mathbf{0}$, but $dim\langle\mathcal{P}_3\rangle$ (as *unity block* $+I$, see in sect. 5.7) depends on the relation between n and $rankA$. The dimension is either $(n-2r)$, or $(2r-n)$, or $\langle\mathcal{P}_3\rangle = \mathbf{0}$.

Fix an original Cartesian base $\tilde{E}$ in $\langle\mathcal{E}^n\rangle$. The canonical W-forms for a real orthogonal matrix of the rotation at the motive tensor angle Φ (or Θ) and for its cosine and sine in the trigonometric base of diagonal cosine $\tilde{E}_1 = R_W\cdot\tilde{E} = \{I\}$ are following (if $2r < n$):

$$Rot\ \Phi = \cos\Phi + i\sin\Phi = \exp(i\Phi) = Rot'\ (-\Phi) = Rot^{-1}(-\Phi) =$$

$$= \underbrace{\begin{bmatrix} \ddots & & & \\ & \cos\varphi_j & 0 & \\ & 0 & \cos\varphi_j & \\ & & & \ddots \\ & & & & \boxed{+1} \end{bmatrix}}_{\cos\Phi} + i\underbrace{\begin{bmatrix} \ddots & & & \\ & 0 & +i\sin\varphi_j & \\ & -i\sin\varphi_j & 0 & \\ & & & \ddots \\ & & & & \boxed{0} \end{bmatrix}}_{\sin\Phi}.$$

The tensor sine and angle eigenvalues $\mathbf{0}$ correspond to the subspace $\langle\mathcal{P}_3\rangle$.

Rule 4. After an change in $Rot\ \Phi$ of the principal angle Φ by its complement $\Xi = \Pi/2 - \Phi$ (or by compatible modal rotation $Rot\ \Pi/4$ of $Rot\ \Phi$), the new sine-cosine function $\overline{Rot}\ \Phi$ gives a rotation at Ξ. (The analogous property relates to orthogonal eigenreflectors too.

$$Rot\ \Xi = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \cos\xi_k & -\sin\xi_k & \\ & +\sin\xi_k & \cos\xi_k & \\ \hline & & & \ddots \end{array}\right] = \overline{Rot}\ \Phi = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \sin\varphi_k & -\cos\varphi_k & \\ & +\cos\varphi_k & \sin\varphi_k & \\ \hline & & & \ddots \end{array}\right]. \tag{267}$$

Describe the canonical forms of a motive angle and its motive functions. At first, we use the complex-valued base of the sine and angle D-form, then return to the original *real-valued* trigonometric base of the diagonal cosine. Such identical modal transformation gives the canonical form of a motive angle $\mathbf{\Phi}$ in the trigonometric base:

$$\overset{\bar{E}_D}{\Rightarrow} \cos \overset{D(\Phi)}{\begin{bmatrix} \ddots & & & & \\ & +\varphi_j & 0 & & \\ & 0 & -\varphi_j & & \\ & & & \ddots & \\ & & & & \boxed{0} \end{bmatrix}} + i \sin \overset{D(\Phi)}{\begin{bmatrix} \ddots & & & & \\ & +\varphi_j & 0 & & \\ & 0 & -\varphi_j & & \\ & & & \ddots & \\ & & & & \boxed{0} \end{bmatrix}} \overset{\bar{E}_D^{-1}}{\Rightarrow}$$

$$\overset{\bar{E}_D^{-1}}{\Rightarrow} \cos \overset{\Phi}{\begin{bmatrix} \ddots & & & & \\ & 0 & +i\varphi_j & & \\ & -i\varphi_j & 0 & & \\ & & & \ddots & \\ & & & & \boxed{0} \end{bmatrix}} + i \sin \overset{\Phi}{\begin{bmatrix} \ddots & & & & \\ & 0 & +i\varphi_j & & \\ & -i\varphi_j & 0 & & \\ & & & \ddots & \\ & & & & \boxed{0} \end{bmatrix}} =$$

$$= \exp \overset{i\Phi}{\begin{bmatrix} \ddots & & & & \\ & 0 & -\varphi_j & & \\ & +\varphi_j & 0 & & \\ & & & \ddots & \\ & & & & \boxed{0} \end{bmatrix}} = \exp i \cdot \overset{\Phi}{\begin{bmatrix} \ddots & & & & \\ & 0 & +i\varphi_j & & \\ & -i\varphi_j & 0 & & \\ & & & \ddots & \\ & & & & \boxed{0} \end{bmatrix}}.$$

The formulae for motive angles follow in addition to (164), (170) for projective ones (with B and B' according to (213), (214) or as independent $n \times n$-lineors of rank r):

$$\mathbf{\Phi}_{12} = -(\mathbf{\Phi}_{12})' = -\mathbf{\Phi}_{21}, \qquad \mathbf{\Phi}_B = -(\mathbf{\Phi}_B)' = -\mathbf{\Phi}_{B'}. \tag{268}$$

Compare them with corresponding formulae for the projective type angles (164) and (170)!

Accordingly, for motive cosine and sine from (264), we obtain such simplest forms in the same trigonometric bases:

$$\cos \overset{\Phi}{\begin{bmatrix} \ddots & & & \\ & 0 & +i\varphi_j & \\ & -i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}} = \cos \overset{D(\Phi)}{\begin{bmatrix} \ddots & & & \\ & +\varphi_j & 0 & \\ & 0 & -\varphi_j & \\ & & & \ddots \end{bmatrix}} = \overset{\cos\Phi}{\begin{bmatrix} \ddots & & & \\ & \cos\varphi_j & 0 & \\ & 0 & \cos\varphi_j & \\ & & & \ddots \end{bmatrix}},$$

$$i \sin \overset{\Phi}{\begin{bmatrix} \ddots & & & \\ & 0 & +i\varphi_j & \\ & -i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}} = \overset{i\sin\Phi}{\begin{bmatrix} \ddots & & & \\ & 0 & -\sin\varphi_j & \\ & +\sin\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}}.$$

5.9 Relations between projective and motive angles and functions

From (236)–(239), and also (277), (278) – see below, we obtain (*in common bases*):

$$Ref\{\cos\tilde{\Phi}\}^{\ominus}\cdot(i\tilde{\Phi}) = \Phi = (-i\tilde{\Phi})\cdot Ref\{\cos\tilde{\Phi}\}^{\ominus}, \quad \tilde{\Phi}^2 = \Phi^2, \tag{269}$$

$$Ref\{\cos\tilde{\Phi}\}^{\ominus}\cdot\begin{Bmatrix} +\cos\tilde{\Phi} \\ -\sin\tilde{\Phi} \\ +\sec\ \tilde{\Phi} \\ -i\tan\tilde{\Phi}\end{Bmatrix} = \begin{Bmatrix} +\cos\Phi \\ +i\sin\Phi \\ +\sec\ \Phi \\ +\tan\Phi\end{Bmatrix} = \begin{Bmatrix} +\cos\tilde{\Phi} \\ +\sin\tilde{\Phi} \\ +\sec\ \tilde{\Phi} \\ +i\tan\tilde{\Phi}\end{Bmatrix}\cdot Ref\{\cos\tilde{\Phi}\}^{\ominus}. \tag{270}$$

Rule 1 – *in generalized form.* (see in sect. 5.3) *Square and any even degrees of all mutual tensor trigonometric functions and angles are equal and commute with any other ones.* (Here it is $\langle\mathcal{P}_4\rangle = \mathbf{0}$, $r_1 = r_2$.)

In the real Cartesian bases $\tilde{E}$, $\tilde{\Phi}$ and $i\Phi$ are real symmetric and antisymmetric bivalent tensors. Find the *complex local trigonometric base* $\tilde{E}_0$ for installing complex *pseudohyperbolic* analogues $\{i\varphi\}_c$ of real spherical angles $\{\varphi\}_r$ as the diagonal square root of reflector-tensor (254), gotten by the modal transformation of $E_1 = \{I\}$ into $E_0 = R_c\cdot E_1 = R_c\cdot\{I\} = \{R_c\}$:

$$R_c = R'_W\cdot\sqrt{Ref\{\cos\tilde{\Phi}\}^{\ominus}}\cdot R_W = (\sqrt{I^{\pm}})_D = \begin{bmatrix}\ddots & & \\ & \begin{matrix}1 & 0\\ 0 & i\end{matrix} & \\ & & \ddots\end{bmatrix}, \ (\tilde{E}_0 = R_c\cdot\tilde{E}_1). \tag{271}$$

Recall (see sect. 5.5, 5.7), that $\tilde{E}_1 = R_W\cdot\tilde{E} = \{I\}$ is the *real local trigonometric base* for W-forms, $\tilde{E}$ is the original Cartesian base. The complex local base $\tilde{E}_0$, unlike the base $\tilde{E}_1$, has imaginary *ordinate* axes, what correspond to the algebraically negative projective cosine eigenvalues ($u_j\to u_j, v_j\to iv_j$). All the real spherical notions are translated into $\tilde{E}_0 = \{R_c\}$ as the pseudohyperbolic ones by modal transformation R_c. Below this is exposed clearly in (272) only for the projective and motive tensor angles. Substituting a base $\tilde{E}_1$ for $\tilde{E}_0$ does not change the cosines and the secants; the angles, their sines and tangents are transformed into the pseudohyperbolic analogues. Further we use indexes "r" and "c" for notions in the real and complex bases $\tilde{E}_1$, $\tilde{E}_0$.

$$R_c^{-1}\cdot\{\tilde{\Phi}\}_r\cdot R_c = \{\tilde{\Phi}\}_c \equiv \{\Phi\}_r \equiv i\{-i\tilde{\Phi}\}_c \Leftrightarrow R_c^{-1}\cdot\{i\Phi\}_r\cdot R_c = -i\{\Phi\}_c \equiv \{-i\tilde{\Phi}\}_r \equiv \{-i\Phi\}_c, \tag{272}$$

$$\{\sin\tilde{\Phi}\}_c \equiv \{\sin\Phi\}_r \equiv \{i\sinh(-i\tilde{\Phi}\}_c \Leftrightarrow \{i\sin\Phi\}_c \equiv i\{\sin\tilde{\Phi}\}_r \equiv \{\sinh(i\Phi)\}_c, \tag{273 – 274}$$

$$\{i\tan\tilde{\Phi}\}_c \equiv i\{\tan\Phi\}_r \equiv \{\tanh(i\tilde{\Phi})\}_c \Leftrightarrow \{\tan\Phi\}_c \equiv \{\tan\tilde{\Phi}\}_r \equiv \{i\tanh(-i\Phi)\}_c. \tag{275 – 276}$$

$$\{\tilde{\Phi}\}_r = \begin{bmatrix}\ddots & & \\ & \begin{matrix}0 & +\varphi_j\\ +\varphi_j & 0\end{matrix} & \\ & & \ddots\end{bmatrix} = \{\tilde{\Phi}\}'_r \to \{\tilde{\Phi}\}_c = \begin{bmatrix}\ddots & & \\ & \begin{matrix}0 & +i\varphi_j\\ -i\varphi_j & 0\end{matrix} & \\ & & \ddots\end{bmatrix} = \{\tilde{\Phi}\}^*_c, \tag{277}$$

$$\{i\Phi\}_r = \begin{bmatrix}\ddots & & \\ & \begin{matrix}0 & -\varphi_j\\ +\varphi_j & 0\end{matrix} & \\ & & \ddots\end{bmatrix} = -\{i\Phi\}'_r \to \{-i\Phi\}_c = \begin{bmatrix}\ddots & & \\ & \begin{matrix}0 & -i\varphi_j\\ -i\varphi_j & 0\end{matrix} & \\ & & \ddots\end{bmatrix} = \{i\Phi\}^*_c. \tag{278}$$

<u>Corollaries</u> $\quad -\{\tilde{\Phi}\}'_c = \{\tilde{\Phi}\}_c \equiv \{\Phi\}_r = -\{\Phi\}'_r \Leftrightarrow \{\Phi\}'_c = \{\Phi\}_c \equiv \{\tilde{\Phi}\}_r = \{\tilde{\Phi}\}'_r$!

Canonical forms of spherical and imaginary pseudohyperbolic trigonometric functions:

$$\{\cos(\pm\tilde{\Phi})\}_r = \begin{bmatrix} \ddots & & & \\ & +\cos\varphi_j & 0 & \\ & 0 & -\cos\varphi_j & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & +\cosh i\varphi_j & 0 & \\ & 0 & -\cosh i\varphi_j & \\ & & & \ddots \end{bmatrix} = \{\cosh(\pm i\tilde{\Phi})\}_c, \qquad (279)$$

$$\{\cos(\pm\Phi)\}_r = \begin{bmatrix} \ddots & & & \\ & +\cos\varphi_j & 0 & \\ & 0 & +\cos\varphi_j & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & +\cosh i\varphi_j & 0 & \\ & 0 & +\cosh i\varphi_j & \\ & & & \ddots \end{bmatrix} = \{\cosh(\pm i\Phi)\}_c, \qquad (280)$$

$$\{\sec(\pm\tilde{\Phi})\}_r = \begin{bmatrix} \ddots & & & \\ & +\sec\varphi_j & 0 & \\ & 0 & -\sec\varphi_j & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & +\operatorname{sech} i\varphi_j & 0 & \\ & 0 & -\operatorname{sech} i\varphi_j & \\ & & & \ddots \end{bmatrix} = \{\operatorname{sech}(\pm i\tilde{\Phi})\}_c, \qquad (281)$$

$$\{\sec(\pm\Phi)\}_r = \begin{bmatrix} \ddots & & & \\ & +\sec\varphi_j & 0 & \\ & 0 & +\sec\varphi_j & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & +\operatorname{sech} i\varphi_j & 0 & \\ & 0 & +\operatorname{sech} i\varphi_j & \\ & & & \ddots \end{bmatrix} = \{\operatorname{sech}(\pm i\Phi)\}_c, \qquad (282)$$

$$\{\sin\tilde{\Phi}\}_r = \begin{bmatrix} \ddots & & & \\ & 0 & +\sin\varphi_j & \\ & +\sin\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & 0 & +\sinh i\varphi_j & \\ & -\sinh i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix} = \{i\sinh(-i\tilde{\Phi})\}_c, \qquad (283)$$

$$\{i\sin\Phi\}_r = \begin{bmatrix} \ddots & & & \\ & 0 & -\sin\varphi_j & \\ & +\sin\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & 0 & -\sinh i\varphi_j & \\ & -\sinh i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix} = \{\sinh(-i\Phi)\}_c, \qquad (284)$$

$$\{i\tan\tilde{\Phi}\}_r = \begin{bmatrix} \ddots & & & \\ & 0 & -\tan\varphi_j & \\ & +\tan\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & 0 & -\tanh i\varphi_j & \\ & -\tanh i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix} = \{\tanh(-i\tilde{\Phi})\}_c, \qquad (285)$$

$$\{\tan\Phi\}_r = \begin{bmatrix} \ddots & & & \\ & 0 & +\tan\varphi_j & \\ & +\tan\varphi_j & 0 & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & 0 & +\tanh i\varphi_j & \\ & -\tanh i\varphi_j & 0 & \\ & & & \ddots \end{bmatrix} = \{i\tanh(-i\Phi)\}_c. \qquad (286)$$

In the next chapter, we shall use these complex canonical pseudohyperbolic forms for the clear introducing real-valued motive and projective hyperbolic tensor angles, trigonometric functions, and reflectors in real pseudo-Euclidean space. *Cosines and secants are real-valued notions, and therefore, they are invariants of R_c!*

With Moivre's and Euler's formulae for the rotational matrix and angles, we get:

$$Rot\{m\Phi\} = \cos\{m\Phi\} + i\sin\{m\Phi\} = Rot^m\Phi =$$

$$= \cosh\{i \cdot m\Phi\} + \sinh\{i \cdot m\Phi\} = \exp\{i \cdot m\Phi\} \to$$

$$\to i \cdot \{m\Phi\} = \ln\ Rot\{m\Phi\} \to i\Phi = \ln\ Rot\ \Phi \to \Phi = -i\ln\ Rot\ \Phi. \tag{287}$$

This give aa motive tensor angle from a rotation tensor! The bond $\Phi \leftrightarrow \tilde{\Phi}$ is in (272)! Canonical foms of rotational matrices are represented below also in the trigonometric base of the diagonal cosine with trigonometric 2×2-cells and only positive unit eigenvalues corresponding to the subspace $\langle \mathcal{P}_3 \rangle$ in the binary W-structures:

$$\{Rot^m\Phi\}_r = \begin{bmatrix} \ddots & & & & \\ & +\cos m\varphi_j & -\sin m\varphi_j & & \\ & +\sin m\varphi_j & +\cos \varphi_j & & \\ & & & \ddots & \\ & & & & \boxed{1} \\ & & & & & \ddots \end{bmatrix} =$$

$$= \exp \begin{bmatrix} \ddots & & & & \\ & 0 & -m\varphi_j & & \\ & +m\varphi_j & 0 & & \\ & & & \ddots & \\ & & & & \boxed{0} \\ & & & & & \ddots \end{bmatrix},$$

$$\{Rot^m\Phi\}_c = \begin{bmatrix} \ddots & & & & \\ & +\cos m\varphi_j & -i\sin m\varphi_j & & \\ & -i\sin m\varphi_j & +\cos \varphi_j & & \\ & & & \ddots & \\ & & & & \boxed{1} \\ & & & & & \ddots \end{bmatrix} =$$

$$= \exp \begin{bmatrix} \ddots & & & & \\ & 0 & -im\varphi_j & & \\ & -im\varphi_j & 0 & & \\ & & & \ddots & \\ & & & & \boxed{0} \\ & & & & & \ddots \end{bmatrix}.$$

The value $m = 1/2$ gives trigonometric square root (251) of the rotational matrix.

5.10 Deformational functions of motive tensor spherical angles

Similarly (249) and due to the principle of binarity, consequent multiplication of two *oblique* eigenreflectors for a pair of equirank lineors (planars) as in (211)–(214) determines tangent-secant motive transformation – the *spherical deformational matrix* function of some its tensor angle, as example, for planars $\langle im\ B\rangle$, $\langle im\ B'\rangle$ ($\langle ker\ B\rangle$, $\langle ker\ B'\rangle$):

$$\overset{Ref\{B'\}}{\begin{bmatrix} \ddots & & \\ & \begin{matrix} +\sec\ \varphi_j & -\tan\varphi_j \\ +\tan\varphi_j & -\sec\ \varphi_j \end{matrix} & \\ & & \ddots \end{bmatrix}} \overset{Ref\{B\}}{\begin{bmatrix} \ddots & & \\ & \begin{matrix} +\sec\ \varphi_j & +\tan\varphi_j \\ -\tan\varphi_j & -\sec\ \varphi_j \end{matrix} & \\ & & \ddots \end{bmatrix}} = \overset{Def\ \alpha_B}{\begin{bmatrix} \ddots & & \\ & \begin{matrix} +\sec\ \alpha_j & +\tan\alpha_j \\ +\tan\alpha_j & +\sec\ \alpha_j \end{matrix} & \\ & & \ddots \end{bmatrix}}.$$

(i. e., $\alpha_B \neq 2\Phi_B$). Note, that α_B and Φ in $Def\ \Phi$ are principal spherical motive angles. (See its exact calculation in sect. 6.2.)

<u>**Rule 5**</u>. *Deformational matrix functions* $Def\ \Phi_{12}$ *and* $Def\ \Phi_{34}$ *only as the trigonometrically compatible are commutative, but their angles-arguments do not form an algebraic sum.*

They, as function $Def\ (\pm\Phi)$, perform the trigonometric deformation at motive angle $\pm\Phi$:

$$Ref\{B'\}\cdot Ref\{B\} = (\sec\ \tilde{\Phi}_B + i\tan\tilde{\Phi}_B)\cdot(\sec\ \tilde{\Phi}_B - i\tan\tilde{\Phi}_B) =$$

$$= \sec^2\tilde{\Phi}_B + \tan^2\tilde{\Phi}_B + 2i\tan\tilde{\Phi}_B\cdot\sec\ \tilde{\Phi}_B = \sec^2\Phi_B + \tan^2\Phi_B + 2\tan\ \Phi_B\cdot\sec\ \Phi_B =$$

$$= sec\ \alpha_B + \tan\alpha_B = Def\ \alpha_B = Def'\alpha_B, \tag{288}$$

$$Ref\{B\}\cdot Ref\{B'\} = sec\ \alpha_B - \tan\alpha_B = Def^{-1}\alpha_B = Def(-\alpha_B). \tag{289}$$

Notation Def is used for the deformational functions of a motive-type spherical tensor angle. For them **Rule 2** (sect. 5.7) does not work entirely, but **Rule 4** (sect. 5.8) works entirely at one-step motion. $Def\ \Phi_{12}$ is based on motive tangent and secant. This binary tensor trigonometric deformation is executed also in the trigonometric subspace (Figure 2) with respect to its complete spherically orthogonal complement in $\langle\mathcal{E}^n\rangle$. Hence, for it the *principle of binarity* works, which can represents its matrix in W-form as above. The tensor secant and tangent were introduced preliminary in sect. 5.8. Now we may give their quite natural definitions in terms of the spherical deformational matrix similarly to (260) and (261):

$$\sec\ \Phi = (Def\ \Phi + Def^{-1}\Phi)/2 = [Def\ \Phi + Def(-\Phi)]/2, \tag{290}$$

$$\tan\Phi = (Def\ \Phi - Def^{-1}\Phi)/2 = [Def\ \Phi - Def(-\Phi)]/2. \tag{291}$$

The tensor cosecant and cotangent are the inverse or quasi-inverse sine and tangent.

Canonical forms of deformational matrices are represented also in the same trigonometric base of the diagonal cosine ($-\pi/2 < \varphi_j < \pi/2$) with 2×2-cells and positive block $+I$ corresponding to the subspace $\langle\mathcal{P}_3\rangle$ in the binary W-structure:

$$Def\ \Phi = Def'\Phi = \sec\ \Phi + \tan\Phi = \begin{bmatrix} \ddots & & & \\ & \begin{matrix} \sec\ \varphi_j & \tan\varphi_j \\ \tan\varphi_j & \sec\ \varphi_j \end{matrix} & & \\ & & \ddots & \\ & & & \boxed{1} \\ & & & & \ddots \end{bmatrix}, \tag{292}$$

$$Def^{-1}\Phi = Def(-\Phi) = \sec\ \Phi - \tan\Phi. \tag{293}$$

The canonical forms for the rotational and deformational matrix functions of a pseudo-hyperbolic angle $i\Phi$ in the complex-valued trigonometric base of the diagonal cosine are realized with the use of formulae (277)–(286). They are following

$$\{Rot\ \Phi\}_c = \cos\Phi + \{-i\sin\Phi\}_c = \begin{bmatrix} \ddots & & & & & \\ & \cosh(-i\varphi)_j & +\sinh(-i\varphi)_j & & & \\ & +\sinh(-i\varphi)_j & \cosh(-i\varphi)_j & & & \\ & & & \ddots & & \\ & & & & \boxed{1} & \\ & & & & & \ddots \end{bmatrix} =$$

$$= \cosh(-i\Phi)_c + \sinh(-i\Phi)_c, \tag{294}$$

$$\{Rot\ \Phi\}_c^{-1} = \{Rot(-\Phi)\}_c = \cos\Phi + \{i\sin\Phi\}_c = \cosh(-i\Phi)_c - \sinh(-i\Phi)_c; \tag{295}$$

$$\{Def\ \Phi\}_c = \sec\ \Phi + \{\tan\Phi\}_c = \begin{bmatrix} \ddots & & & & & \\ & \mathrm{sech}(-i\varphi)_j & -\tanh(-i\varphi)j & & & \\ & +\tanh(-i\varphi)_j & \mathrm{sech}(-i\varphi)_j & & & \\ & & & \ddots & & \\ & & & & \boxed{1} & \\ & & & & & \ddots \end{bmatrix} =$$

$$= \mathrm{sech}(-i\Phi)_c + \tanh(-i\Phi)_c, \tag{296}$$

$$\{Def\ \Phi\}_c^{-1} = \{Def(-\Phi)\}_c = \sec\ \Phi - \{\tan\ \Phi\}_c = \mathrm{sech}(-i\Phi)_c - \tanh(-i\Phi)_c. \tag{297}$$

For the rotational and deformational matrices, their determinants as well as determinants of their binary cells are equal to 1. *Spherical* deformational matrices are symmetric and positive definite. Rotation of their 2×2-cells (on the trigonometric eigenplanes) at angles $\varphi_j = \pm\pi/4$ transforms these cells into diagonal ones. The eigenvalues of the deformational matrix cells are pairs $\mu_{2j} = \sec\ \varphi_j + \tan\varphi_j > 0$, $\mu_{2j+1} = \sec\ \varphi_j - \tan\varphi_j = \mu_{2j}^{-1} > 0$, and here not necessity in the values $\mu_k = +1$ inside the matrix unity block $\langle\mathcal{P}_3\rangle$.

In order to clarify the sense of the binary deformation function, we represent it in the new base spherically rotated by modal transformation in each cells of W-form at angles $\varphi_j = \pm\pi/4$. For example, expose the rotation of some cell at angles $\varphi_j = +\pi/4$.

Namely, on the level of the binary trigonometric cells, we have:

$$\begin{bmatrix} \sec\ \varphi_j & \tan\varphi_j \\ \tan\varphi_j & \sec\ \varphi_j \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix} \begin{bmatrix} \sec\ \varphi_j + \tan\varphi_j & 0 \\ 0 & \sec\varphi_j - \tan\varphi_j \end{bmatrix} \begin{bmatrix} \cos\pi/4 & \sin\pi/4 \\ -\sin\pi/4 & \cos\pi/4 \end{bmatrix}.$$

Now it is seen that the modal spherical deformation in canonical form (292) on the trigonometric eigenplane consists *in Euclidean metric* in:

– extension of the base along the principal diagonal by coefficient $\mu = \sec\ \varphi + \tan\varphi > 0$,
– contraction of the base along the secondary diagonal by coefficient $\mu^{-1} = \sec\ \varphi - \tan\varphi > 0$.

Similarly to the real-valued binary structure (149) for a complex number, the following binary unique representation of an arbitrary real *positive* number by 2×2 deformational matrix in terms of a spherical angle $(-\pi/2 < \varphi < \pi/2)$ holds with two eigenvalue:

$$\mu = \sec\ \varphi + \tan\varphi > 0,\ \mu^{-1} = \sec\ \varphi - \tan\varphi > 0. \tag{298}$$

From here we have $\sec\ \varphi = (\mu + \mu^{-1})/2, \quad \tan\varphi = (\mu - \mu^{-1})/2.$ Numbers (298) are equivalent to analogous ones $\exp\ (+\gamma)$, $\exp\ (-\gamma)$, what will be clear in next chapter.

One more interpretation of a binary deformation is respected to so called *cross bases*. They may be used in relativistic STR-transformations of geometric objects in the Minkowski space-time. Consider two Cartesian bases $\tilde{E}_\imath$ and $\tilde{E}_\jmath$ and the rotational transformation $\tilde{E}_\imath = Rot(-\Phi_{\imath\jmath})\tilde{E}_\jmath$. Cartesian coordinates of a vector $\mathbf{a}$ in the two bases $\tilde{E}_\jmath$ and $\tilde{E}_\imath$ are related as at *passive modal transformations* by the angle $+\Phi_{\imath\jmath}$:

$$\mathbf{a}^{(\imath)} = Rot\ \Phi_{\imath\jmath}\mathbf{a}^{(\jmath)} =$$

$$= \begin{bmatrix} \ddots & & & \\ & \cos\varphi_t & -\sin\varphi_t & \\ & \sin\varphi_t & \cos\varphi_t & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x_1^{(\jmath)} \\ x_2^{(\jmath)} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \cos\varphi_t x_1^{(\jmath)} - \sin\varphi_t x_2^{(\jmath)} \\ \sin\varphi_t x_1^{(\jmath)} + \cos\varphi_t x_2^{(\jmath)} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ x_1^{(\imath)} \\ x_2^{(\imath)} \\ \vdots \end{bmatrix}.$$

In 2×2-cells, the base $\tilde{E}_\imath$ is the result of rotating $\tilde{E}_\jmath$ at the clockwise angles $+\varphi_t$. Introduce *cross bases* $\tilde{E}_{\imath,\jmath}$ with mixed axes $\langle x_1^{(\imath)}, x_2^{(\jmath)}\rangle$ and $\tilde{E}_{\jmath,\imath}$ with mixed axes $\langle x_1^{(\jmath)}, x_2^{(\imath)}\rangle$. These both bases are related by the *cross modal transformation*:

$$\tilde{E}_{\imath,\jmath} = Def(-\Phi_{\imath\jmath})\tilde{E}_{\jmath,\imath}. \tag{299}$$

In t-th cells, so called *cross coordinates* of a vector $\mathbf{a}$ in the cross bases $\tilde{E}_{\imath,\jmath}$ and $\tilde{E}_{\jmath,\imath}$ are related as at *passive cross modal transformations* by the angle $+\Phi_{\imath\jmath}$:

$$\mathbf{a}^{(\imath,\jmath)} = Def(+\Phi_{\imath\jmath})\mathbf{a}^{(\jmath,\imath)} =$$

$$= \begin{bmatrix} \ddots & & & \\ & \sec\ \varphi_t & \tan\varphi_t & \\ & \tan\varphi_t & \sec\ \varphi_t & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x_1^{(\jmath,\imath)} \\ x_2^{(\imath,\jmath)} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ x_1^{(\imath,\jmath)} \\ x_2^{(\jmath,\imath)} \\ \vdots \end{bmatrix}. \tag{300}$$

Then the cross coordinates of vector $\mathbf{a}^{(\imath,\jmath)}$ are determined here by *cross projecting* with the use of deformational matrix-function of a principal motive angle compatible with a reflector tensor of the space (see sect. 5.7).

Most widely, the usefulness of introduced above tensor deformations is manifested in so-called universal bases, for example, $\tilde{E}_1$, where they produce their *quasi-invariants*, which are transformed into *one-step invariants* when the plane or space metric is changed, which at the general matrix level embodies the main specific spherical-hyperbolic analogy with its nomerous applications in geometries with quadratic metrics and STR. See about last in detail in next Ch. 6 and further.

5.11 Transformations of orthogonal and oblique eigenprojectors

In an Euclidean space, there exists an one-to-one correspondence between a centralized planar and a symmetric projector of the same rank. There exists an one-to-one correspondence between the planar and its orthocomplement too. Any planar may be transformed into each other one of the same rank with tensor rotation as well as with tensor mid-reflector (mid-reflector give only the single motive angle Φ!). Formulae for such transformations may be derived, for example, of (256), (226), (176) and (177) or with direct applying the principle of binarity.

$$\overleftrightarrow{A_2A_2'} = Rot\ \Phi_{12} \cdot \overleftrightarrow{A_1A_1'} \cdot Rot'\ \Phi_{12} = Ref\{\cos\tilde{\Phi}_{12}\}^{\ominus} \cdot \overleftrightarrow{A_1A_1'} \cdot Ref\{\cos\tilde{\Phi}_{12}\}^{\ominus}, \tag{301}$$

$$\overleftrightarrow{B'B} = Rot\ \Phi_B \cdot \overleftrightarrow{BB'} \cdot Rot'\ \Phi_B = Ref\{\cos\tilde{\Phi}_B\}^{\ominus} \cdot \overleftrightarrow{BB'} \cdot Ref\{\cos\tilde{\Phi}_B\}^{\ominus}. \tag{302}$$

These are rotation and reflection transformations of 2-valent orthogonal tensors inside of the symbolic octahedron (Figure 1). Use the octahedron for illustration. The diagonal PQ generates isosceles triangles PZQ and PIQ, where $\angle PZQ \equiv \angle PIQ \equiv \Phi_B$.

Moreover, in an Euclidean space, there exists a one-to-one correspondence due to (217)–(220) between a pair of *equirank* centralized planars $\langle im\ A_1\rangle$, $\langle im\ A_2\rangle$ (then we have $k(A_1A_2', r) = det(A_1'A_2) \neq 0$) and a pair of the oblique eigenprojectors $\overleftarrow{B}$, $\overleftarrow{B'}$ (because in definition (213) $B = A_1A_2'$). Then $\overleftarrow{B}$ and $\overleftarrow{B'}$ ($\overrightarrow{B}$ and $\overrightarrow{B'}$) are transformed into each other with tensor deformation as well as with tensor mid-reflector. Analogous formulae for such transformations (as formulae (301), (302)) may be derived too with the principle of binarity.

$$\overset{\rightleftarrows}{B'} = Def\ \Phi_B \cdot \overset{\rightleftarrows}{B} \cdot Def(-\Phi_B) = Ref\{\cos\tilde{\Phi}_B\}^{\ominus} \cdot \overset{\rightleftarrows}{B} \cdot Ref\{\cos\tilde{\Phi}_B\}^{\ominus}. \tag{303}$$

(These non-similarity and similarity with 1st and 2nd parts of (302) are quite logical.)

Following formulae are similar to rotational prototypes (255) and (256):

$$Ref\{B'\} = Def\ \Phi_B \cdot Ref\{B\} \cdot Def(-\Phi_B) =$$

$$= Ref\{\cos\tilde{\Phi}_B\}^{\ominus} \cdot Ref\{B\} \cdot Ref\{\cos\tilde{\Phi}_B\}^{\ominus}, \tag{304}$$

$$Ref\{\cos\tilde{\Phi}_B\}^{\ominus} = Def\ \Phi_B \cdot Ref\{B\} = Def(-\Phi_B) \cdot Ref\{B'\}. \tag{305}$$

If the original matrix B is null-prime, then, for example, from formulae (186)-(189) and relation $\cos\tilde{\Phi}_B \cdot \sec\ \tilde{\Phi}_B = I$ one may get the mutual modal transformations:

$$\left\{ \begin{array}{c} \overleftrightarrow{B'B} \\ \overset{\rightleftarrows}{B'} \end{array} \right\} = \cos\tilde{\Phi}_B \cdot \left\{ \begin{array}{c} \overleftrightarrow{BB'} \\ \overset{\rightleftarrows}{B} \end{array} \right\} \cdot \sec\ \tilde{\Phi}_B = \sec\ \tilde{\Phi}_B \cdot \left\{ \begin{array}{c} \overleftrightarrow{BB'} \\ \overset{\rightleftarrows}{B} \end{array} \right\} \cdot \cos\tilde{\Phi}_B. \tag{306}$$

The formulae may be checked with the use of the Table of multiplication for eigenprojectors (185) in sect. 5.2. too. Formulae indicated above represent the modal transformations found by different manners, but *the results are the same*. Express all the eigenprojectors in terms of corresponding projective trigonometric functions pairs according to (176)–(179):

$$\left. \begin{array}{l} \overleftarrow{A_1A_1'} = (I + \cos\tilde{\Phi} - \sin\tilde{\Phi})/2 = \overleftarrow{BB'}, \\ \overrightarrow{A_1A_1'} = (I - \cos\tilde{\Phi} + \sin\tilde{\Phi})/2 = \overrightarrow{BB'}, \\ \overleftarrow{A_2A_2'} = (I + \cos\tilde{\Phi} + \sin\tilde{\Phi})/2 = \overleftarrow{B'B}, \\ \overrightarrow{A_2A_2'} = (I - \cos\tilde{\Phi} - \sin\tilde{\Phi})/2 = \overrightarrow{B'B}, \end{array} \right\} \tag{307}$$

$$\left.\begin{array}{l} \overleftarrow{B} = (I + \sec\ \tilde{\Phi} - i\,\mathrm{tg}\,\tilde{\Phi})/2 = \overleftarrow{A_1 A_2'}, \\ \overrightarrow{B} = (I - \sec\ \tilde{\Phi} + i\,\mathrm{tg}\,\tilde{\Phi})/2 = \overrightarrow{A_1 A_2'}, \\ \overleftarrow{B'} = (I + \sec\ \tilde{\Phi} + i\,\mathrm{tg}\,\tilde{\Phi})/2 = \overleftarrow{A_2 A_1'}, \\ \overrightarrow{B'} = (I - \sec\ \tilde{\Phi} - i\,\mathrm{tg}\,\tilde{\Phi})/2 = \overrightarrow{A_2 A_1'}. \end{array}\right\} \qquad (308)$$

These expressions show that *the principle of binarity is valid for projectors too.* There exists a bijection between the complete set of eigen orthoprojectors and the complete set of symmetric idempotent matrices of the same size and rank. Iff the matrix B is null-prime, then $det \cos\tilde{\Phi} \neq 0$, and there exists a bijection between the pairs $\langle \overleftrightarrow{BB'}, \overleftrightarrow{B'B} \rangle$ and $\langle \overleftrightarrows{B}, \overleftrightarrows{B'} \rangle$.

Represent orthoprojectors in the trigonometric W-form according to (307). Principle of binarity enable one to evaluate the modal matrices for constructing D-forms. For example, consider this for orthoprojector $\overleftarrow{BB'}$. In i-cells of matrices, there holds:

$$\overset{Rot\ \Phi_B/2}{\begin{bmatrix} \ddots & & & \\ & \cos\varphi/2 & -\sin\varphi/2 & \\ & \sin\varphi/2 & \cos\varphi/2 & \\ & & & \ddots \end{bmatrix}} \frac{1}{2} \overset{\overleftarrow{BB'}}{\begin{bmatrix} \ddots & & & \\ & 1+\cos\varphi & -\sin\varphi & \\ & -\sin\varphi & 1-\cos\varphi & \\ & & & \ddots \end{bmatrix}} \overset{Rot'\Phi_B/2}{\begin{bmatrix} \ddots & & & \\ & \cos\varphi/2 & \sin\varphi/2 & \\ & -\sin\varphi/2 & \cos\varphi/2 & \\ & & & \ddots \end{bmatrix}} =$$

$$= \overset{D(\overleftarrow{BB'})}{\begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & 0 & \\ & & & \ddots \end{bmatrix}}, \text{ i. e., } V_{col}^{-1} \cdot \overleftrightarrow{BB'} \cdot V_{col} = D(\overleftrightarrow{BB'}).$$

This matrix is expressed in the original orthogonal base $\tilde{E}$ as $\overleftarrow{BB'}$, but in D-form they is expressed as above in the base:

$$\tilde{E}_D = V_{col} \cdot \tilde{E} = Rot\ (-\Phi_B/2) \cdot \tilde{E} = Rot\ (-\Phi_B/2) \cdot R_W' \tilde{E}_1 = \{Rot\ (-\Phi_B/2)\}, \qquad (309)$$

(here $R_W \cdot \tilde{E} = \tilde{E}_1 = \{I\}$ is the base of W-forms).

The following orthogonal eigenvector-columns of the same modal matrix correspond to the subspaces $\langle im\ B \rangle$ and $\langle ker\ B \rangle$:

$$\mathbf{b}_{i_1} = R_W \cdot \begin{bmatrix} \vdots \\ 0 \\ +\cos\varphi_i/2 \\ -\sin\varphi_i/2 \\ 0 \\ \vdots \end{bmatrix}, \quad \mathbf{d}_{i_1} = R_W \cdot \begin{bmatrix} \vdots \\ 0 \\ +\sin\varphi_i/2 \\ +\cos\varphi_i/2 \\ 0 \\ \vdots \end{bmatrix}.$$

By analogy, ones find the modal matrix for getting the base for the eigen orthoprojector $\overleftarrow{B'B}$ diagonal form, i. e., for $D(\overleftrightarrow{B'B}) = V_{col}^{-1} \cdot \overleftrightarrow{B'B} \cdot V_{col}$. This base is

$$\tilde{E}_D = V_{col} \cdot \{\tilde{E}\} = Rot\ (+\Phi_B/2) \cdot R_W \{\tilde{E}\} = R_W \cdot \{R_W' \cdot Rot\ (+\Phi_B/2) \cdot R_W\}\{\tilde{E}\}. \qquad (310)$$

The following orthogonal eigenvector-columns of the other modal matrix, gotten by (310), correspond to the subspaces $\langle im\ B'\rangle$ and $\langle ker\ B'\rangle$:

$$\mathbf{b}_{t\prime\prime} = R_W \cdot \begin{bmatrix} \vdots \\ 0 \\ +\cos\varphi_t/2 \\ +\sin\varphi_t/2 \\ 0 \\ \vdots \end{bmatrix}, \quad \mathbf{d}_{t\prime\prime} = R_W \cdot \begin{bmatrix} \vdots \\ 0 \\ -\sin\varphi_t/2 \\ +\cos\varphi_t/2 \\ 0 \\ \vdots \end{bmatrix}.$$

The modal matrices for constructing D-forms of mutual oblique eigenprojectors will be derived in Chapter 6 with the use of spherical-hyperbolic analogy. Here, we present only preliminary two expressions in terms of arithmetic roots of the same deformational matrix, though they have no *spherical* trigonometric sense:

$$\{R'_W \cdot \sqrt{Def\ \Phi_B}\} \cdot \overset{\rightleftarrows}{B} \cdot \{\sqrt{Def\ (-\Phi_B)}\} \cdot R_W\} = D(\overset{\rightleftarrows}{B}), \tag{311}$$

$$\{R'_W \cdot \sqrt{Def\ (-\Phi_B)}\} \cdot \overset{\rightleftarrows}{B'} \cdot \{\sqrt{Def\ \Phi_B}\} \cdot R_W\} = D(\overset{\rightleftarrows}{B'}). \tag{312}$$

Thus we can see much common in various modal transformations of the mutual eigenprojectors and eigenreflectors from one into another with the trigonometric rotational and deformational modal matrices. This is usually obviously, when the operations are executed in their same bases of W-forms. What's more, in the middle of these modal transformations we have their diagonal forms. So, for mutual eigenreflectors, we obtain their mid-reflector.

5.12 Spherical tensors of rotation and deformation with frame axis

Consider the set of centralized principal spherical rotations in the oriented along frame axis $\overrightarrow{y}$ quasi-Euclidean space $\langle Q^{n+1}\rangle$ with simplest diagonal reflector tensor $\{I^{\pm}\}$ of index $q = 1$ (see about general $\langle Q^{n+q}\rangle$ in sect. 5.7). They are realized by special matrix functions $rot\ \Phi$. These matrices with the minimal trigonometric subspace for homogeneous motions-rotations in own quasi-Cartesian base have the unique trigonometric 2×2-cell with one eigen scalar motive principal angle φ and the unique rotation's frame axis. Such trigonometric matrix functions are called *elementary*. Initially we conserve counterclockwise angles φ.

$$\{rot(\pm\Phi)\}_{3\times 3} = \cos\Phi \pm i\sin\Phi = \cos(\pm\Phi) + i\sin(\pm\Phi)$$

$$\begin{array}{|c|c|c|} \hline 1-(1-\cos\varphi)\cos^2\alpha_1 & -(1-\cos\varphi)\cos\alpha_1\cos\alpha_2 & \mp\sin\varphi\cos\alpha_1 \\ \hline -(1-\cos\varphi)\cos\alpha_1\cos\alpha_2 & 1-(1-\cos\varphi)\cos^2\alpha_2 & \mp\sin\varphi\cos\alpha_2 \\ \hline \pm\sin\varphi\cos\alpha_1 & \pm\sin\varphi\cos\alpha_2 & \cos\varphi \\ \hline \end{array}, \tag{313}$$

$$\{rot(\pm\Phi)\}_{(n+1)\times(n+1)}$$

$$\begin{array}{|c|c|} \hline I_{n\times n} - (1-\cos\varphi)\cdot \mathbf{e}_\alpha\mathbf{e}'_\alpha & \mp\sin\varphi\cdot\mathbf{e}_\alpha \\ \hline \pm\sin\varphi\cdot\mathbf{e}'_\alpha & \cos\varphi \\ \hline \end{array}, \quad (\mathbf{e}_\alpha\mathbf{e}'_\alpha = \overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}). \tag{314}$$

The coordinates of the matrices are expressed with respect to right quasi-Cartesian bases $\tilde{E}_{1u}$ of their canonical E-*form* (313), (314). The *oriented* straight line $\langle x_{n+1}\rangle$ is the frame (polar) axis for the eigen rotation angle φ, the angle is positive for $rot\{+\Phi\}$, and has the directional cosines $\cos\alpha_k$, $k = 1, \ldots, n$ in the base $\langle x_1, \ldots, x_n\rangle$ – the frame axis orthocomplement.

In particular, matrix function $rot\ \Phi$ realizes the full set of principal motions on the Hyperspheroid embedded and oriented in $\langle \mathcal{Q}^{n+1}\rangle$ with reflector tensor $\{I^{\pm}\}$ and Euclidean metric. And they are identical to such rotations in this space! (See in Ch. 8A). In what follows, the option of the unique rotation's frame axis $\langle x_{3+1}\rangle$ will be very important for us also in the hyperbolic non-Euclidean geometries from the external point of view and in Theory of Relativity considered in the pseudo-Euclidean space $\langle \mathcal{P}^{3+1}\rangle$ – see in Appendix.

At first, prove formula (313). Find a rotational transformation of the complement base $\langle x_1, x_2\rangle$ into new same base $\langle x_1', x_2'\rangle$ such that the axis $\langle x_1'\rangle$, $\mathbf{e}_\alpha = (\cos\alpha_1, \cos\alpha_2)$ (where $\cos^2\alpha_1 + \cos^2\alpha_2 = 1$), and the frame axis $\langle x_3\rangle$ should be coplanar. This transformation is the spherical rotation matrix at a certain tensor angle β_{12}. If $n = 2$, then it has the scalar eigen angle α_1, and the rotational matrix demanded is

$$rot\ \beta_{12}$$

$\cos\alpha_1$	$-\sin\alpha_1$	0
$+\sin\alpha_1$	$\cos\alpha_1$	0
0	0	1

.

This matrix function executes the rotation on the plane $\langle x_1, x_2\rangle$ at the angle α_1.

Further, in this new 3-dimensional base $\tilde{E}$ we use the elementary principal rotational matrix function $rot\ \Phi$, but in the 2×2-cell corresponding to the plane $\langle x_1', x_3\rangle$, with following condition: if the frame axis is $\langle x_1'\rangle$, then the angle of rotation is counter-clockwise; if the frame axis is $\langle x_3\rangle$, then this angle is clockwise. So, the last form of this elementary spherical rotational matrix is

$$\{rot(\pm\Phi)\}$$

$\cos\varphi$	0	$-\sin\varphi$
0	1	0
$\sin\varphi$	0	$\cos\varphi$

. (315)

Then we transform the matrix in E-form applying the inverse base rotation

$$\{rot(\pm\Phi)\}_{3\times 3} = rot\ \beta_{12}\cdot\{rot(\pm\Phi)\}\cdot rot\ \beta_{12}\,'.$$

The result is rotational matrix function (313) with the frame axis $\langle x_3\rangle$ for the motive tensor angle Φ in 3-dimensional Cartesian base $\tilde{E}_{1u}$.

General formula (314) is inferred similarly. Now find a rotational transformation of $\langle x_1, \ldots, x_n\rangle$ into $\langle x_1', \ldots, x_n'\rangle$ such that the axis $\langle x_1'\rangle$, the directional cosines vector $\mathbf{e}_\alpha = \{\cos\alpha_k\}$ $(\sum_{k=1}^{n}\cos^2\alpha_k = 1)$, and the frame axis $\langle x_{n+1}\rangle$ should be coplanar. Use consequently tensor angles of the radius-vector rotation with their spherical coordinates: β_{12} in the plane $\langle x_1, x_2\rangle$, $\beta_{1'3}$ in the plane $\langle x_1', x_3\rangle$, ..., $\beta_{1''\ldots' n}$ in the plane $\langle x_1''\ldots', x_n\rangle$. Due to the trigonometric nature of the transformations, we have the following formulae:

$$\left.\begin{aligned}
\cos\beta_{12} &= \cos\alpha_1/\sqrt{\cos^2\alpha_1 + \cos^2\alpha_2},\\
\cos\beta_{1'3} &= \sqrt{\cos^2\alpha_1 + \cos^2\alpha_2}/\sqrt{\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3},\\
&\vdots\\
\cos\beta_{1''\ldots' n} &= \sqrt{\cos^2\alpha_1 + \cdots + \cos^2\alpha_{n-1}} = \sin\alpha_n.
\end{aligned}\right\}$$

The consequent rotations are executed with the matrices $rot\ \beta_{12}$, $rot\ \beta_{1'3}$, ... :

$$rot\ \beta_{12} \qquad\qquad rot\ \beta_{1'3}$$

$$\left[\begin{array}{cc|c} \cos\beta_{12} & -\sin\beta_{12} & Z \\ \sin\beta_{12} & \cos\beta_{12} & \\ \hline \multicolumn{2}{c|}{Z'} & I_{n-1} \end{array}\right], \quad \left[\begin{array}{ccc|c} \cos\beta_{1'3} & 0 & -\sin\beta_{1'3} & Z \\ 0 & 1 & 0 & \\ \sin\beta_{1'3} & 0 & \cos\beta_{1'3} & \\ \hline \multicolumn{3}{c|}{Z'} & I_{n-2} \end{array}\right], \ldots .$$

The result is the base of the simplest 2×2-cell form for the elementary rotation $\tilde{E} = rot\ \beta \cdot \tilde{E}_1$, where $rot\ \beta = rot\ \beta_{12} \cdot rot\ \beta_{1'3} \cdots rot\ \beta_{1''...'n}$. Then construct the 2-dimensional form for this elementary rotation in the hyperplane $\langle x_1', \ldots, x_n', x_{n+1}\rangle$ with respect to the base $\tilde{E}$:

$$\{rot\ \Phi\}$$

$$\left[\begin{array}{c|c|c} \cos\varphi & \mathbf{0}' & -\sin\varphi \\ \hline \mathbf{0} & I_{n-1} & \mathbf{0} \\ \hline \sin\varphi & \mathbf{0}' & \cos\varphi \end{array}\right]. \tag{316}$$

Further we transform the matrix in E-form applying the inverse base rotation

$$\{rot\ \Phi\}_{(n+1)\times(n+1)} = rot\ \beta \cdot \{rot\ \Phi\}_{can} \cdot rot'\ \beta.$$

The result is rotational tensor function (314) with the frame axis $\langle x_{n+1}\rangle$ for the motive tensor angle Φ in $(n+1)$-dimensional quasi-Cartesian base $\tilde{E}_{1u}$ of $\langle \mathcal{Q}^{n+1}\rangle$ *with naturally appearing (thanks to the reflector tensor) eigen binary angles – primary and mutual, as sign alternative* $\mp\varphi_i$ *or* $\pm\varphi_i$ *depending of the choose of counterclockwise or clockwise angle's type!*

In $\langle \mathcal{Q}^{n+1}\rangle \equiv \langle \mathcal{E}^n\rangle \boxplus \overrightarrow{y}$, **due to structures of (313), (314)**, the function $rot\ (+\Phi)$ rotates at adopted counterclockwise primary angle $+\varphi$ in the direction to the frame axis $\overrightarrow{y}$ from the Euclidean hyperplane $\langle \mathcal{E}^n\rangle$. The function $rot\ (-\Phi)$ rotates in contrary direction. These structures explain: why $rot\ (\pm\Phi)$ in realizes rotations at $\pm\Phi$ – similar to acting S-arm! We see that due to **Rule 4** with (267), after change in $rot\ \Phi$ of the principal angle Φ by its complement $\Xi = \Pi/2 - \Phi$, the new function $\overline{rot}\ \Phi$ gives the complementary rotation at Ξ:

$$rot\ \Xi \quad (\xi \in [+\pi/2; 0]) \qquad = \qquad \overline{rot}\ \Phi \quad (\varphi \in [0; +\pi/2])$$

$$\left\| \begin{array}{c|c|c} \dfrac{\cos\xi \cdot \overleftarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha}'}{+\sin\ \xi \cdot \mathbf{e}_\alpha'} & \dfrac{-\sin\ \xi \cdot \mathbf{e}_\alpha}{\cos\xi} & \begin{array}{c}\cdots\\ \cdots\end{array} \end{array} \right| \begin{array}{c|c} \dfrac{\sin\varphi \cdot \overleftarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha}'}{+\cos\varphi \cdot \mathbf{e}_\alpha'} & \dfrac{-\cos\varphi \cdot \mathbf{e}_\alpha}{\sin\varphi} \end{array} \right\|. \tag{317}$$

We will distinguish the two kinds of principal angles: counterclockwise φ, measured off the frame axis $\overrightarrow{y}$ as *cooriented* ones, and clockwise ξ, measured off the Euclidean hyperplane $\langle \mathcal{E}^n\rangle$ as *counteroriented* ones (both named here with respect to direction of $\overrightarrow{y}$). Obviously, φ and ξ are complementary iff $\varphi + \xi = \pi/2$ (for compatible tensor angles iff $\Phi + \Xi = \Pi/2$). The angles φ in $rot\ \Phi$ and in $\overline{rot}\ \Phi$, seeming as the same, are distinguished geometrically. In functions $rot\Phi$ and $\overline{rot}\ \Phi$ – both with their identical trigonometric quadratic invariants

$$rot\ \Phi \cdot rot(-\Phi) = \sin^2\Phi + \cos^2\Phi = I = \cos^2\Xi + \sin^2\Xi = rot\ \Xi \cdot rot(-\Xi), \tag{318}$$

such angles φ play roles in (317) either as adjacent to the frame axis $\overrightarrow{y}$ or as adjacent to the Euclidean hyperplane $\langle \mathcal{E}^n\rangle$. Hence, both variants of rotations are realized also in the counterclockwise direction from $\langle \mathcal{E}^2\rangle$ to $\overrightarrow{y}$, but at ξ. This will manifest itself much more important for the more complex hyperbolic rotations in Chs. 6, 8A and 10A. In quasi-Euclidean spaces, in particular, $\Pi/2$ is the angle between $\overrightarrow{y}$ and $\langle \mathcal{E}^n\rangle$. See further in Ch. 8A!

In addition, finally we consider briefly elementary spherical deformational matrix functions $\boldsymbol{def}\ \Phi$ (292), but also with frame axis, similar (213), (214). The deformational matrices with the minimal trigonometric subspace for homogeneous deformation of a vector, a straight line, and a hyperplane in an Cartesian base have too the unique trigonometric 2×2-cell. Notation $\boldsymbol{def}\ \Phi$ is used for them as the particular case of $\boldsymbol{Def}\ \Phi$. Elementary deformations have also one eigen scalar deformation angle φ and accordingly the same unique deformation frame axis. The more important variant, if the frame axis is $\langle x_{n+1}\rangle$ in $\langle Q^{n+1}\rangle$. Then the matrices in the special Cartesian base $\tilde{E}_{1u} = \{I\}$ have the canonical structure in E-form:

$$\{def(\pm\Phi)\}_{3\times 3} = \sec\ \Phi + \tan\Phi$$

$$\begin{array}{|c|c|c|} \hline 1 + (\sec\varphi - 1)\cos^2\alpha_1 & (\sec\varphi - 1)\cos\alpha_1\cos\alpha_2 & \pm\tan\varphi\cdot\cos\alpha_1 \\ \hline (\sec\varphi - 1)\cos\alpha_1\cos\alpha_2 & 1 + (\sec\varphi - 1)\cos^2\alpha_2 & \pm\tan\varphi\cdot\cos\alpha_2 \\ \hline \pm\tan\varphi\cdot\cos\alpha_1 & \pm\tan\varphi\cdot\cos\alpha_2 & \sec\varphi \\ \hline \end{array}\ , \qquad (319)$$

$$\{def(\pm\Phi)\}_{(n+1)\times(n+1)}$$

$$\begin{array}{|c|c|} \hline I_{n\times n} + (\sec\varphi - 1)\cdot\mathbf{e}_\alpha\mathbf{e}'_\alpha & \pm\tan\varphi\cdot\mathbf{e}_\alpha \\ \hline \pm\tan\varphi\cdot\mathbf{e}'_\alpha & \sec\varphi \\ \hline \end{array}\ , \quad (\mathbf{e}_\alpha\mathbf{e}'_\alpha = \overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha})\ . \qquad (320)$$

The coordinates of the deformational matrices are expressed, as usually, with respect to the right Cartesian base $\tilde{E}_{1u}$. The *oriented* straight line $\langle x_{n+1}\rangle$ is the frame axis for the angle φ of trigonometric deformation, this angle is positive for $\boldsymbol{def}\ {+\Phi}$ and has the directional cosines $\cos\alpha_k$, $k = 1, \ldots, n$, with respect to the base $\langle x_1, \ldots, x_n\rangle$ as of $\langle x_{n+1}\rangle$ orthocomplement. The canonical E-forms (319), (320) are inferred by similar way *with their quasi invariants*.

Note, that $\{I^\pm\} = -\{I^\mp\}$. Furthermore, in the quasi-Euclidean space $\langle Q^{n+1}\rangle$ $(q = 1)$ as initially *not axes-oriented* and having the complete quasi-Cartesian base $\tilde{E} = \{\mathbf{e}_k\} = R_W\tilde{E}_1$ with the selected last ordinate $\mathbf{e}_{n+1}$, its reflector tensor may be defined as follows:

$$\{\sqrt{I}\}_S = R'_W\{I^\pm\}R'_W = \overleftarrow{\mathbf{e_{n+1}e'_{n+1}}} - \overrightarrow{\mathbf{e_{n+1}e'_{n+1}}} = 2\cdot\mathbf{e_{n+1}e'_{n+1}} - I, \qquad (321)$$

where $\mathbf{e_{n+1}}$ is the frame axis $\langle x_{n+1}\rangle$ and simultaneously the *orthogonal reflector's mirror* – see (176), and therefore, here we have: $\{\sqrt{I}\}_S = ref\{\mathbf{e_{n+1}e'_{n+1}}\}$. (In the most general case, $n \times q$ quasi-orthogonal matrix Rq from (129) in Ch. 3 instead of $\mathbf{e_{n+1}}$ may be used.)

Thus this chapter represented complete fundamentals of the Tensor Trigonometry in its affine, Euclidean and quasi-Euclidean versions, which are realized and act in the same spaces. Latter two spaces have the quadratic Euclidean metric. In different quasi-Euclidean spaces, their reflector tensor may be given either by the simplest *sign-alternating unity form* $\{I^\pm\}$ $(q \le n)$ and $\{I^\mp\}$ $(n < q)$ or in the *general form* $\{\sqrt{I}\}_S = \{R_W I^\pm R'_W\}$. Further (!) number q is an index of any similar binary space as a quantity of negative eigenvalues, here -1, of a reflector tensor of a given binary space – see in Chs. 6, 10–12 and in Appendix. Above it is $q = 1$. In the quasi-Euclidean space, its reflector tensor generates the continuous group of own quasi-Euclidean rotations including the *set of principal spherical rotations* and the *subgroup of secondary orthospherical rotations*, and, in addition, the *set of own quasi-Euclidean reflections* including the *set of principal spherical orthogonal reflections* and the *set of secondary orthospherical orthogonal reflections*. This continuous group of the admissible own rotations together with the full set of the admissible own reflections form the complete group of quasi-Euclidean motions of the given quasi-Euclidean space. So, the reflector tensor of the index q with the quadratic Euclidean metric are main attributes of any quasi-Euclidean spaces. In particular, the n-dimensional Euclidean geometry, when $q = 0$, and the q-dimensional anti-Euclidean geometry, when $n = 0$, are two extreme cases of the general quasi-Euclidean geometry with unity $\{+I\}$ and antiunity $\{-I\}$ reflector tensors.

Chapter 6

Pseudo-Euclidean tensor and scalar trigonometry

6.1 Hyperbolic tensor angles, trigonometric functions, reflectors

Passive transformation R_c (271) of spherical angle Φ gives in (272) pseudohyperbolic angle $\{-i\Phi\}_c$ in *complex pseudo-Euclidean space* $\langle\mathcal{P}^{n+q}\rangle_c$ with its *Pseudo-hyperbolic trigonometry* and the space *reflector tensor* $\{I^{\pm}\}$. as in $\langle\mathcal{Q}^{n+q}\rangle$! The scalar product is always invariant:

$$\mathbf{x}'\mathbf{x} = (R_c\mathbf{z}_{01})'\cdot(R_c\mathbf{z}_{01}) = [(\sqrt{I^{\pm}})\mathbf{z}_{01}]'\cdot[(\sqrt{I^{\pm}})\mathbf{z}_{01}] = \mathbf{z}'_{01}\{\sqrt{I^{\pm}}\}^2\mathbf{z}_{01} = \mathbf{z}'_{01}\{I^{\pm}\}\mathbf{z}_{01},$$

where $\mathbf{z}_{01} = R_c^{-1}\cdot\mathbf{x}$ in $\tilde{E}_{01} = R_c\cdot\tilde{E}_1$, according to (271); $\tilde{E}_1 = \{I\}$. Thus in $\langle\mathcal{P}^{n+q}\rangle_c$ we have $\{I^{\pm}\} = R'_c\cdot R_c = R_c^2 = \{\sqrt{I^{\pm}}\}_D^2$ as the *metric tensor too.* With respect to the original base $\tilde{E}_1$ the latter may have the form $\{R_W\cdot I^{\pm}\cdot R'_W\} = \{\sqrt{I}\}_S$. Hence, in $\langle\mathcal{P}^{n+q}\rangle_c$ reflector tensor and metric tensor are equivalent! Importance of this complex pseudo-Euclidean space consists in simple transition off intermediate pseudoanalogues (277), (278) into hyperbolic. Thus, for motive tensor angles, this is tealized by two ways with generalization in (341):

$$\Phi \leftrightarrow -i\Phi \leftrightarrow \Gamma, \quad \varphi_j \leftrightarrow -i\varphi_j \leftrightarrow \gamma_j, \ (\mathbf{x}\ in\ \tilde{E}_1 \leftrightarrow \mathbf{z}_{01}\ in\ \tilde{E}_{01} \leftrightarrow \mathbf{u}\ in\ \tilde{E}_1), \tag{322}$$

$$\Gamma \leftrightarrow +i\Gamma \leftrightarrow \Phi, \quad \gamma_j \leftrightarrow i\gamma_j \leftrightarrow \varphi_j, \ (\mathbf{u}\ in\ \tilde{E}_1 \leftrightarrow \mathbf{z}_{02}\ in\ \tilde{E}_{02} \leftrightarrow \mathbf{x}\ in\ \tilde{E}_1). \tag{323}$$

This transition between imaginary and real angles is called *spherical–hyperbolic analogy of abstract type* with preserving binary spaces structure and reflector tensor. Applying abstract analogy (322) to relations (277)–(286), (294)–(297), one obtains hyperbolic analogs of angles, trigonometric functions and reflectors in a *real-valued binary pseudo-Euclidean space* $\langle\mathcal{P}^{n+q}\rangle$ *with the metric reflector tensor* $\{\sqrt{I}\}_S$, also in their W-forms, with paired eigen angles $\pm\gamma_j$ – in their D-forms, in the *trgonometric base of diagonal cosine* $\tilde{E}_1 = R_W\cdot\tilde{E}$ (as in Ch. 5):

$$\begin{bmatrix} \cdots & & & \\ & \cosh\gamma_j & \sinh\gamma_j & \\ & \sinh\gamma_j & \cosh\gamma_j & \\ & & & \cdots \end{bmatrix} = \cosh\Gamma + \sinh\Gamma = Roth\ \Gamma = Roth'\ \Gamma = \exp\ \Gamma, \tag{324}$$

$$Roth\ (-\Gamma) = \cosh\ \Gamma - \sinh\ \Gamma = Roth^{-1}\ \Gamma = \exp(-\Gamma). \tag{325}$$

$$\Rightarrow\ Roth\ (+\Gamma)\cdot Roth\ (-\Gamma) = \exp\ (+\Gamma)\cdot\exp(-\Gamma) = \cosh^2\Gamma - \sinh^2\Gamma = I.$$

It is the hyperbolic rotational function of the motive angle Γ or $-\Gamma$ and its tensor Invariant.

$$\begin{bmatrix} \cdots & & & \\ & \operatorname{sech}\ \gamma_j & -\tanh\gamma_j & \\ & +\tanh\gamma_j & \operatorname{sech}\ \gamma_j & \\ & & & \cdots \end{bmatrix} = \operatorname{sech}\ \Gamma + i\tanh\Gamma = Defh\ (+\Gamma), \tag{326}$$

$$Defh\ (-\Gamma) = \operatorname{sech}\ \Gamma - i\tanh\ \Gamma = Defh^{-1}\ \Gamma = Defh'\ \Gamma. \tag{327}$$

$$\Rightarrow\ Defh\ (+\Gamma)\cdot Defh\ (-\Gamma) = Defh\ \Gamma\cdot Defh'\ \Gamma = \operatorname{sech}^2\Gamma + \tanh^2\Gamma = I.$$

It is the hyperbolic deformational function of the motive angles with tensor quasi-Invariant.

$$\begin{bmatrix} \cdots & & & \\ & \cosh\gamma_j & \pm\sinh\gamma_j & \\ & \mp\sinh\gamma_j & -\cosh\gamma_j & \\ & & & \cdots \end{bmatrix} = \cosh\tilde{\Gamma} \mp \sinh\tilde{\Gamma} \quad \to (339), (340), \tag{328}$$

$$\begin{bmatrix} \cdots & & & \\ & \text{sech}\ \gamma_j & \mp\tanh\gamma_j & \\ & \mp\tanh\gamma_j & -\text{sech}\ \gamma_j & \\ & & & \cdots \end{bmatrix} = \text{sech}\ \tilde{\Gamma} \mp \tanh\tilde{\Gamma} \quad \to (337), (338). \tag{329}$$

$$\cosh^2\tilde{\Gamma} - \sinh^2\tilde{\Gamma} = I, \quad \text{sech}^2\tilde{\Gamma} + \tanh^2\tilde{\Gamma} = I$$

They are the hyperbolic orthogonal and oblique reflectors with the projective angle $\tilde{\Gamma}$ and their Invariant and quasi-Invariant.

In pseudo-Euclidean trigonometry, the general *reflector tensor* (253) is identical by form to a metric reflector tensor of the *non-coaxially oriented* pseudo-Euclidean space:

$$Ref\ \{\cosh\tilde{\Gamma}\}^{\ominus} = \{\sqrt{I}\}_S = \{R_W \cdot I^{\pm} \cdot R'_W\}, \quad (\tau = \tau_{max} = q). \tag{330}$$

Apply the principle of binarity and take into account (271) and (324), the result are the following conditions of annihilation similar to (257) for secondary orthospherical rotations $Rot\ \Theta$ and for quasi-Euclidean principal rotations as

$$Roth\ \Gamma \cdot \{\sqrt{I}\}_S \cdot Roth\ \Gamma = \{\sqrt{I}\}_S.$$

Further the inverse passive modal transformation R_c^{-1} converts hyperbolic angles and functions into pseudo-spherical ones. Angles $\{i\Gamma\}_c$ have spherical form. *Pseudo-spherical trigonometry* is realizable in isometric to original $\langle\mathcal{P}^{n+q}\rangle$ a *complex quasi-Euclidean space* $\langle\mathcal{Q}^{n+q}\rangle_c$. The scalar product is also invariant in both these isometric spaces:

$$\mathbf{u}' \cdot \{I^{\pm}\} \cdot \mathbf{u} = (R_c\mathbf{u})'(R_c\mathbf{u}) = \mathbf{z}'_{02} \cdot \mathbf{z}_{02}.$$

Such space $\langle\mathcal{Q}^{n+q}\rangle_c$ is a complex isomorphism of the real pseudo-Euclidean space by Minkowski. It was introduced by H. Poincaré in 1905 [63] as the 3-dimensional model of a relativistic space-time with the Lorentz transformations group called so also by Poincaré.

Abstract analogy (323) applied to pseudo-spherical angles and their functions gives finally the original spherical notions in our quasi-Euclidean space $\langle\mathcal{Q}^{n+q}\rangle$. **The whole closed cycle (322)–(323) with abstract spherical–hyperbolic analogy is described.**

The analogy with spherical formulae (269) and (270) connects hyperbolic projective and motive angles and their functions *in common bases* in terms of mid-reflectors:

$$-i\tilde{\Gamma}_{12} \cdot Ref\ \{\cosh\tilde{\Gamma}_{12}\}^{\ominus} = \Gamma_{12} = Ref\ \{\cosh\tilde{\Gamma}_{12}\}^{\ominus} \cdot i\tilde{\Gamma}_{12}, \quad (\ \{\tilde{\Gamma}_{12}\}^2 = \{\Gamma_{12}\}^2\ ).$$

6.2 Covariant and countervariant spherical–hyperbolic analogies

Abstract analogies in (322), (323) give no any quantitative relation between real-valued spherical and hyperbolic angles or functions. However such relations may be determined if a one-to-one specific correspondence in the original (universal) Cartesian base $\tilde{E}_1 = \{I\}$ between both these arguments-angles is fixed. Spherical and hyperbolic angles, functions and transformations with this isomorphic correspondence in all eigen quasiplanes and pseudoplanes in $\tilde{E}_1$ may be represented clearly at the general trigonometric diagram (Figure 3).

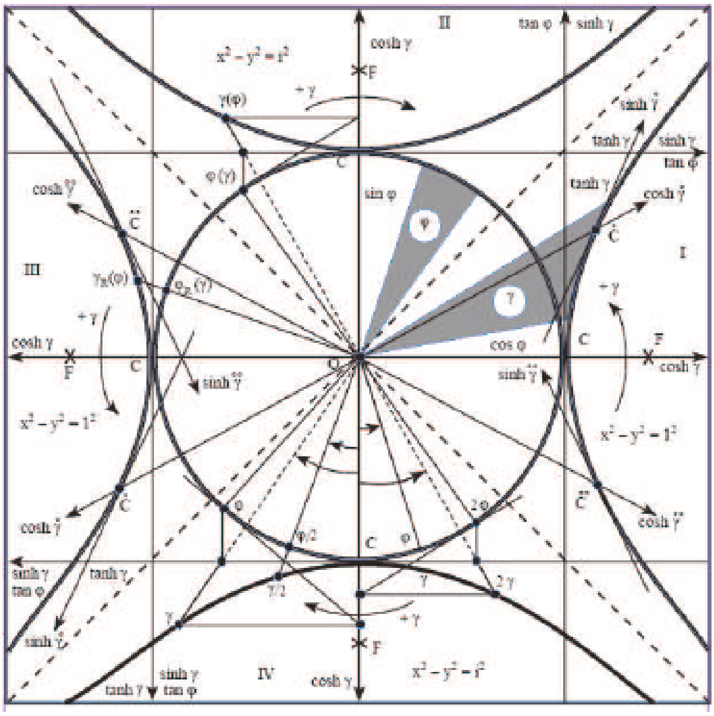

Figure 3. The Trigonometric Diagram on the base of unity circle and quadrohyperbola with spherical–hyperbolic analogies in an eigen plane – pseudoplane of a tensor angle with respect to the right universal base $\tilde{E}_1$. (The angle φ is spherical, the angle γ is hyperbolic.)

Here we use the following notations:
I, II, III, IV are the hyperbolic quadrants of a pseudoplane with conjugate hyperbolae (I, III and II, IV) and hyperbolic angles dividing by the two asymptotic diagonals.
$\overset{\circ}{\gamma}$ and $\overset{\circ\circ}{\gamma}$ are the positive and negative angles of hyperbolic rotations determined along hyperbolae, they are shown in I and III.
$\varphi(\gamma)$ and $\gamma(\varphi)$ are the examples of specific sine-tangent spherical–hyperbolic analogy, they are shown in II; in hyperbolae focus we define especial angle $\omega = \gamma_F = \gamma(\pi/4) \approx 0.881$ rad.
$\varphi_R(\gamma)$ and $\gamma_R(\varphi)$ are the examples of specific tangent-tangent analogy, they are shown in III. Besides, bisection and duplication of an hyperbolic angle with respect to the base $\tilde{E}_1$, with the use of these two analogies, are shown in the left and right parts of IV, and in sect. 6.4.

The identical ranges of certain trigonometric functions of two complementary hyperbolic angles γ and υ and of two complementary spherical angles φ and $\xi = \pi/2 - \varphi(\gamma)$ allows us to define covariant (or *sine-tangent*) and countervariant (or *sine-cotangent*) specific analogies, *but correctly only in the so called universal bases*, usually in the simplest base $\tilde{E}_1 = \{I\}$:

$$\sinh(\gamma, \upsilon) \equiv \tan(\varphi, \xi), \quad \tanh(\gamma, \upsilon) \equiv \sin(\varphi, \xi) \quad [\gamma(\varphi) \;\Leftrightarrow\; \varphi(\gamma),\; \upsilon(\xi) \;\Leftrightarrow\; \xi(\upsilon)]. \quad (331 - I)$$

$$\sinh(\gamma, \upsilon) \equiv \cot(\xi, \varphi) \quad \tanh(\gamma, \upsilon) \equiv \cos(\xi, \varphi) \quad [\gamma(\xi) \Leftrightarrow \xi(\gamma)],\; \upsilon(\varphi) \;\Leftrightarrow\; \varphi(\upsilon)]. \quad (331 - II)$$

(See more strict and descriptive justification of this below in sect. 6.4.)
Then, on the basis of (331A), argument angles are connected by the following equalities:

$$\gamma = \gamma(\varphi) = \operatorname{artanh}(\sin\varphi) = \operatorname{arsinh}(\tan\varphi) = \ln(\sec\varphi + \tan\varphi),$$

$$\varphi = \varphi(\gamma) = \arctan(\sinh\gamma) = \arcsin(\tanh\gamma) = -i\ln(\operatorname{sech}\gamma + i\tanh\gamma).$$

Function $\gamma(\varphi)$ (Lambertian) as in (331-I) was introduced by Johann Lambert in 1760 [36]. In 1830 Christoph Gudermann added to it the inverse function $\gamma(\varphi)$ (Gudermannian) [51]. Besides, at Figure 3, in an illustrative sense, we show only covariant analogy (331) between the hyperbola with hyperbolic angle-argument and the semi-circle with spherical angle in the angular interval or sector with $R = 1$ indicated above beginning from the zero points C.

For application in our Tensor Trigonometry, we added else in (331-II) two functions of the types – direct $\gamma(\xi)$ and inverse $\xi(\gamma)$. So, the latter gives the spherical parallel angle of Lobachevsky $\xi = \pi/2 - \varphi(\gamma)$ as one-step in $\tilde{E}_1$ and in hyperbolic geometry. Trigonometric definitions of four functions for covariant and countervariant specific spherical-hyperbolic analogies, emphasize the fact, that in Geometry, including Tensor Trigonometry, *they can be used only in the universal bases* $\tilde{E}_1$ or widely $\tilde{E}_{1u} = \{rot\ \theta \cdot \{I\}$ and from zero point O!

According to our Trigonometric Diagram at Figure 3, the main values of spherical angles are in $[-\pi/2;\ +\pi/2]$, as in Ch. 5. For this range of the angles their cosines and sines are nonnegative, thus formulae (331) may be supplemented by two analogs:

$$\text{sech}\ \gamma \equiv \cos\varphi \geq 0\ , \qquad \cosh\gamma \equiv \sec\ \varphi \geq 0\ . \qquad (332-I)$$

$$\cosh\gamma \equiv \csc\xi \geq 0\ , \qquad \text{sech}\ \gamma \equiv \sin\xi \geq 0\ ; \qquad (332-II)$$

$$d\gamma(\varphi) = \sec\varphi\ d\varphi = \cosh\gamma\ d\varphi,\ d\varphi(\gamma) = \text{sech}\gamma\ d\gamma = \cos\varphi\ d\gamma;\ (d\xi = -d\varphi). \qquad (332-III)$$

Differentials and derivatives will be useful in the instantaneous bases $\tilde{E}_m$ – see more further.

The range $[-\pi/2;\ +\pi/2]$ of spherical angles is sufficient for trigonometric transformations (rotations, deformations) of lineors and bivalent tensors. Identities (331) generate specific *sine–tangent spherical–hyperbolic analogy*, represented in vector-scalar form usually in $\tilde{E}_1$:

$$\left.\begin{matrix}\sin\Phi \equiv \tanh\Gamma,\ \tan\Phi \equiv \sinh\Gamma, \\ \cos\Phi \equiv \text{sech}\ \Gamma,\ \sec\ \Phi \equiv \cosh\Gamma,\end{matrix}\right\}\ \pm\varphi_j \in [-\pi/2;\ +\pi/2],\ \pm\gamma_j \in (-\infty; +\infty). \qquad (333)$$

Most generally covariant specific analogy is expresed using spherical and hyperbolic tensors of rotation and deformation with respect to the universal base $\tilde{E}_1 = \{I\}$ *in two directions* (with identities in 2×2-cells for *binary primary and mutual eigen angles*) as follows:

$$Roth\ \Gamma \equiv Def\ \Phi \quad (\text{in the base } \tilde{E}_1 = \{I\}) \qquad (334-I)$$

$$Roth\ \Gamma = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \cosh\gamma_j & \sinh\gamma_j & \\ & \sinh\gamma_j & \cosh\gamma_j & \\ \hline & & & \ddots \end{array}\right] \equiv \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \sec\varphi_j & \tan\varphi_j & \\ & \tan\varphi_j & \sec\varphi_j & \\ \hline & & & \ddots \end{array}\right] = Def\ \Phi(\Gamma),$$

$$Defh\ \Gamma \equiv Rot\ \Phi \quad (\text{in the base } \tilde{E}_1 = \{I\}) \qquad (334-II)$$

$$Defh\ \Gamma = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \text{sech}\ \gamma_j & -\tanh\gamma_j & \\ & +\tanh\gamma_j & \text{sech}\ \gamma_j & \\ \hline & & & \ddots \end{array}\right] \equiv \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \cos\varphi_j & -\sin\varphi_j & \\ & +\sin\varphi_j & \cos\varphi_j & \\ \hline & & & \ddots \end{array}\right] = Rot\ \Phi(\Gamma).$$

Analogy (334) infers the Rule 5 for spherical deformational matrices in sect. 5.10. Functional relations between both these motive tensor angles in the base $\tilde{E}_1$ follow from (331-I):

$$\Gamma(\Phi) = \ln Def\ \Phi = \ln(\sec\ \Phi + \tan\Phi), \quad i\Phi(\Gamma) = \ln Defh\ \Gamma = \ln(\text{sech}\ \Gamma + i\tanh\Gamma).$$

Countervariant analogies are realised by replacing principal angle by complementary one.

Most generally countervariant specific analogy is expresed using spherical and hyperbolic tensors of rotation and deformation also in the universal base $\tilde{E}_1 = \{I\}$ in two directions:

$$Roth\ \Gamma \equiv \overline{Def}\ \Xi \quad (\text{in the base } \tilde{E}_1 = \{I\}) \qquad (335-I)$$

$$Roth\ \Gamma = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \cosh\gamma_j & \sinh\gamma_j & \\ & \sinh\gamma_j & \cosh\gamma_j & \\ \hline & & & \ddots \end{array}\right] \equiv \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \csc\xi_j & \cot\xi_j & \\ & \cot\xi_j & \csc\xi_j & \\ \hline & & & \ddots \end{array}\right] = \overline{Def}\ \Xi(\Gamma),$$

$$Defh\ \Gamma \equiv \overline{Rot}\ \Xi(\Gamma) \quad (\text{in the base } \tilde{E}_1 = \{I\}) \qquad (335-II)$$

$$Defh\ \Gamma = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \text{sech}\ \gamma_j & -\tanh\gamma_j & \\ & +\tanh\gamma_j & \text{sech}\ \gamma_j & \\ \hline & & & \ddots \end{array}\right] \equiv \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \sin\xi_j & -\cos\xi_j & \\ & +\cos\xi_j & \cos\xi_j & \\ \hline & & & \ddots \end{array}\right] = \overline{Rot}\ \Xi(\Gamma).$$

Rules 1; 2, 3 (sects. 5.3; 5.7) stay valid also for trigonometrically compatible hyperbolic rotational matrices and orthogonal reflectors. **Rules 3** is *foundation for principal spherical and hyperbolic rotations in the quasi- and pseudo-Euclidean geometries* – see sect. 6.3. Here **Rule 3** holds for trigonometric functions and transformations in pseudoplane. In particular,

$$\prod_{j=1}^{m}(\sec\ \varphi_j \pm \tan\varphi_j)^{h_j} \equiv \prod_{j=1}^{m}(\cosh\gamma_j \pm \sinh\gamma_j)^{h_j} = \exp\left(\sum_{j=1}^{m} \pm h_j\gamma_j\right) = \qquad (336)$$

$$= \exp\gamma = \cosh\gamma + \sinh\gamma \equiv \sec\ \varphi + \tan\varphi, \quad \varphi \in [-\pi/2; +\pi/2], \ (\text{see sect. } 5.10).$$

If the mid-reflector (330) for $\mathbf{\Gamma}_B$ is used as a reflector tensor, then hyperbolic reflectors (328), (329) are hyperbolic analogies of spherical ones (178), (179) and (211), (212):

$$Ref\{BB'\} = \text{sech}\ \bar{\Gamma}_B - \tanh\bar{\Gamma}_B, \ Ref\{B'B\} = \text{sech}\ \bar{\Gamma}_B + \tanh\bar{\Gamma}_B; \qquad (337), (338)$$

$$Ref\{B\} = \cosh\ \bar{\Gamma}_B - i\sinh\bar{\Gamma}_B, \ Ref\{B'\} = \cosh\ \bar{\Gamma}_B + i\sinh\bar{\Gamma}_B. \qquad (339), (340)$$

Spherical–hyperbolic analogies of the two types – abstract in $\bar{E}$ and covariant specific in the base $\tilde{E}_1$ generates the following *Quart-Circle* of motive matrix functions–transformations:

$$\begin{array}{ccccccc} Rot\ (i\Gamma) & \equiv & Defh\ (-i\Phi) & \Leftrightarrow & Roth\ \Gamma & \equiv & Def\ \Phi \\ & \Updownarrow & & & & \Updownarrow & \\ Rot\ \Phi & \equiv & Defh\ \Gamma & \Leftrightarrow & Roth\ (-i\Phi) & \equiv & Def\ (i\Gamma). \end{array} \qquad (341)$$

The sine-tangent analogy generates hyperbolically orthogonal forms of affine projectors, quasi-inverse matrices, and reflectors considered before, if the mid-reflector for $\mathbf{\Gamma}_B$ is used as a reflector tensor. Then hyperbolic relations are similar to spherical ones (249):

$$Ref\{B'\}\cdot\ Ref\{B\} = (\cosh\bar{\Gamma}_B + \sinh\bar{\Gamma}_B)(\cosh\bar{\Gamma}_B - \sin\bar{\Gamma}_B) = Roth\ 2\Gamma_B. \qquad (342)$$

This means here that reflection $\{(\sqrt{I})'_h(\sqrt{I})_h\}$, where $(\sqrt{I})_h \neq (\sqrt{I})'_h$ – the prime non-symmetric square root, is the double hyperbolic rotation similar to (251). Hence rotational matrix $Roth\ \mathbf{\Gamma}_B$ is a *trigonometric* hyperbolic square root of the *symmetric* matrix in square brackets similar to spherical one in (251); but in this case it is also an *arithmetic* root:

$$Roth\ \mathbf{\Gamma}_B = [(\pm Ref\{B\})' \cdot (\pm Ref\{B\})]_S^{1/2} = [Roth\ 2\Gamma_B]_S^{1/2}. \qquad (343)$$

If $\mathbf{a_1}, \mathbf{a_2}$ are the non-oriented vectors or planars $im\ \mathbf{a_1}, im\ \mathbf{a_2}$ of rank 1 and $\mathbf{a_1a_2} \neq \mathbf{0}$, then they may also determine *the elementary rotational hyperbolic matrix* with $\tau = 1$:

$$Roth\ \Gamma_{12} = \left[(I - 2\overleftarrow{\mathbf{a_2a'_1}})(I - 2\overleftarrow{\mathbf{a_1a'_2}})\right]^{1/2} = \qquad (344)$$

$$= \left[I - 2\left(\frac{\mathbf{a_1a'_2}}{\mathbf{a'_2a_1}} + \frac{\mathbf{a_2a'_1}}{\mathbf{a'_1a_2}}\right) + 4\cosh^2\gamma_{12}\cdot\frac{\mathbf{a_2a'_2}}{\mathbf{a'_2a_2}}\right]^{1/2},$$

$$\text{where: } \overleftarrow{\mathbf{a_2a'_1}} = \frac{\mathbf{a_2a'_1}}{\mathbf{a'_1a_2}}, \quad \overleftarrow{\mathbf{a_1a'_2}} = \frac{\mathbf{a_1a'_2}}{\mathbf{a'_2a_1}}.$$

(In particular, $\mathbf{a_1} = \mathbf{e_1}, \ \ \mathbf{a_2} = \mathbf{e_2}, \rightarrow \mathbf{e'_2e_1} = \mathbf{e'_1e_2} = \cos\varphi_{12} \equiv \text{sech}\ \gamma_{12}, \ \ \mathbf{e_2e'_1} = \overleftarrow{\mathbf{e_2e'_1}}$.)

Recall, that $\{\overleftarrow{\mathbf{a_2 a_1'}}\}$ is a projector into $\langle im\ \mathbf{a_2}\rangle$ parallel to $\langle ker\ \mathbf{a}_1'\rangle \equiv \langle im\ \mathbf{a_1}\rangle^{\perp}$.
The *spherical angle* α_B in (288) is evaluated quantitatively with the use of (331) and (324):

$$\boxed{Def\ \alpha_B \equiv Roth\ 2\Gamma_B \ \Rightarrow\ \alpha_B = \arctan(\sinh\ 2\gamma_B).}$$

This relation comments commutativity of the trigonometrically compatible matrix functions of their motive angles – either Φ or Γ according to the quart circle (341)!

The sine-tangent analogy leads to the following four expressions for the mid-reflector:

$$Ref\{\cos\tilde{\Phi}\}^{\ominus} = Ref\{sec\ \tilde{\Phi}\}^{\ominus} \equiv Ref\{\cosh\tilde{\Gamma}\}^{\ominus} = Ref\{sech\ \tilde{\Gamma}\}^{\ominus}. \quad (345)$$

Right multiply the matrices in quart circle (341) by the mid-reflector, we obtain the *similar quart circle for the reflectors.* Repeat this operation once more and we return to their original motive type. Definitions of projective hyperbolic angles $\tilde{\Gamma}_B$ and functions may be obtained from the spherical ones with the use of sine-tangent analogy (331), if the mid-reflector (345) for Γ_B is used as the pseudo-Euclidean space metric reflector tensor.

Application of spherical formulae (255), (256) and (303) gives the similar hyperbolic modal relations:

$$\left.\begin{array}{r} Ref\{B'\} = Roth\ (+\Gamma_B)\cdot Ref\{B\}\cdot Roth\ (-\Gamma_B);\\ Ref\{B'\} = Ref\{\cosh\tilde{\Gamma}_B\}^{\ominus}\cdot Ref\{B\}\cdot Ref\{\cos\tilde{\Gamma}_B\}^{\ominus},\\ Ref\{B\} = Ref\{\cosh\tilde{\Gamma}_B\}^{\ominus}\cdot Ref\{B'\}\cdot Ref\{\cos\tilde{\Gamma}_B\}^{\ominus};\\ \overrightarrow{\overleftarrow{B'}} = Roth\ (+\Gamma_B)\cdot \overrightarrow{\overleftarrow{B}}\cdot Roth\ (-\Gamma_B) = Ref\{\cosh\tilde{\Gamma}_B\}^{\ominus}\cdot \overrightarrow{\overleftarrow{B}}\cdot Ref\{\cos\tilde{\Gamma}_B\}^{\ominus}. \end{array}\right\} \quad (346)$$

Add to them the set $\langle T_B\rangle \equiv \langle Roth\ \Gamma_B\cdot Rot\ \Theta_B\rangle$ of modal rotational matrices performing operations (346). Here the matrix $Roth\ \Gamma_B$ determined by (343) has the trigonometric subspace of the minimal dimension among all matrices of $\langle T_B\rangle$. In particular, it enables one to evaluate the rotation variant of modal matrices for transforming affine projectors into D-forms, i. e., developing further relations (311), (312):

$$\left.\begin{array}{r} R_W'\cdot\ Roth\ (+\Gamma_B/2)\cdot \overrightarrow{\overleftarrow{B}}\cdot Roth\ (-\Gamma_B/2)\cdot R_W = D\{\overrightarrow{\overleftarrow{B}}\},\\ R_W'\cdot\ Roth\ (-\Gamma_B/2)\cdot \overrightarrow{\overleftarrow{B'}}\cdot Roth\ (+\Gamma_B/2)\cdot R_W = D\{\overrightarrow{\overleftarrow{B'}}\}. \end{array}\right\} \quad (347)$$

* * *

As in the quasi-Euclidean space and geometry – see (257), (258) in sect. 5.7, in pseudo-Euclidean ones only the compatible rotations and reflections of two types as principal hyperbolic and orthospherical are used. In the *motive version*, these rotations satisfy relations:

$$\left.\begin{array}{l} Roth\ \{\pm\Gamma_{12}\}\cdot Ref\{\cosh\tilde{\Gamma}\}^{\ominus}\cdot Roth\ \{\pm\Gamma_{12}\} = Ref\{\cosh\tilde{\Gamma}\}^{\ominus},\\ Rot'\ \{\pm\Theta_{12}\}\cdot Ref\{\cosh\tilde{\Gamma}\}^{\ominus}\cdot Rot\ \{\pm\Theta_{12}\} = Ref\{\cosh\tilde{\Gamma}\}^{\ominus}. \end{array}\right\} \quad (348)$$

For the *projective version*, we use analogue of (256) in (356) with reflectors in angular variant from (211, 212) and adding to them one orthospherical reflector; and we obtain the compatible hyperbolic reflections of two types, which all satisfy relations:

$$\left.\begin{array}{l} Ref_{\boxtimes}\{\mp\tilde{\Gamma}_{12}\}\}\cdot Ref\{\cosh\tilde{\Gamma}\}^{\ominus}\cdot Ref_{\boxtimes}\{\pm\tilde{\Gamma}_{12}\}\} = Ref\{\cosh\tilde{\Gamma}\}^{\ominus},\\ Ref_{\boxtimes}\{\pm\tilde{\Theta}_{12}\}\cdot Ref\{\cosh\tilde{\Gamma}\}^{\ominus}\cdot Ref_{\boxtimes}\{\pm\tilde{\Theta}_{12}\} = Ref\{\cosh\tilde{\Gamma}\}^{\ominus}. \end{array}\right\} \quad (349)$$

Transferring through the reflector-tensor $Ref\{\cosh\tilde{\Gamma}\}^{\ominus}$, the principal reflector is transformed into its mutual one annihilating; the secondary reflector is transferring through both unity parts of the reflector tensor without changes and annihilating too!

Relations (348, 349) are pseudo-Euclidean analogues of quasi-Euclidean ones (257, 258). They produce the *pseudo-Euclidean space* $\langle\mathcal{P}^{n+q}\rangle$, *its geometry and tensor trigonometry* with the given reflector metric tensor, introduced independently, as well as for the *external type of the two non-Euclidean hyperbolic geometries with an index* q. The latters at $q = 1$ are two *hyperbolic geometries on two hyperbolic hypersurfaces of the constant negative curvature*. These two and general hyperbolic geometries realized on twq hyperboloids of the radius-parameter R embedded into the *pseudo-Euclidean space* $\langle\mathcal{P}^{n+q}\rangle$ with a set reflector metric tensor. We will continue this topic in Chs. 11, 12, 6A, 7A and 10A with complete tensor and differential trigonometric descriptions of such motions with the laws of their summation.

6.3 Reflector tensor in quasi- and pseudo-Euclidean interpretation

Applications of hyperbolic and spherical matrices of the two principal motive types and reflective ones in tensor trigonometry need in correct theoretical justification including a choice of *binary spaces* with their reflector metric tensors, admissible transformations and coordinate bases. Fix an initial arithmetic (affine) space with the universal unity base $\tilde{E}_1 = \{I\}$. Then introduce in this space by quite independent way the reflector tensor for beginning in its general form as $\{\sqrt{I}\}_S$ (see in sect. 5.7). In this initial base, it determines the *non-coaxial* orientation of the quasi-Euclidean and pseudo-Euclidean spaces and, for example, the tensor rotations of three types defined before in (257) and (348):

in the space $\langle \mathcal{Q}^{n+q} \rangle \equiv \langle \mathcal{E}^n \rangle \boxplus \langle \mathcal{E}^q \rangle \equiv CONST$, set of principal spherical rotations $\langle Rot\ \Phi \rangle$

$$Rot\ \Phi \cdot \{\sqrt{I}\}_S \cdot Rot\ \Phi = \{\sqrt{I}\}_S = Rot\ (-\Phi) \cdot \{\sqrt{I}\}_S \cdot Rot\ (-\Phi);$$

in the space $\langle \mathcal{P}^{n+q} \rangle \equiv \langle \mathcal{E}^n \rangle \boxtimes \langle \mathcal{E}^q \rangle \equiv CONST$, set of principal hyperbolic rotations $\langle Roth\ \Gamma \rangle$

$$Roth\ \Gamma \cdot \{\sqrt{I}\}_S \cdot Roth\ \Gamma = \{\sqrt{I}\}_S = Roth\ (-\Gamma) \cdot \{\sqrt{I}\}_S \cdot Roth\ (-\Gamma);$$

in both spaces, common group of induced or independent orthospherical rotations $\langle Rot\ \Theta \rangle$

$$Rot'\ \Theta \cdot \{\sqrt{I}\}_S \cdot Rot\ \Theta = \{\sqrt{I}\}_S = Rot\ \Theta \cdot \{\sqrt{I}\}_S \cdot Rot'\ \Theta.$$

The new *quasi-Euclidean space* $\langle \mathcal{Q}^{n+q} \rangle$ (see initially in the end of sect. 5.7) is determined by Euclidean quadratic metric and the set reflector tensor $\{\sqrt{I}\}_S$. They define the admissible transformations, forming the *new complete group of quasi-Euclidean rotations (motions)*:

$$\tilde{E}_2 = Rot\ \Phi \cdot Rot\ \Theta \cdot \tilde{E}_1 \ or\ \tilde{E}_3 = Rot\ \Theta \cdot Rot\ \Phi \cdot \tilde{E}_1. \tag{350}$$

$\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$ are called *rotationally connected quasi-Cartesian bases*. The quasi-Euclidean scalar, vector and tensor trigonometries are realized in spaces $\langle \mathcal{Q}^{n+q} \rangle$, with respect here to the right quasi-Cartesian bases such as (350).

The well-known *pseudo-Euclidean Minkowski space* $\langle \mathcal{P}^{n+q} \rangle$ (see in Chs. 10 and 11) is determined by pseudo-Euclidean quadratic metric with the set reflector metric tensor $\{\sqrt{I}\}_S$. They define the admissible transformations, forming the complete *Lorentzian group of pseudo-Euclidean rotations (motions)*:

$$\tilde{E}_2 = Roth\ \Gamma \cdot Rot\ \Theta \cdot \tilde{E}_1 \ or\ \tilde{E}_3 = Rot\ \Theta \cdot Roth\ \Gamma \cdot \tilde{E}_1. \tag{351}$$

$\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$ are called *rotationally connected pseudo-Cartesian bases*. The pseudo-Euclidean scalar, vector and tensor trigonometries are realized in spaces $\langle \mathcal{P}^{n+q} \rangle$, with respect here to the right pseudo-Cartesian bases such as (351).

Introduce the right so-called **universal bases** including the original base $\tilde{E}_1 = \{I\}$:

$$\langle \tilde{E}_{Iu} \rangle \equiv \langle Rot\ \Theta \cdot \tilde{E}_1 \rangle \quad (\tilde{E}'_{Iu}\tilde{E}_{Iu} = I,\ \tilde{E}'_{Iu}\{\sqrt{I}\}_S\tilde{E}_{Iu} = \{\sqrt{I}\}_S,\ det\ \tilde{E}_{Iu} = +1). \tag{352}$$

The transformations $\langle Rot\ \Theta \rangle$ form the *orthospherical subgroup*, what s the intersection of these Quasi-Euclidean and Lorentz groups, but only with respect to universal bases $\langle \tilde{E}_{Iu} \rangle$!

A reflector tensor and a choice of the principal trigonometry from two kinds (spherical or hyperbolic) determine the spaces quadratic metric with internal and external multiplications from their two kinds (either Euclidean or pseudo-Euclidean); and vice versa! The two complete sets of admissible rotations in these two binary spaces (quasi-Euclidean or pseudo-Euclidean) contain the subsets of Special quasi-Euclidean or Lorentzian pseudo-Euclidean rotations with the parallel translations, which stipulate the spaces isotropy and homogeneity!

Recall (!), that q is a quantity of negative unity eigenvalues of the reflector tensor of any binary spaces with such a reflector tensor – see in Chs. 10–12 and in Appendix.

The original base $\tilde{E}_1 = \{I\}$ is the simplest universal base by its form. Universal bases enable one to jointly realize quasi-Euclidean and pseudo-Euclidean trigonometries on the basis of concrete spherical–hyperbolic analogy, but only with *one-step* motions. Note, in particular, that in STR (special theory of relativity) physical *one-step* motions with respect to relatively fixed Observer are described in terms of universal bases.

Consider how a reflector tensor acts on matrices eigenprojectors in both spaces. Let B be a null-prime matrix, used initially in an affine space $\langle \mathcal{A}^n \rangle$. Introduce in the space the reflector tensor as the mid-reflector of the tensor angle for the matrix B in two following variants (with introducing metrics for external and internal products):

$$\{\sqrt{I}\}_S = Ref\ \{\cos \tilde{\Phi}_B\}^{\ominus} \equiv Ref\ \{\cosh \tilde{\Gamma}_B\}^{\ominus}. \tag{353}$$

We got quasi- and pseudo-Euclidean spaces. Here the identity is true only in $\langle \tilde{E}_{Iu} \rangle$.

In the first case, the symmetric projectors $\overleftarrow{BB'}$ and $\overrightarrow{BB'}$ are spherically orthogonal each to another in Euclidean and quasi-Euclidean spaces with a metric tensor $\{I^+\}$, i. e., reflector tensor (353) does not determine here internal and external products.

In the second case, the non-symmetric projectors $\overleftarrow{B}$ and $\overrightarrow{B}$ are hyperbolically orthogonal each to another in a pseudo-Euclidean space with *metric* reflector tensor $\{I^{\pm}\}$ according to (353). This fact follows taking into account last formula in (346):

$$(\overleftarrow{B})' Ref\ \{\cosh \tilde{\Gamma}_B\}^{\ominus} \overrightarrow{B} = Ref\ \{\cosh \tilde{\Gamma}_B\}^{\ominus} \cdot \overleftarrow{B} \cdot \overrightarrow{B} = Z.$$

Consequently, B^- (sect. 2.1) is a *hyperbolically orthogonal quasi-inverse matrix* with respect to reflector tensor (353) if B is null-prime. Also, in this case, the direct sum $\langle im\ B \rangle \oplus \langle ker\ B \rangle$ is hyperbolically orthogonal. Then for non-symmetric projectors $\overleftarrow{B}$ and $\overrightarrow{B}$, their eigen subspaces corresponding to the eigenvalues 0 and 1 are hyperbolically orthogonal too.

Equirank projectors $\overleftarrow{B}$ and $\overleftarrow{B'}$ as well as $\overrightarrow{B}$ and $\overrightarrow{B'}$ and their planars are transformed into each other with hyperbolic rotation in (346) as hyperbolically orthogonal 2-valent tensors and tensor objects. Projectors $\overleftarrow{B}$ and $\overrightarrow{B}$ hyperbolically orthogonally project respectively into $\langle im\ B \rangle$ and $\langle ker\ B \rangle$. Projective formulae (186)–(197) are transformed. In the symbolic octahedron (Figure 1) the diagonal RS generates two pseudo-isosceles triangles RZS and RIS with equal hyperbolic angles $\angle RZS \equiv \angle RIS \equiv \Gamma_B$. These facts are responses to the introduction of reflector tensor (353) in the hyperbolic form.

As hyperbolic analogues of Moivre and Euler formulae, we have the motive tensor angle from the rotation tensor in cell-forms – see in (287):

$$Roth\{m\Gamma\} = \cosh\{m\Gamma\} + \sinh\{m\Gamma) = Roth^m\ \Gamma =$$

$$= \exp\{m\Gamma\} \to \{m\Gamma\} = \ln\ Roth\{m\Gamma\} \to \Gamma = \ln\ Roth\ \Gamma.$$

$$\{Roth^m\Gamma\} = \begin{bmatrix} \ddots & & & & \\ & \cosh m\gamma_j & \sinh m\gamma_j & & \\ & \sinh m\gamma_j & \cosh \gamma_j & & \\ & & & \ddots & \\ & & & & \boxed{1} \\ & & & & & \ddots \end{bmatrix} = \exp \begin{bmatrix} \ddots & & & & \\ & 0 & m\gamma_j & & \\ & m\gamma_j & 0 & & \\ & & & \ddots & \\ & & & & \boxed{0} \\ & & & & & \ddots \end{bmatrix},$$

In particular, here the value $m = 1/2$ gives arithmetic and trigonometric square root (343) of the rotational matrix.

Properties of the matrix, no depending on the rotation angle, are the same as of a spherical deformational matrix. This matrix is symmetric and positive definite, its eigenvalues are $\mu_{2j} = \cosh\gamma_j + \sinh\gamma_j > 0$, $\mu_{2j+1} = \mu_{2j}^{-1} = \cosh\gamma_j - \sinh\gamma_j > 0$, and also may be in addition $\mu_k = +1$.

Any pair of positive numbers x and x^{-1} may be uniquely represented in terms of a scalar hyperbolic angle, in particular, as a 2×2-matrix (see sect. 5.10).

In order to establish compatibility of certain transformations for some tensor angle with the space reflector tensor in both kinds tensor trigonometries on the basis of quadratic metrics, one may use the defining relations (257), (258) and (348), (349) as criterions.

6.4 Scalar trigonometry in a pseudoplane with main relations

A diagonal reflector tensor $\{I^{\pm}\}$ produces a *coaxially oriented* pseudo-Euclidean space $\langle\mathcal{P}^{n+q}\rangle$, which has binary structure and admissible to it the pseudo-Cartesian bases $\tilde{E}$. Represent the hyperbolic rotational matrix $Roth\,\Gamma$ at a level of the j-th 2×2 cell in W-form (324) with respect to the trigonometric base $\tilde{E}_1 = \{I\}$, where the rotation realizes along the *characteristic quadrohyperbola* of coupled hyperbolae (Figure 3). In the j-th eigen pseudoplane, two axes – ordinate and abscissa are the eigenvectors $\mathbf{u}$ and $\mathbf{v}$ for the D-forms of $\cosh\,\Gamma$ (with $\pm\cosh\gamma$) and $\{I^{\pm}\}$ (with ± 1); two asymptotes of the quadrohyperbola with respect to any admissible base $\tilde{E}$ are the main and lateral *invariant diagonals* – lines with zero quadratic metric. Hence, these two asymptotes for all similar quadrohyperbolae are invariant under hyperbolic rotations of the base. If a pseudo-Euclidean space dimension n is greater than 2, then the diagonals correspond to an *invariant dividing hypersurface*. At $q = 1$, $n > 2$, it is an asymptotic hypersurface for the embedded *hyperboloids* I an II (pseudospheres of radii $R = \pm 1$ and $R = \pm i$) – see more in Ch. 12. If the j-th pseudoplane cuts such hyperboloids, then, on it in the base $\tilde{E}_1$, the rotation $\{Roth\,\Gamma\}_{2\times 2}$ is performed along the quadrohyperbola. In hyperbolic quadrants, *positive scalar angle γ_j is measured off the coordinate axis till another side of the angle in direction to the nearest main invariant diagonal, and vice versa*. Hence, the angle cannot be visually more, than $\pi/4$ in the universal base $\tilde{E}_1$ at Figure 3. Accordingly, for the pseudo-Euclidean scalar and tensor trigonometry, we adopt that the complementary to γ_j hyperbolic angle υ_j are measured off the second side of γ_j till this invariant diagonal. The identical definition is done at front Cover of our book. Then visually their sum is equal $\pi/4$ and last is right! See this further more and in detail.

The pseudo-Euclidean length is noted further as λ with Lambert angular measure γ. The Euclidean length l of the arc is obviously greater its pseudo-Euclidean length λ:

$$\lambda = R\int_{\gamma_1}^{\gamma_2}\sqrt{(d\sinh\gamma)^2 - (d\cosh\gamma)^2} = R(\gamma_2 - \gamma_1) < R\int_{\gamma_1}^{\gamma_2}\sqrt{(d\sinh\gamma)^2 + (d\cosh\gamma)^2}.$$

The area of a hyperbolic sector is $S = R^2(\gamma_2 - \gamma_1)/2$. In these four quadrants the *radius-vectors of pseudocurvature* $\pm R$ or $\pm iR$ is hyperbolically orthogonal to the hyperbola tangent at the point of tangency in an admissible base $\tilde{E}$ and $d\lambda = Rd\gamma$. These vector and tangent determine local hyperbolically connected coordinate axes. The focus of the hyperbola corresponds to the *especial hyperbolic angle* $\omega \approx 0.881\ rad$:

$$\sinh\omega = 1, \quad \cosh\omega = \sqrt{2}, \quad \tanh\omega = \sqrt{2}/2, \quad \coth\omega = \sqrt{2}. \tag{354}$$

By sine-tangent analogy, $\varphi(\omega) = \pi/4$, $\gamma(\pi/4) = \omega$. Thus the angle or the number ω is the hyperbolic analogue of the angle or the number $\pi/4$. We shall often use the angle ω in the sequel. For example, $\sin(\pi/4 \pm i\omega) = 1 \pm (\sqrt{2}/2)i; \cos(\pi/4 \pm i\omega) = 1 \mp (\sqrt{2}/2)i.$

There exist infinitely many kinds of specific analogies. Let us consider some of them. Introduce specific *tangent-tangent analogy* for the *visual angle* φ_r (or φ_R) also with respect to the base $\tilde{E}_1$ (i. e., at $\gamma_0 = 0$) with this simmetric condition (see at Figure 3, quadrant III):

$$\tan\varphi_r \equiv \tanh\gamma \;\rightarrow\; \varphi_r = \varphi_r(\gamma) = \arctan(\tanh\gamma), \quad (-\pi/4 \le \varphi_r \le +\pi/4). \tag{355}$$

This angle-analog φ_r of the angle γ are determined by the same radius-vector $\mathbf{r}$. That is why the angle $\varphi_r(\gamma)$ is called here *visual*. Thus, this visual angle φ_r may be is used for *descriptivity* in STR (see Ch. 1A), what is useful only with respect to the universal base $\tilde{E}_1$.

This mapping leads to other relations between spherical and hyperbolic functions:

$$\left.\begin{array}{ll} \sin\varphi_r \equiv \sinh\gamma/\sqrt{\cosh 2\gamma}, & \cos\varphi_r \equiv \cosh\gamma/\sqrt{\cosh 2\gamma}, \\ \sinh\gamma \equiv \sin\varphi_r/\sqrt{\cos 2\varphi_r}, & \cosh\gamma \equiv \cos\varphi_r/\sqrt{\cosh 2\varphi_r}. \end{array}\right\} \tag{356}$$

$$\varphi_r(\gamma) < \varphi(\gamma) < \gamma < \gamma_r. \;\; \text{(For example, } \varphi_r(\omega) \approx 35^\circ, \;\; \gamma_r(\pi/4) = \infty.)$$

Generally, such visually obvious specific spherical–hyperbolic analogies (in that number for angles ξ and υ) are reduced to their identities in various angular intervals similar to

$$\tan(k_1\varphi/2) \equiv \tanh(k_2\gamma/2) \Leftrightarrow \left\{ \begin{array}{ll} \sin(k_1\varphi) = \tanh(k_2\gamma), & \tan(k_1\varphi) = \sinh(k_2\gamma), \\ \cos(k_1\varphi) = \text{sech}\;(k_2\gamma), & \sec\;(k_1\varphi) = \cosh(k_2\gamma), \end{array}\right. \tag{357}$$

where spherical interval is limited: $-\pi/4 \le k_1\varphi/2 \le \pi/4$. These two variants are important:
1) $k_1 = k_2 = 1$ (this corresponds to (331-I) with Lambertian and Gudermannian functions),
2) $k_1 = k_2 = 2$ (this corresponds to main visual analogy (355)) also in $\tilde{E}_1$.

The joint application of (2) and (1) gives pure geometric (using a compass and a ruler!) duplication and bisection of hyperbolic angle γ with respect to the base $\tilde{E}_1$, because we don't have a hyperbolic compass (see at Figure 3). Under their joint acting, with preliminary duplication of φ_r and bisection of φ, we get:

$$\left.\begin{array}{l} (a)\;\gamma \rightarrow \varphi_r(\gamma) \rightarrow 2\varphi \rightarrow\;\; \tan 2\varphi = 2\tan\varphi/(1-\tan^2\varphi) \equiv 2\tanh\gamma/(1-\tanh^2\gamma) = \sinh 2\gamma; \\ (b)\;\gamma \rightarrow \varphi(\gamma) \rightarrow \varphi/2 \;\rightarrow\;\; \tan\varphi/2 = \sin\varphi/(1+\cos\varphi) \equiv \tanh\gamma/(1+\text{sech}\;\gamma) = \tanh\gamma/2. \end{array}\right\} \tag{358}$$

$$\Rightarrow |\varphi_R(\gamma)| < |\varphi(\gamma)| < |2\varphi_R(\gamma)|.$$

Indeed for the inequality, if $\cos\varphi \equiv \text{sech}\;\gamma$ and $\cos(2\varphi_R) \equiv \text{sech}\;(2\gamma)$, then $\cos\varphi > \cos(2\varphi_R)$; but if $\tan\varphi \equiv \sinh\gamma$ and $\tan\varphi_R \equiv \tanh\gamma$, then $|\tan\varphi| > |\tan\varphi_R|$.

In 1763, J. Lambert, using abstract analogy of his hypothetical imaginary sphere with the real-valued sphere, revealed the angular defect of a hyperbolic triangle and connected it with its area and radius of this sphere [33], in addition to the Th. Harriot connection from 1603 for the angular excess of a spherical triangle. Great creators of the hyperbolic non-Euclidean geometry, as a holistic axiomatic system, N. Lobachevsky and J. Bolyai used the abstract and specific spherical–hyperbolic analogies with respect to the spherical geometry for inferences of the hyperbolic non-Euclidean geometry all metric relations.

Thus, for the Lobachevsky spherical angle of parallelism, there holds: $\Pi(a) = \xi$, where the latter is complementary to spherical motion angle $\varphi(\gamma)$. According to countervariant analogy (331-II), we obtain the well-known Lobachevsky formula $\Pi(a) = \arccos\tanh\gamma$ [40]. It was by the first manner of introducing in the hyperbolic geometry of its principal motion angle γ, namely, through the finite and visual *spherical* countervariant parallel angle of Lobachevsky $\Pi(a) = \xi$, but it is correct only in the universal base, for example, simplest $\tilde{E}_1$. Contrary, the angle of motion γ generates by direct way (331-I) the covariant finite spherical angle of parallelism $\varphi = \arcsin\tanh\gamma = \pi - \xi$ also in $\tilde{E}_1$.

Tensor Trigonometry given us the opportunities to found the fundamental exact and correct connection in any own base $\tilde{E}_k$ *between principal* γ *and complementary* υ *hyperbolic angles in* [16] *– see further in (360 IY, Y) and on the book Cover.* Their bond is significantly more complex. Moreover, υ is the parallel angle correct also in any own base $\tilde{E}_k$!

The analogy (331) gives all trigonometric formulae for a pseudo-Euclidean right triangle ABC (plane trigonometry began with solving a right triangle!). See visually at front and back Covers of the book! Its legs a and b lie in two hyperbolic quadrants (suppose $a \leq b$). The principal angle γ at the vertex A is contrary to the leg $a < b$. Denote the pseudo-hypotenuse as g. The common pseudo-Euclidean Pythagorean Theorem is $g^2 = b^2 - a^2$, because $a \leq b$. If the angle γ is in hyperbolic quadrant I (Figure 3), then the triangle ABC is exterior, g is "space-like", i. e., is outside of two isotropic, or light in the relativistic physics, diagonals. If $|a| = |b|$, then γ is infinite, g is situated onto the diagonal with zero pseudo-Euclidean length. If the angle γ is in hyperbolic quadrant II, then the triangle is interior, g is "time-like", i. e., is inside of two isotropic diagonals. Below for determinacy we choose the *exterior triangle* ABC, i. e., at back Cover. Its legs a and b belong to distinct eigen subspaces of reflector tensor with eigenvalues -1 and $+1$. In order to infer all trigonometric formulae between hyperbolic angles of the right triangle ABC, we consider preliminary the locations and behavior together of all its hyperbolic angles and sides with Euclidean analogs, preliminary, according to sine–tangent analogy (331-I) in the universal base $\tilde{E}_1$.

The *hyperbolically acute angle* γ at the vertex A is contrary to the leg $a < b$ and adjacent to the leg b. Positive scalar values of the angle are measured in direction to the main invariant diagonal off the leg $b = AC$ (i. e., *Cartesian* axis x). The *hyperbolically acute complementary angle* υ is defined by us correctly as the angle at the vertex B between the pseudo-hypotenuse $g = AB$ and the internal isotropic diagonal passing through the vertex B. Its spherical analogs are $(\pi/4 - \varphi_R)$ by (355) and $(\pi/2 - \varphi)$ by (331-I). Positive values of the angle υ are measured also in direction to the isotropic diagonal off the pseudo-hypotenuse. Identically both these acute complementary anges are defined together as it is shown at Figure 4, Ch. 12, and in Appendix at Figure 1A, Ch. 3A.

The infinite angles $\delta = \infty$ (with analog $\varphi_R = \pi/4$) are disposed, for example in the universal base $\tilde{E}_1$, between the pseudo-Cartesian axes and the internal isotropic diagonals.

The as if *hyperbolically right angle* ν is disposed between the legs a and b within of both hyperbolic quadrants I and II. The angle is equal to zero in hyperbolic metric, because it consists from two infinite antithetical angles $+\delta$ and $-\delta$ ($\varphi_R = \pm\pi/4$) (directions of these angles measurement are to one side, i. e., off $b = AC$ to $a = BC$).

The combined hyperbolic *obtuse* angle ABC at the vertex B is contrary to the leg $b > a$. Geometrically it consists from the hyperbolic *intrinsic acute angle* υ with the hyperbolic *infinite angle* $\delta = +\infty$ (it is as if the *geometric sum* $\upsilon + \delta$). *Such an obtuse angle appears on the pseudo-Euclidean graph with a compensatory angular excess due to the fact that we can use only Euclidean geometry for visualization, but* γ *and* υ *in the right triangle are acute and quasi-acute angles and with equal rights.* In the right triangle, they complement each other up to infinite angle δ. **In the plane and cylindrical hyperbolic geometries, on the Minkowski hyperbolids (Ch. 12), these complementary hyperbolic angles relate to the lengths of the hyperbolic figures legs, including of right triangles!**

When the pseudoplane is convoluted into a hyperbolic surface, these original hyperbolic angles now express the lengths of the opposite legs instead of their previous pseudo-Euclidean lengths, and the original hyperbolic angles are transformed into spherical analogs with the angular defect of Lambert in hyperbolic triangles [36]. On the tangent plane and pseudoplane to the hyperbolic surfaces, the Euclidean and pseudo-Euclidean pictures are restored!

The sine-tangent analogy determines one-to-one correspondence between 3 hyperbolic angles: γ, υ, $\delta = \infty$ (as principal of motion, complemetary and infinite with right one) and their spherical analogues: φ, ξ, $d = \pi/2$; and for three sides of the right triangle in pseudoplane and in quasiplane with respect to the **universal base** $\tilde{E}_1$ for specific analogy. Accordingly, classification of hyperbolic angles differs from spherical angles. This relates for all angles as we saw above. We add to them the independent orthospherical angle θ (or Θ).

Under this map the first *Euclidean* axis (with the leg b) is invariant, now as the first Cartesian axis; the main invariant diagonal is transformed into the second Cartesian axis (under the angle $\varphi(\delta) = \pi/2$). The leg $a = CB$ is rotated to the left at spherical angle $\varphi(\gamma)$ into the new leg $a_E = CB'$, i. e., up to its contact with the central circle of radius $g_E = g = \sqrt{b^2 - a^2}$ at the point of tangency B', now as the new vertex of the triangle $AB'C$ in the quasiplane; the pseudo-hypotenuse $g = AB$ is transformed into the new leg $AB' = g_E$ with the same length. Now the principal angle $\varphi(\gamma)$ at the vertex A is contrary to the rotated leg $a_E = a$, the complementary angle $\xi(\upsilon)$ at the vertex C is contrary to the new leg $AB' = g_E = g$, and the new right angle $d = \pi/2$ (from the infinite angle $\pm\delta$) at the vertex B' is contrary to the new hypotenuse $AC = b_E = b$. The quasi-Euclidean Pythagorean theorem is $b^2 = g^2 + a^2$. And we have two pseudo-Euclidean Pythagorean theorems with hypotenuses g and legs a. Now we can realize the covariant sine-tangent (at $\varphi \neq \pm\pi/2$) and countervariant sine-cotangent (at $\varphi \neq 0$) analogies, *together with various functional bonds of two complementary hyperbolic anges* ! Under metric tensor $\{I^{\pm}\}$ for 4 angles we obtain:

$$\left.\begin{array}{c} \sinh\gamma = a/g \equiv \tan\varphi, \quad \tan\xi = g/a \equiv \sinh\upsilon \to \sinh\gamma\cdot\sinh\upsilon = 1,\\ \cosh\gamma = b/g \equiv \sec\varphi, \quad \sin\xi = g/b \equiv \tanh\upsilon \to \cosh\gamma\cdot\tanh\upsilon = 1,\\ \cosh\gamma\cdot\tanh\upsilon = \cosh\upsilon\cdot\tanh\gamma = 1 = \mathrm{sech}\ \gamma\cdot\coth\upsilon = \mathrm{sech}\ \upsilon\cdot\coth\gamma;\\ (\gamma,\upsilon = 0 \Leftrightarrow \upsilon,\gamma = \pm\infty) \quad \Leftrightarrow \quad (\varphi,\xi = 0 \Leftrightarrow \xi,\varphi = \pm\pi/2). \end{array}\right\} \qquad (359)$$

$$\left.\begin{array}{c} \sinh\gamma = \mathrm{csch}\ \upsilon = a/g = \tan\varphi = \cot\ \xi, \ [\sinh(\gamma,\upsilon) = \mathrm{csch}\ (\upsilon,\gamma)],\\ \cosh\gamma = \coth\ \upsilon = b/g = \sec\ \varphi = \csc\ \xi, \ [\pm\cosh(\gamma,\upsilon) = \coth\ (\upsilon,\gamma)],\\ \tanh\gamma = \mathrm{sech}\ \upsilon = |a|/|b| = \sin\varphi = \cos\xi, \ [\tanh(\gamma,\upsilon) = \pm\mathrm{sech}\ (\upsilon,\gamma)];\\ (\sinh\gamma = \sinh\upsilon = 1 \ \Leftrightarrow \ \cosh\omega = \coth\omega = \sqrt{2} \ \Leftrightarrow \ \gamma = \upsilon = \omega\ !) \end{array}\right\} \qquad (360-I)$$

$$\cosh^2(\gamma,\upsilon) - \sinh^2(\gamma,\upsilon) = +1 = \coth^2(\upsilon,\gamma) - \mathrm{csch}^2(\upsilon,\gamma) - \text{two invariants !}$$

$$\tanh^2(\gamma,\upsilon) + \mathrm{sech}^2(\gamma,\upsilon) = 1 = \mathrm{sech}^2(\upsilon,\gamma) + \tanh^2(\upsilon,\gamma) - \text{two one-step quasi-invariants !}$$

$$-1 < \tanh(\gamma+\upsilon) \equiv \sin\sigma < +1 \ \Leftrightarrow \ \{-\infty = -\delta < \gamma+\upsilon < +\delta = +\infty\} - \textbf{the Theorem!}$$

For γ, υ, with (357), var. (1) and $\pm d\xi = \mp d\varphi$, we obtain useful functional and specific bonds:

$$\left.\begin{array}{c} d\upsilon,\gamma = -d\gamma,\upsilon/\sinh\gamma,\upsilon \equiv \ -d\varphi,\xi/\sin\varphi,\xi = +d\xi,\varphi/\cos\xi,\varphi \leftrightarrow\\ \leftrightarrow +\gamma,+\upsilon = \ln\coth(\upsilon/2,\gamma/2) \equiv \ln\cot(\xi/2,\varphi/2) \quad \leftrightarrow (331)\\ \leftrightarrow -\gamma,-\upsilon = \ln\tanh(\upsilon/2,\gamma/2) \equiv \ln\tan(\xi/2,\varphi/2) \quad \leftrightarrow (331)\\ \leftrightarrow \exp(-\gamma,-\upsilon) = \tanh(\upsilon/2,\gamma/2) \equiv \tan(\xi/2,\varphi/2); \ (\varphi,\xi) \in [0 \div \pi/2]. \end{array}\right\} \qquad (360-II)$$

One of applications of all analogies (360) is a natural introduction of the **various parallel angles** in non-Euclidean geometries – see above and in the end of Ch. 1A. The as if visual parallel angle of Lobachevsky $\Pi(a)$ [40, 49] is not an angle acting in hyperbolic geometry, because it has a principal spherical nature and correct only in the universal bases $\tilde{E}_{1u}$, so, in $\tilde{E}_1$. It is equal to complementary spherical angle $\xi = \Pi(a)$ in (331-II). In Lobachevsky form it is gotten by a brief way, with countervariant analogy (360-II) and only in the universal base $\tilde{E}_1$ (in fact, in the enveloping binary space $\langle\mathcal{P}^{n+1}\rangle$) as follows

$$\tan\xi/2 \equiv \exp(-\gamma) \Rightarrow \xi \equiv 2\arctan[\exp(-\gamma)] \equiv \pi/2 - \varphi \equiv \arccos(\tanh\gamma) \qquad (360-III).$$

(See it in well-known monograph by H. S. M. Coxeter [49, p. 208] with more complex infer.) From (360-II) we get the parallel and complementary angle $\upsilon = P(a)$ correct in any bases $\tilde{E}_k$, with their exact connection:

$$\tanh\upsilon/2 = \exp(-\gamma) \Rightarrow \boxed{\upsilon = 2\,\mathrm{artanh}[\exp(-\gamma)]} \ \Leftrightarrow \ \boxed{\gamma = 2\,\mathrm{artanh}[\exp(-\upsilon)]} \quad (360-IY).$$

$$\boxed{\sinh\gamma\cdot\sinh\upsilon = 1} \ \Leftrightarrow \ \boxed{\cosh^2\gamma\cdot\cosh^2\upsilon = \cosh^2\gamma + \cosh^2\upsilon} \qquad (360-Y).$$

Rule 4 (sect. 5.8) stays valid also for hyperbolic principal rotations, reflections and one-step deformations. For instance, after an change in (324) of angle Γ by complementary angle Υ with the use of formulae in (360), the new rotational function of Γ gives the rotation at Υ:

$$Roth\ \Upsilon = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \cosh \upsilon_i & \sinh \upsilon_i & \\ & \sinh \upsilon_i & \cosh \upsilon_i & \\ \hline & & & \ddots \end{array}\right] = \overline{Roth}\ \Gamma = \left[\begin{array}{c|cc|c} \ddots & & & \\ \hline & \coth \gamma_i & \operatorname{csch} \gamma_i & \\ & \operatorname{csch} \gamma_i & \coth \gamma_i & \\ \hline & & & \ddots \end{array}\right]. \quad (361)$$

And two invariant relations above correspond to these two types of rotations!

6.5 Hyperbolic tensors of rotation and deformation with frame axis

Consider matrices of quart circle (341). If a certain matrix structure in this quart circle is known, then other ones (spherical and hyperbolic) may be quickly evaluated with the use of abstract or specific spherical–hyperbolic analogies (for the latter, initially in the common universal base $\tilde{E}_1$). So, with the same natural reflector tensor as (17A-I), from spherical rotations (313), (314) or deformations (319), (320) obtained in Ch. 5 in canonical E-forms, the analogous structures for such hyperbolic matrices in canonical E-forms follow (with their useful Invariants and quasi-Invariants as above in (360)):

$$\{roth\ (\pm\Gamma)\}_{4\times4} \quad (362)$$

$1+(\cosh\gamma-1)\cos^2\alpha_1$	$(\cosh\gamma-1)\cos\alpha_1\cos\alpha_2$	$(\cosh\gamma-1)\cos\alpha_1\cos\alpha_3$	$\pm\sinh\gamma\cos\alpha_1$
$(\cosh\gamma-1)\cos\alpha_1\cos\alpha_2$	$1+(\cosh\gamma-1)\cos^2\alpha_2$	$(\cosh\gamma-1)\cos\alpha_2\cos\alpha_3$	$\pm\sinh\gamma\cos\alpha_2$
$(\cosh\gamma-1)\cos\alpha_1\cos\alpha_3$	$(\cosh\gamma-1)\cos\alpha_2\cos\alpha_3$	$1+(\cosh\gamma-1)\cos^2\alpha_3$	$\pm\sinh\gamma\cos\alpha_3$
$\pm\sinh\gamma\cos\alpha_1$	$\pm\sinh\gamma\cos\alpha_2$	$\pm\sinh\gamma\cos\alpha_3$	$\cosh\gamma$

$$\{roth\ (\pm\Gamma)\}_{(n+1)\times(n+1)} \quad (363)$$

$I_{n\times n}+(\cosh\gamma-1)\cdot\mathbf{e}_\alpha\mathbf{e}'_\alpha$	$\pm\sinh\gamma\cdot\mathbf{e}_\alpha$
$\pm\sinh\gamma\cdot\mathbf{e}'_\alpha$	$\cosh\gamma$

$(\mathbf{e}_\alpha\mathbf{e}'_\alpha = \overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha})$.

Such tensor function realizes the hyperbolic rotations at $\pm\Gamma$ also similar to acting S-arm!

$$\{defh\ (\pm\Gamma)\}_{4\times4} \quad (364)$$

$1+(\operatorname{sech}\gamma-1)\cos^2\alpha_1$	$(\operatorname{sech}\gamma-1)\cos\alpha_1\cos\alpha_2$	$(\operatorname{sech}\gamma-1)\cos\alpha_1\cos\alpha_3$	$\mp\tanh\gamma\cos\alpha_1$
$(\operatorname{sech}\gamma-1)\cos\alpha_1\cos\alpha_2$	$1+(\operatorname{sech}\gamma-1)\cos^2\alpha_2$	$(\operatorname{sech}\gamma-1)\cos\alpha_2\cos\alpha_3$	$\mp\tanh\gamma\cos\alpha_2$
$(\operatorname{sech}\gamma-1)\cos\alpha_1\cos\alpha_3$	$(\operatorname{sech}\gamma-1)\cos\alpha_2\cos\alpha_3$	$1+(\operatorname{sech}\gamma-1)\cos^2\alpha_3$	$\mp\tanh\gamma\cos\alpha_3$
$\pm\tanh\gamma\cos\alpha_1$	$\pm\tanh\gamma\cos\alpha_2$	$\pm\tanh\gamma\cos\alpha_3$	$\operatorname{sech}\gamma$

$$\{defh\ (\pm\Gamma)\}_{(n+1)\times(n+1)} \quad (365)$$

$I_{n\times n}+(\operatorname{sech}\gamma-1)\cdot\mathbf{e}_\alpha\mathbf{e}'_\alpha$	$\mp\tanh\gamma\cdot\mathbf{e}_\alpha$
$\pm\tanh\gamma\cdot\mathbf{e}'_\alpha$	$\operatorname{sech}\gamma$

$(\mathbf{e}_\alpha\mathbf{e}'_\alpha = \overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha})$.

Indicated 4×4 E-forms (362), (364) with frame axes as hyperbolic analogs of (313), (319) may be also inferred directly from their original 2×2-cells (324), (326) as the same analogs of (259), (292) with the scheme similar to (315), (316).

An inversion of E-forms (363), (365) of elementary rotational and deformational matrices consists in application of the simplest reflective operations: $\mathbf{e}_\alpha \to (-\mathbf{e}_\alpha)$ equivalent here to rotation $rot\ \Pi\cdot\mathbf{e}_\alpha = -\mathbf{e}_\alpha$ and analogical $\Gamma\to(-\Gamma)$. Generally, orthospherical rotational change of an universal base $rot\ \Theta\cdot\tilde{E}_1 = \tilde{E}_{1u}$ leads only to change of the directional cosines unity vector: $rot'\ \Theta_{n\times n}\cdot\mathbf{e}_\alpha = rot\ (-\Theta_{n\times n})\cdot\mathbf{e}_\alpha = \mathbf{e}_{\alpha'}$ within the same Euclidean subspace.

Chapter 7

Tensor trigonometric interpretation of prime matrices commutativity and anticommutativity

7.1 Commutativity of prime matrices

Two *biorthogonal* prime matrices $P_1P_2 = P_2P_1 = Z$ are commutative and anticommutative simultaneously: $P_1P_2 = +P_2P_1 = -P_2P_1 = Z$. By the reason, they always are singular: $r_1 + r_2 \leq n$. Due to commutativity, the biorthogonal matrices P_1, P_2 as prime ones may be converted also into their D-forms D_1, D_2 in a certain common base, where $D_1D_2 = Z$. Consequently, such multiplications P_1P_2 may be analyzed from the trigonometric point of view enough only for *nonsingular* prime matrices (they have not such biorthogonal blocks!).

Commutative prime matrices P_1 and P_2 are diagonalized always in some common base:

$$\overset{D(P_1)}{\begin{bmatrix} \ddots & & & \\ & a_j & & \\ & & a_k & \\ & & & \ddots \end{bmatrix}}, \quad \overset{D(P_2)}{\begin{bmatrix} \ddots & & & \\ & b_j & & \\ & & b_k & \\ & & & \ddots \end{bmatrix}}, \quad D(P) = V_{col}^{-1} P V_{col}.$$

Indeed, if $\mathbf{u}$ is any eigenvector of the matrix P_1 with the eigenvalue μ, i. e., $P_1 \cdot \mathbf{u} = \mu \cdot \mathbf{u}$, then the commutativity of the matrices P_1 and P_2 implies the equalities:

$$P_1P_2 \cdot \mathbf{u} = P_2P_1 \cdot \mathbf{u} = P_2 \cdot (\mu \cdot \mathbf{u}) = \mu \cdot (P_2 \cdot \mathbf{u}) \rightarrow P_2 \cdot \mathbf{u} = \nu \cdot \mathbf{u}.$$

From where the same eigenvector $\mathbf{u}$ of P_1 relates to P_2 too. Further, we must continue this process onto the rest invariant subspaces as the direct complements to $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{(n-1)}$. This property determines a set of common k transections of invariant subspaces of P_1, P_2 (see them in sect. 2.2) with the eigenvalues of the prime matrix $P = P_1P_2$ as $\mu_k \cdot \nu_k$, which uniquely determines the set of common bases of diagonal forms for commutative P_1 and P_2. Contrary, if prime matrices of the same size P_1 and P_2 in some common basis have their diagonal forms, then they are commutative in it and, therefore, in the original basis too.

This diagonal structures in the common base with commutativity of prime matrices are invariant under the following modal transformations of the pair binary (j, k)-th cells, that are compatible, in their affine three W-forms:

$$\overset{W_1}{\begin{bmatrix} \ddots & & & \\ & \pm c & 0 & \\ & 0 & \mp c & \\ & & & \ddots \end{bmatrix}}, \quad \overset{W_2}{\begin{bmatrix} \ddots & & & \\ & 0 & \mp d & \\ & \pm d^{-1} & 0 & \\ & & & \ddots \end{bmatrix}}, \quad \overset{W_3}{\begin{bmatrix} \ddots & & & \\ & 0 & \pm id & \\ & \pm id^{-1} & 0 & \\ & & & \ddots \end{bmatrix}}.$$

The first matrix is similar to affine reflection, it merely changes *pairly* directions of the coordinate axes, in general with their deformation. The second and third matrices are similar to rotations, they permute *pairly* the diagonal elements as well as coordinate axes (with their compression-stretching).

All compositions of such transformations of these three types form the complete set of modal matrices with respect to the invariant D-forms given above. Eigenvalues of P_1 and P_2 are supposed to be distinct, otherwise the set should be widen, it should contain base changes in the intersection of P_1 and P_2 invariant eigen subspaces with multiple eigenvalues.

The three affine types of modal matrices indicated above give rise to their admissible *trigonometric* W-forms in $\langle \mathcal{E}^n \rangle$ (i. e., at $d = 1$):

$$
\underset{Ref\{I^{\pm}\}}{\begin{bmatrix} \ddots & & & \\ & \pm 1 & 0 & \\ & 0 & \mp 1 & \\ & & & \ddots \end{bmatrix}}, \quad
\underset{Rot\ (\pm\Pi/2)}{\begin{bmatrix} \ddots & & & \\ & 0 & \mp 1 & \\ & \pm 1 & 0 & \\ & & & \ddots \end{bmatrix}}, \quad
\underset{Roth\ (\pm\, i\Pi/2)}{\begin{bmatrix} \ddots & & & \\ & 0 & \pm i & \\ & \pm i & 0 & \\ & & & \ddots \end{bmatrix}}. \tag{366}
$$

Main corollaries

1. *According to (366), the following trigonometric* **Rule** *is valid: for commutative prime matrices* $P_1, P_2, \ldots$ *their common bases of* $D(P_1)$, $D(P_2), \ldots$ *may differ in* $\langle \mathcal{E}^n \rangle$ *namely by admissible and compatible modal reflections or rotations at spherical angle-arguments* $k \cdot \Pi/2$ *or at pseudohyperbolic angle-arguments* $k \cdot i\Pi/2$, $(k = 0, \pm 1, \pm 2, \ldots)$ *under their modal transformation as bivalent tensors.*

2. *According to* **Corollary 3** *in sect. 5.7, such modal transformations are identical for the coordinates axes (similar to reflectors) to their real rotations or reflections at double spherical angle-arguments* $k \cdot \Pi$ *or pseudohyperbolic angle-arguments* $k \cdot i\Pi$, $(k = 0, \pm 1, \pm 2, \ldots)$ *under their modal transformation as monovalent tensors (either from the left or from the right).*

7.2 Anticommutativity of prime matrices pairs

If a pair of prime matrices P_1 and P_2 are anticommutative, i. e., $P_1 P_2 = -P_2 P_1$, then

$$P_1^2 P_2 = P_2 P_1^2, \quad P_1 P_2^2 = P_2^2 P_1, \quad P_1^2 P_2^2 = P_2^2 P_1^2.$$

Suppose that the pair of anticommutative prime matrices P_1, P_2 have no biorthogonal blocks (see sect. 7.1). Thus, in first, sizes of these nonsingular matrices are even and, in second, the matrices and their multiplications are nonsingular. According to the principle of binarity (sect. 5.7), they may be converted into the compatible W-forms in a certain common base $\tilde{E} = V_W\{E_1\}$ with the result:

$$
\underset{W(P_1)}{\begin{bmatrix} \ddots & & & & \\ & \cdot & \cdot & & \\ & \cdot & \cdot & & \\ & & & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & & & \ddots \end{bmatrix}}, \quad
\underset{W(P_2)}{\begin{bmatrix} \ddots & & & & \\ & \cdot & \cdot & & \\ & \cdot & \cdot & & \\ & & & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & & & \ddots \end{bmatrix}}. \qquad W(P_i) = V_W^{-1} P_i V_W, \quad i = 1, 2.
$$

Execute such modal transformation V_W of $W(P_1)$ and $W(P_2)$ *together*, in order to convert P_1 into its diagonal form. In the new common base, P_1 and P_2 as before are anticommutative.

Now the property is valid iff their compatible j-th 2×2-cells have diagonal and contradiagonal forms (it is proved by the action $D(P_1)P_2 = -P_2D(P_1)$):

$$\begin{bmatrix} \ddots & & & \\ & +a & 0 & \\ & 0 & -a & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & 0 & b_{12} & \\ & b_{21} & 0 & \\ & & & \ddots \end{bmatrix}. \tag{367}$$

If the matrix P_2 rather than P_1 is diagonalized, then 2×2-cells in the new base are

$$\begin{bmatrix} \ddots & & & \\ & 0 & a_{12} & \\ & a_{21} & 0 & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & +b & 0 & \\ & 0 & -b & \\ & & & \ddots \end{bmatrix}. \tag{368}$$

In addition in the both cases there holds: $a = \sqrt{a_{12}a_{21}}$, $b = \sqrt{b_{12}b_{21}}$ at all indices j. (The special case when both the matrices may be in contradiagonal forms – see later.)

Indeed, for the variant $\Pi_1 = P_1 \cdot P_2$, in general case, we have:

$$\Pi_1 = \begin{bmatrix} \ddots & & & \\ & a_1 & 0 & \\ & 0 & a_2 & \\ & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \ddots & & & \\ & p_{11} & p_{12} & \\ & p_{21} & p_{22} & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & a_1p_{11} & a_1p_{12} & \\ & a_2p_{21} & a_2p_{22} & \\ & & & \ddots \end{bmatrix}.$$

And, for the variant $\Pi_2 = P_2 \cdot P_1 = -\Pi_1$, in general case, we have:

$$\Pi_2 = \begin{bmatrix} \ddots & & & \\ & p_{11} & p_{12} & \\ & p_{21} & p_{22} & \\ & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \ddots & & & \\ & a_1 & 0 & \\ & 0 & a_2 & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & a_1p_{11} & a_2p_{12} & \\ & a_1p_{21} & a_2p_{22} & \\ & & & \ddots \end{bmatrix}.$$

We give P_1 in its diagonal form, and then it is necessary to find the form of P_2. Obviously, we have the initial conditions: $a_1 \neq 0, a_2 \neq 0$ (as well as $b_1 \neq 0, b_2 \neq 0$).

Further, there hold:

$$a_1p_{11} = -a_1p_{11},\ a_2p_{22} = -a_2p_{22} \to p_{11} = p_{22} = 0,$$

$$a_1p_{12} = -a_2p_{12},\ a_2p_{21} = -a_1p_{21} \to a_1 = -a_2 = +a;\ p_{12} \neq 0, p_{21} \neq 0.$$

Analogously, for diagonal elements of P_2 there hold: $b_1 = -b_2 = +b$.

After permutation of a_j in (367), for its two contradiagonal elements there holds: $\det\ P_1 = -a^2 = -a_{12} \cdot a_{21}$. Analogously, there holds: $\det\ P_2 = -b^2 = -b_{12} \cdot b_{21}$!

The covariant column matrix converting the contradiagonal form in (367) or (368) into D-form may be evaluated, for example, with the use of results in sect. 2.2.

This modal matrix may be represented in the following general *affine trigonometric form*, for example, for contradiagonal form of P_2 in (367) as its j-th 2×2-cell:

$$W_{col}^{-1}\cdot W(P)\cdot W_{col} =$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & +\frac{\sqrt{2}}{2}\sqrt{\frac{b_{12}}{b_{21}}} \\ -\frac{\sqrt{2}}{2}\sqrt{\frac{b_{21}}{b_{12}}} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\sqrt{\frac{b_{12}}{b_{21}}} \\ +\frac{\sqrt{2}}{2}\sqrt{\frac{b_{21}}{b_{12}}} & \frac{\sqrt{2}}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} +\sqrt{b_1 b_2} & 0 \\ 0 & -\sqrt{b_1 b_2} \end{bmatrix} = \begin{bmatrix} +b & 0 \\ 0 & -b \end{bmatrix} = D(P), \tag{369}$$

$$W_{col} = \{Rot\ \pi/4\}_{af} = W^{-1}\cdot\{Rot\ \pi/4\}\cdot W, \tag{370}$$

$$D(P) = W_{col}^{-1}\cdot W(P)\cdot W_{col} = W_{col}^{-1}\cdot V_W^{-1}\cdot P\cdot V_W\cdot W_{col} = V_{col}^{-1}\cdot P\cdot V_{col}. \tag{371}$$

Here $det\{Rot\ \pi/4\}_{af} = 1$, $\mu_{1,2} = \cos\pi/4 \pm i\sin\pi/4$. Formula (370) determines a spherical rotational matrix in a certain *affine base*. In the real Cartesian base, this matrix is $Rot\ \pi/4$; in complex binary Cartesian base (271), it is $Roth\ (-i\pi/4)$. Besides, due to (366)–(368), the diagonal and contradiagonal W-structures are preserved under the base rotations and reflections of their W-forms as in (366), i. e., at compatible right tensor angles.

Consider most important special cases of normal matrices anticommutativity what are related to the tensor trigonometry in $\langle\mathcal{E}^n\rangle$. In general, $a_{12} = \pm a_{21}$, $b_{12} = \pm b_{21}$, and then $V_W = R_W$. Suppose $P_1 = M_1, P_2 = M_2$ are anticommutative real-valued *normal* matrices (or *complex-valued adequately normal* ones – sect. 4.2). They may be either symmetric (S), or skew-symmetric (K). Three trigonometric variants (S_1 and S_2, S and K, K_1 and K_2) are exposed with the use of (367) and (368). One else variant corresponds to the case when the matrices S and K may be together in contradiagonal forms. (But it is combination of two simple variants.) All these variants are:

A) $a_{12} = a_{21} = +a$, $b_{12} = b_{21} = +b$; $P_1 = S_1$, $P_2 = S_2$, $S_1\cdot S_2 = -S_2\cdot S_1$. This corresponds in (183) to $S_1 = \cos\tilde\Phi$, $S_2 = \sin\tilde\Phi$ ($a^2+b^2 = 1$, $S_1^2 + S_2^2 = I$). Then

$$V_{col} = R_W\cdot\begin{bmatrix} \ddots & & & \\ & \sqrt{2}/2 & -\sqrt{2}/2 & \\ & +\sqrt{2}/2 & \sqrt{2}/2 & \\ & & & \ddots \end{bmatrix} = Rot\ \pi/4\cdot R_W.$$

B) $a_{12} = a_{21} = +a$, $-b_{12} = +b_{21} = +b/i$; $P_1 = S$, $P_2 = K$, $S\cdot K = -K\cdot S$. This corresponds in (209) to $S = \sec\tilde\Phi$, $K = i\tan\tilde\Phi$ ($a^2 - b^2 = 1$, $S^2 - K^2 = I$). Then

$$V_{col} = R_W\cdot\begin{bmatrix} \ddots & & & \\ & \sqrt{2}/2 & -i\sqrt{2}/2 & \\ & -i\sqrt{2}/2 & \sqrt{2}/2 & \\ & & & \ddots \end{bmatrix} = Roth\ i\pi/4\cdot R_W, \text{(see in scheme (322))}.$$

We have in (204) $S = \cos\tilde\Phi$, $K = i\tan\tilde\Phi$; and the unusual pair $S = \cos\tilde\Phi$, $K = i\sin\Phi$ (in the last case: $\cos\tilde\Phi\sin\Phi = (\cos\tilde\Phi\sin\Phi)' = \sin'\Phi\cos\tilde\Phi = -\sin\Phi\cos\tilde\Phi$).

C) $a_{12} = a_{21} = +a = ic$, $-b_{12} = +b_{21} = +b/i$; $K_1 \cdot K_2 = -K_2 \cdot K_1$; $-c^2 - b^2 = 1$, $-K_1^2 - K_2^2 = I$. This variant is given for completeness.

D) Begin with conditions from (B), then transform the base $\tilde{E}$ for both the matrices by $Rot\ \pi/4$. Now the matrix P_1 and the matrix P_2 (invariant to this rotation) have two *different contradiagonal forms* with the j-entries $a_{12} = a_{21} = +a$,, $-b_{12} = +b_{21} = +b/i$. This corresponds in (204) to $S = \sin\tilde{\Phi}$, $K = i\tan\tilde{\Phi}$ (or $S = \sin\bar{\Phi}$, $K = i\sin\Phi$). The bases of such anticommutative trigonometric matrices *in their diagonal forms* are differed by amalgamated rotation $Roth\ i\pi/4 \cdot Rot\ \pi/4$ or $Rot\ \pi/4 \cdot Roth\ i\pi/4$ (or by the tensor angles algebraic sum). For the matrices there hold $a_{12}b_{21} = -a_{21}b_{12}$.

The main result in the trigonometric forms is the following.

1. *Nonsingular prime matrices P_1, P_2 are anticommutative iff bases of their D-forms are connected by compatible rotations or reflections at tensor angles $\pm\pi/4$ or / and $\pm i\pi/4$.*

2. *Sizes of nonsingular anticommutative prime matrices P_1, P_2 are even.*

3. *Anticommutative singular prime matrices P_1, P_2 have compatible biorthogonal blocks, what may be converted into biorthogonal D-forms in their common sub-base.*

Note, as in the end of sect. 7.1, that the rotation angles $\pm\pi/4$ and $\pm i\pi/4$ correspond to the deformational angle $\pm\omega$ or $\pm i\omega$ (*nonperiodic*) in universal bases – see in Ch. 6.

* * *

Further consider some trigonometric examples of the complex-valued Hermitean normal matrices N_1, N_2 corresponding to examples A, B, C, exposed above. We have

$$b_1 = \rho_1(\cos\beta_1 + i\sin\beta_1), \quad b_2 = \rho_2(\cos\beta_2 + i\sin\beta_2), \quad \rho_1 > 0, \rho_2 > 0, \beta_1, \beta_2 \in [0; 2\pi],$$

$$b = \sqrt{b_1 b_2} = \sqrt{\rho_1\rho_2}\exp[i(\beta_1 + \beta_2)/2],$$

$$\sqrt{b_2/b_1} = \sqrt{\rho_2/\rho_1}\exp(i\beta_{12}), \quad \sqrt{b_1/b_2} = \sqrt{\rho_1/\rho_2}\exp(-i\beta_{12}), \quad \beta_{12} = \beta_2 - \beta_1.$$

As above, variants $b_{1j} = \pm b_{2j}$, $a_{1j} = \pm a_{2j}$, $V_W = R_W$ are possible. And more complicated cases $|b_{1j}| = |b_{2j}| = \rho_b$, $|a_{1j}| = |a_{2j}| = \rho_a$; $V_W = U_W$ are possible too.

Let $P_1 = N_1$, $P_2 = N_2$ be anticommutative *Hermitean normal* matrices. Here they may be Hermitean or skew-Hermitean, this corresponds to three anticommutative pairs: H_1 and H_2, H and Q, Q_1 and Q_2. The affine spherical unitary modal matrix V_{col} is

$$\begin{bmatrix} \ddots & & & \\ & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\cdot\exp(-i\beta_{12}) & \\ & +\frac{\sqrt{2}}{2}\cdot\exp(+i\beta_{12}) & \frac{\sqrt{2}}{2} & \\ & & & \ddots \end{bmatrix} = \left\{Exp\frac{-i\beta_{12}}{2}\cdot Rot\ \pi/4\cdot Exp\frac{+i\beta_{12}}{2}\right\}, \quad (372)$$

(it is more general rotational complex modal matrix, then ones used above), where

$$Exp(+i\beta_{12}/2) = \{Rot\ (+\beta_{12}/2)\}_c = U_W \cdot \begin{bmatrix} \ddots & & & \\ & \exp(+i\beta_{12}/2) & 0 & \\ & 0 & \exp(-i\beta_{12}/2) & \\ & & & \ddots \end{bmatrix} \cdot U_W^*.$$

And if $\beta_{12} = \beta_2 - \beta_1 = \pi/2$, then the modal matrix is $Roth\ i\pi/4$ in variant (B) above. It corresponds to the *complex-valued binary Cartesian base* – see (287) in sect. 5.9.

More generally, formula (372) expresses *Rot* $\pi/4$ in a Hermitean orthogonal base with imaginary shift at the angle $i\beta_{12}$ in formulae (367) and (368):

$$\overset{N_1}{\begin{bmatrix} \ddots & & & \\ & +\rho_a \exp[i(\alpha_1+\alpha_2)/2] & 0 & \\ & 0 & -\rho_a \exp[i(\alpha_1+\alpha_2)/2] & \\ & & & \ddots \end{bmatrix}}, \quad \overset{N_2}{\begin{bmatrix} \ddots & & & \\ & 0 & \rho_b \exp(i\beta_1) & \\ & \rho_b \exp(i\beta_2) & 0 & \\ & & & \ddots \end{bmatrix}};$$

$$\overset{N_1}{\begin{bmatrix} \ddots & & & \\ & 0 & \rho_a \exp(i\alpha_1) & \\ & \rho_a \exp(i\alpha_2) & 0 & \\ & & & \ddots \end{bmatrix}}, \quad \overset{N_2}{\begin{bmatrix} \ddots & & & \\ & +\rho_a \exp[i(\beta_1+\beta_2)/2] & 0 & \\ & 0 & -\rho_a \exp[i(\beta_1+\beta_2)/2] & \\ & & & \ddots \end{bmatrix}}.$$

For the pair $N_1 \cdot N_2 = -N_2 \cdot N_1$ three important special cases as above are possible.

A) $\beta_{1j} + \beta_{2j} = \alpha_{1j} + \alpha_{2j} = 0$. Then N_1 and N_2 are the anticommutative Hermitean matrices $P_1 = H_1$, $P_2 = H_2$. In the special case $a_j^2 + b_j^2 = 1$, then these matrices are the projective Hermiteized cosine and sine, and $H_1^2 + H_2^2 = I$, $H_1 \cdot H_2 = -H_2 \cdot H_1$.

B) $\beta_{1j} + \beta_{2j} = \pi$, $\alpha_{1j} + \alpha_{2j} = 0$. Then N_1 and N_2 are the anticommutative Hermitean and skew-Hermitean matrices $P_1 = H$, $P_2 = Q$. In the special case $a_j^2 - b_j^2 = 1$, then these matrices are the projective Hermiteized secant and skew-Hermiteized tangent, and $H^2 + Q^2 = I$, $H \cdot Q = -Q \cdot H$.

C) $\beta_{1j} + \beta_{2j} = \alpha_{1j} + \alpha_{2j} = \pi$. Then N_1 and N_2 are the anticommutative skew-Hermitean matrices $P_1 = Q_1$, $P_2 = Q_2$., and $-Q_1^2 - Q_2^2 = I$.

Thus, all most important types of anticommutative prime matrices types are described!

Of course, *according to Corollary 3 in sect. 5.7, such modal transformations are identical for the coordinates axes (similar to reflectors) also to their real rotations or reflections at the double spherical angle-arguments $k \cdot \Pi/2$ or the pseudohyperbolic angle-arguments $k \cdot i\Pi/2$, $(k = 0, \pm 1, \pm 2, \dots)$ under their modal transformation as monovalent tensors (either from the left or from the right).*

Therefore, in any case, anticommutativity of prime matrices P_1 and P_2 is fixed at the trigonometric divergence of their diagonal forms with the principal angle exactly half that for the case of their commutativity!

Chapter 8

Tensor trigonometric spectra with general inequalities

8.1 Trigonometric spectrum of a null-prime matrix

Matrix characteristic coefficients of higher orders, as well as eigenprojectors, are prime singular matrices with a unique eigenvalue (see Ch. 1 and 2). Consider a null prime matrix B with its coefficient $K_2(B,r)$ of the highest order r and angle $\tilde{\Phi}_B$. Represent $K_2(B,r)$ as an algebraic orthogonal sum over eigen trigonometric subspaces of $\tilde{\Phi}_B$:

$$K_2(B,r) = \sum_{i=1}^{r-\nu'} \overrightarrow{S_i} \cdot K_2(B,r) \cdot \overrightarrow{S_i} + \overrightarrow{S_m} \cdot K_2(B,r) \cdot \overrightarrow{S_m}, \tag{373}$$

where $\overrightarrow{S_i} = \overrightarrow{\cos^2 \tilde{\Phi}_B - \cos^2 \varphi_i \cdot I}$ is the orthogonal projector into the i-th trigonometric eigen plane $\langle \mathcal{P}_i \rangle$ – see (240), $\overrightarrow{S_m} = \overrightarrow{\cos \tilde{\Phi}_B - I}$ is the orthogonal projector into the subspace $\langle \mathcal{P}_m \rangle \equiv \langle im\ B \rangle \cap \langle im\ B' \rangle$ of dimension ν' (see Figure 2). *Here $\nu'' = 0$ as the matrix B is null-prime!* The orthoprojectors form too the complete algebraic sum;

$$\sum_{i=1}^{r-\nu'} \overrightarrow{S_i} + \overrightarrow{S_m} + \overrightarrow{S_q} = I,$$

where $\overrightarrow{S_q} = \overrightarrow{\cos \tilde{\Phi}_B + I}$ is the orthogonal projector into the subspace $\langle \mathcal{P}_q \rangle \equiv \langle ker\ B \rangle \cap \langle ker\ B' \rangle$ of dimension $n - 2r + \nu'$ (Figure 2). The entire sum of these dimensions $2(r-\nu') + \nu' + (n - 2r + \nu') = n$ is equal to dimension of the whole Euclidean space. In the direct sum, according to the principle of binarity (see sect. 5.7), we have the following. The coefficient $K_2(B,r)$ in the subspace $\langle \mathcal{P}_i \rangle$ is a singular matrix of rank 1 and of size 2×2, the coefficient $K_2(B,r)$ in the subspace $\langle \mathcal{P}_m \rangle$ is a nonsingular matrix of size $\nu' \times \nu'$, and the coefficient $K_2(B,r)$ in the space $\langle \mathcal{P}_q \rangle$ is the zero $(n-2r+\nu') \times (n-2r+\nu')$-matrix. Thus,

$$K_2(B,r) = \sum_{i=1}^{r-\nu'} \boxplus\, B_i^{2\times 2} \boxplus det\ B_m^{\nu'\times\nu'} \cdot I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}, \tag{374}$$

where mark $\boxplus$ stands for *direct orthogonal summation*; $r - \nu' \geq 0$, $n - 2r + \nu' \geq 0$ and, consequently, there hold:

$$2r - n \leq \nu' \leq r. \tag{375}$$

If B is a null-normal matrix (see sect. 2.4), then formula (374) is the simplest:

$$K_2(B,r) = det\ B_m^{r\times r} \cdot I^{r\times r} \boxplus Z^{(n-r)\times(n-r)}.$$

We used especial notation beginning with formula (374):

$B_i^{2\times 2}$ for a 2×2-matrix of rank 1, its highest matrix coefficient is, according to (29), the matrix itself, its highest scalar coefficient is the trace of the matrix;

$B_m^{\nu'\times\nu'}$ stands for a $\nu' \times \nu'$-matrix of rank ν', its highest matrix and scalar coefficients are $det\ B_m^{\nu'\times\nu'} \cdot I^{\nu'\times\nu'}$ and $det\ B_m^{\nu'\times\nu'}$ respectively

$Z^{(n-2r+\nu')\times(n-2r+\nu')}$ is the zero matrix of indicated size not intersecting with $B_i^{2\times 2}$.

The total singularity of B and of $K_2(B,r)$ is $(r-\nu') + (n-2r+\nu') = n - r$.

Formula (374) may be transformed, with the use of (62) for $r = 2$ and $r = n$, into the direct trigonometric spectrum of the eigen oblique projector $\overleftarrow{B}$, it is called the *trigonometric spectrum of a null-prime matrix* B:

$$\overleftarrow{B} = \frac{K_2(B,r)}{k(B,r)} = \sum_{i=1}^{r-\nu'} \boxplus \frac{B_i^{2\times 2}}{tr\ B_i^{2\times 2}} \boxplus I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}. \tag{376}$$

Similar algebraic representation of the coefficient $K_2(BB',r)$ of the highest order and the eigen orthoprojector $\overleftarrow{BB'}$, as the *trigonometric spectrum of a multiplicative matrix* BB', are derived, according to the principle of binarity (see sect. 5.7):

$$K_2(BB',r) = \sum_{i=1}^{r-\nu'} \overrightarrow{S_i} \cdot K_2(BB',r) \cdot \overrightarrow{S_i} + \overrightarrow{S_m} \cdot K_2(BB',r) \cdot \overrightarrow{S_m}, \tag{377}$$

$$K_2(BB',r) = \sum_{i=1}^{r-\nu'} \boxplus B_i^{2\times 2}(B')_i^{2\times 2} \boxplus det^2\ B_m^{\nu'\times\nu'} \cdot I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}, \tag{378}$$

$$\overleftarrow{BB'} = \frac{K_2(BB',r)}{k(BB',r)} = \sum_{i=1}^{r-\nu'} \boxplus \frac{B_i^{2\times 2}(B')_i^{2\times 2}}{tr\ [B_i^{2\times 2}(B')_i^{2\times 2}]} \boxplus I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}. \tag{379}$$

Note, that for a null-prime matrix B'. we use similar algebraic representations of the coefficient $K_2(B'B,r)$ and the eigen orthoprojector $\overleftarrow{B'B}$.

From direct spectra (374), (376) and (378), (379) we infer multiplicative formulae for the highest scalar coefficients for matrices B (or B') and BB' (or $B'B$):

$$k(B,r) = \prod_{i=1}^{r-\nu'} tr\ B_i^{2\times 2} det\ B_m^{\nu'\times\nu'} = \prod_{i=1}^{r-\nu'} tr\ (B')_i^{2\times 2} det\ (B')_m^{\nu'\times\nu'} = k(B',r), \tag{380}$$

$$k(BB',r) = \prod_{i=1}^{r-\nu'} tr\ [B_i^{2\times 2} \cdot (B')_i^{2\times 2}] det^2\ B_m^{\nu'\times\nu'} = k(B'B,r). \tag{381}$$

8.2 The general Cosine inequality

For null-prime matrices $rank\{\cos\tilde{\Phi}_B\} = n$ $(\nu'' = 0)$, and due to (186), (194) we have

$$\overleftarrow{BB'} = \overleftarrow{B} \cdot \overleftarrow{B'} \cdot \cos^2\tilde{\Phi}_B = (\overleftarrow{B} \cdot \cos\tilde{\Phi}_B) \cdot (\overleftarrow{B} \cdot \cos\tilde{\Phi}_B)'. \tag{382}$$

In $\overleftarrow{B} \cdot \overleftarrow{B'} \cdot \cos^2\tilde{\Phi}_B$, represent all the matrices as direct spectra, obtain the following inequalities for each trigonometric cell with the use of the principle of binarity:

$$0 \le \cos^2\varphi_i = \frac{tr^2\ B_i^{2\times 2}}{tr\ [B_i^{2\times 2} \cdot (B')_i^{2\times 2}]} \le 1. \tag{383}$$

From (380), (381), and (383) the *general cosine inequality in the normalized form* for a square matrix (where $\varphi_i \in (0;\pi/2]$), i. e., in variant (138), follows:

$$0 \le \prod_{i=1}^{r-\nu'} \cos^2\varphi_i = |\{B\}|^2_{\cos} = |det\ \cos\tilde{\Phi}_B| = \frac{k^2(B,r)}{k(BB',r)} \le 1. \tag{384}$$

Here $|\{B\}|_{\cos}$ defines the *cosine norm* of $\tilde{\Phi}_B$ and Φ_B. Its extremal special cases are: $|\{B\}|_{\cos} = 0$ if B is a null-defected matrix, $|\{B\}|_{\cos} = 1$ if B is a null-normal matrix. In terms of the dianal and the minorant of B (see Ch. 3) the general cosine inequality and the cosine norm of $\tilde{\Phi}_B$ and Φ_B (or the cosine ratio for B) are expressed as

$$0 \le \frac{|\mathcal{D}l(r)B|}{\mathcal{M}t(r)B} = |\{B\}|_{\cos} = \frac{|\mathcal{D}l(r)B|}{\sqrt{\mathcal{D}l(r)BB'}} \le 1.$$

Consider $(\overleftarrow{B} \cdot \cos\tilde{\Phi}_B) \cdot (\overleftarrow{B} \cdot \cos\tilde{\Phi}_B)'$ in (382) and obtain similar cosine inequalities in *the sign form* (where $\varphi_t \in (0;\pi]$):

$$-1 \le \cos\varphi_t = \frac{tr\ B_t^{2\times 2}}{\sqrt{tr\ \{B_t^{2\times 2} \cdot (B')_t^{2\times 2}\}}} \le +1. \tag{385}$$

The cosine ratio $|\{B\}|_{\cos}$ is supplemented by the *signed cosine ratio* as in variant (137):

$$-1 \le \prod_{t=1}^{r-\nu'} \cos\varphi_t = \{B\}_{\cos} = \frac{k(B,r)}{\sqrt{k(BB',r)}} = \frac{\mathcal{D}l(r)B}{\mathcal{M}t(r)B} = \frac{\mathcal{D}l(r)B}{\sqrt{\mathcal{D}l(r)BB'}} \le +1. \tag{386}$$

The extreme cases (at values ± 1) correspond to the null-normal matrices B with the positive or negative dianals – see in (138), Ch. 3. Note (!), that Inequality (386), as new one, is independent on the Inequality of Hermann Weyl for the eigen and singular numbers of $n\times n$-matrix B [7]. Both Inequalities intersect in the trivial case of non-singular matrix B.

The *cosine distinct ranges* of the angles is similar to that for the angle between two undirected vectors and the angle between two directed vectors (or straight lines). (But the *sine distinct ranges* of the angles give algebraically $\varphi_t \in [-\pi/2; +\pi/2]$ – Ch. 3.)

Corollary. *For spherical functions of tensor angles $\tilde{\Phi}_B$ and Φ_B, their eigen angles φ_t have the following trigonometric sense: they are the scalar angles between planars or lineors, given by matrices $B_t^{2\times 2}$ and $B_t'^{2\times 2}$ of rank 1 in the trigonometric spectra of the eigen projectors $\overleftarrow{B}$ and $\overleftarrow{B'}$* (see (186)–(189), (190)-(193) and Figure 1).

$|\{B\}|_{\cos}$ is the cosine ratio for the *planars* $\langle im\ B\rangle$, $\langle im\ B'\rangle$ as well as the planars $\langle ker\ B\rangle$, $\langle ker\ B'\rangle$; but $\{B\}_{\cos}$ is the cosine ratio for *lineors* determined by B and B'.

If a binary tensor angle $\tilde{\Phi}_{12}$ is determined by equirank lineors A_1, A_2 or planars $\langle im\ A_1\rangle$, $\langle im\ A_2\rangle$, then scalar angles φ_t in cells have the similar sense. Suppose, for $B = A_1A_2'$ condition (224) holds, and consequently bijection (226) between eigen orthoprojectors takes place. The trigonometric spectra for external multiplications are

$$K_2(AA',r) = \sum_{t=1}^{r-\nu'} \overrightarrow{S_t} \cdot K_2(AA',r) \cdot \overrightarrow{S_t} + \overrightarrow{S_m} \cdot K_2(AA',r) \cdot \overrightarrow{S_m} \equiv$$

$$\equiv \sum_{t=1}^{r-\nu'} \boxplus\ (AA')_t^{2\times 2} \boxplus det\ (AA')^{\nu'\times\nu'} \cdot I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}, \tag{387}$$

$$K_2(A_1A_2',r) = \sum_{t=1}^{r-\nu'} \overrightarrow{S_t} \cdot K_2(A_1A_2',r) \cdot \overrightarrow{S_t} + \overrightarrow{S_m} \cdot K_2(A_1A_2',r) \cdot \overrightarrow{S_m} \equiv$$

$$\equiv \sum_{t=1}^{r-\nu'} \boxplus\ (A_1A_2')_t^{2\times 2} \boxplus det\ (A_1A_2')^{\nu'\times\nu'} \cdot I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}, \tag{388}$$

$$\overleftarrow{AA'} = \frac{K_2(AA',r)}{k(AA',r)} = \sum_{t=1}^{r-\nu'} \boxplus\ \frac{(AA')_t^{2\times 2}}{tr\ ((AA')_t^{2\times 2}} \boxplus I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}, \tag{389}$$

$$\overleftarrow{A_1A_2'} = \frac{K_2(A_1A_2', r)}{k(A_1A_2', r)} = \sum_{i=1}^{r-\nu'} \boxplus \frac{(A_1A_2')_i^{2\times 2}}{tr\ ((A_1A_2')_i^{2\times 2}} \boxplus I^{\nu'\times\nu'} \boxplus Z^{(n-2r+\nu')\times(n-2r+\nu')}, \qquad (390)$$

$$k(AA', r) = \prod_{i=1}^{r-\nu'} tr\ (AA')_i^{2\times 2} det\ (AA')^{\nu'\times\nu'} = det\ (A'A), \qquad (391)$$

$$k(A_1A_2', r) = \prod_{i=1}^{r-\nu'} tr\ (A_1A_2')_i^{2\times 2} det\ (A_1A_2')^{\nu'\times\nu'} = det\ (A_1'A_2). \qquad (392)$$

According to (132) there holds $det^2\ (A_1A_2')_i^{\nu'\times\nu'} = det\ (A_1A_1')_i^{\nu'\times\nu'} \cdot det\ (A_2A_2')_i^{\nu'\times\nu'}$. Then further, from (186), (187), (196), and (226) we obtain

$$\overleftarrow{A_1A_1'} \cdot \overleftarrow{A_2A_2'} = \overleftarrow{A_1A_2'} \cdot \cos^2 \tilde{\Phi}_{12} = (\overleftarrow{A_1A_2'} \cos \tilde{\Phi}_{12}) \cdot (\overleftarrow{A_2A_1'} \cdot \cos \tilde{\Phi}_{12}). \qquad (393)$$

In addition, intermediately, by (68) in its special case for $n = 2$, and the obvious relation $[A_2'A_1]_i = [A_1'A_2]_i$, for the i-th 2×2-cells of rank 1 there holds

$$(A_1A_2')_i^{2\times 2} \cdot (A_1A_2')_i^{2\times 2} = tr\ (A_1A_2')_i^{2\times 2} \cdot (A_1A_2')_i^{2\times 2} = (A_1A_1')_i^{2\times 2} \cdot (A_2A_2')_i^{2\times 2}. \qquad (394)$$

Represent the matrices in (393) as direct spectra and apply (394) in all the i-th cells, obtain the i-th elementary cosine inequalities

$$0 \le \cos^2 \varphi_i = \frac{tr^2\ (A_1A_2')_i^{2\times 2}}{tr\ (A_1A_1')_i^{2\times 2} tr\ (A_2A_2')_i^{2\times 2}} \le 1, \qquad (395)$$

and the general cosine inequality for equirank lineors A_1, A_2 in *the normalized form*:

$$0 \le \prod_{i=1}^{r-\nu'} \cos^2 \varphi_i = |\{A_1A_2'\}|^2_{\cos} = |det\ \cos \tilde{\Phi}_{12}| = \frac{\mathcal{D}l^2(r)(A_1A_2')}{\mathcal{M}t^2(r)A_1 \cdot \mathcal{M}t^2(r)A_2} \le 1, \qquad (396)$$

where φ_i are the scalar angles between the planars $\langle im\ (A_1A_1')_i^{2\times 2}\rangle \equiv \langle im\ (A_1A_2')_i^{2\times 2}\rangle$ and $\langle im\ (A_2A_2')_i^{2\times 2}\rangle \equiv \langle im\ (A_2A_1')_i^{2\times 2}\rangle$. Under condition (224) there holds (sect. 3.3):

$$0 \le \prod_{i=1}^{r-\nu'} \cos^2 \varphi_i = |\{A_1A_2'\}|^2_{\cos} = |det\ \cos \tilde{\Phi}_{12}| = \frac{det^2\ (A_1'A_2)}{det\ (A_1'A_1) \cdot det\ (A_2'A_2)} \le 1,$$

(for non-orthogonal lineors: $rank\{\cos \tilde{\Phi}_{12}\} = n\ (\nu'' = 0)$). The extremal cases are
$|\{A_1A_2'\}|_{\cos} = 1$ if the lineors are entirely parallel, $\{A_1A_2'\}$ is null-normal;
$|\{A_1A_2'\}|_{\cos} = 0$ if the lineors are orthogonal, may be partially, $\{A_1A_2'\}$ is null-defected.
This general cosine inequality is a direct product of the particular Cauchy Inequalities [24]. It is inferred through the external or internal multiplications of cosine type of two lineors.

The signed forms of these inequalities and the cosine ratio are

$$-1 \le \cos \varphi_i = \frac{tr\ (A_1A_2')_i^{2\times 2}}{\sqrt{tr\ (A_1A_1')_i^{2\times 2} \cdot tr\ (A_2A_2')_i^{2\times 2}}} \le +1, \qquad (397)$$

$$-1 \le \prod_{i=1}^{r-\nu'} \cos \varphi_i = \{A_1A_2'\}_{\cos} = \frac{\mathcal{D}l(r)(A_1A_2')}{\mathcal{M}t(r)A_1 \cdot \mathcal{M}t(r)A_2} \le +1. \qquad (398)$$

The numerators and denominators in (384) and (396) under condition (224) are the same in accordance with (132). (If $r_1 \ne r_2$, then the cosine ratio formally is 0.)

In general cosine inequality (396), the value $|\{A_1A_2'\}|_{\cos}$ determines the *cosine norm* of $\tilde{\Phi}_{12}$ and Φ_{12}. In the special case $r = 1$, formula (396) is *the module form* of the geometric Cauchy Inequality for two vectors [24]. The Cauchy Inequality is used in analytical geometry for normalizing the angle between two vectors in $[0;\pi/2]$. *The sign form* of the inequality similar to (141) determines the signed cosine of the angle between two directed vectors in $[0;\pi]$. It is the same special case of (398). Initially, the Cauchy Inequality had the pure algebraic character. General inequalities (384), (386), and (396), (398) may be considered from the algebraic point of view too if they are applied to scalar elements of matrices.

From (229), (230) the following *internal multiplication criterion for at least the partial orthogonality of two equirank $n \times r$-lineors* is inferred:

$$det\ C_{12} = det\ (A_1'A_2) = 0 \ \Leftrightarrow\ \{A_1A_2'\}_{\cos} = 0. \tag{399}$$

So, we see (or those who wish to see it) that the generalization of the classic algebraic and trigonometric Inequality of Cauchy for a pair of vectors onto the above matrix geometric objects, and which we previously anticipated in Chapter 3, is now strictly justified for such general objects in fundamental (386) and (398), thanks to our discovery and application of the tensor trigonometric spectrum of matrices.

We also revealed that the cosine inequality (386) in its form using eigenvalues for a zero-normal matrix, in fact, only very successfully complements the well-known Inequality of Hermann Weyl [7] for the eigen and singular numbers of $n \times n$-matrix B [7], but at the same time, it is completely independent on it and has the relation not only to matrix algebra, but also to the geometry of lineors, as a generalization of vectors or vector-columns. of a matrix in our Tensor Trigonometry. However, both Inequalities intersect in the trivial case of non-singular matrix B.

8.3 Spectral-cell presentations of tensor trigonometric functions

Now it is possible to consider in details the structures of tensor trigonometric functions at the level of elementary 2×2-cells. It was shown in Ch. 5 that the eigen trigonometric planes corresponding to 2×2-cells are the same for projective and motive tensor angles. That is why from the left side of (301) and spectral formula (389) we obtain the following rotational connection between two equirank planars

$$\begin{bmatrix} \cos\varphi_i & -\sin\varphi_i \\ +\sin\varphi_i & \cos\varphi_i \end{bmatrix} \cdot \frac{(A_1A_1')_i^{2\times2}}{tr\ (A_1A_1')_i^{2\times2}} \cdot \begin{bmatrix} \cos\varphi_i & +\sin\varphi_i \\ -\sin\varphi_i & \cos\varphi_i \end{bmatrix} = \frac{(A_2A_2')_i^{2\times2}}{tr\ (A_2A_2')_i^{2\times2}}.$$

Further, represent the 2×2-cell $[\overleftarrow{AA'}]_i^{2\times2}$ of rank 1 for the eigen projector $\overleftarrow{AA'}$ as the following exterior multiplication of the unity 2×1-vector $\mathbf{e}_i$:

$$[\overleftarrow{AA'}]_i^{2\times2} = \frac{(AA')_i^{2\times2}}{tr\ (AA')_i^{2\times2}} = \mathbf{e}_i\mathbf{e}_i' = \overleftarrow{\mathbf{e}_i\mathbf{e}_i'}.$$

Here the unity 2×1-vector $\mathbf{e}_i$ determines the i-th basic line of the planar $\langle im\ A\rangle$ in the i-th eigen plane of the binary tensor angle $\tilde{\Phi}_{12}$. Respectively the two sides of this tensor angle between planars $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ of rank r at the level of 2×2-cells may be represented as two unity eigenvectors (straight lines). They may be transformed into each other with rotation or reflection due to (301). Express the Cartesian coordinates of these vectors as

$$\mathbf{e}_1 = \begin{bmatrix} \cos\varphi_1 \\ \sin\varphi_1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} \cos\varphi_2 \\ \sin\varphi_2 \end{bmatrix}.$$

Then their rotational transformation is

$$\mathbf{e}_2 = \begin{bmatrix} \cos\varphi_{12} & -\sin\varphi_{12} \\ +\sin\varphi_{12} & \cos\varphi_{12} \end{bmatrix} \cdot \mathbf{e}_1, \quad \varphi_{12} = \varphi_2 - \varphi_1.$$

The vector $\mathbf{e_1}$ and each of two its orthoprojections are rotated at the same angle. And according to definition (171), the tensor cosine at the level of elementary 2×2-cells is

$$[\cos\tilde{\Phi}_{12}]^{2\times 2} = \overleftarrow{\mathbf{e_1 e_1'}} + \overleftarrow{\mathbf{e_2 e_2'}} - I^{2\times 2} = \mathbf{e_1 e_1'} + \mathbf{e_2 e_2'} - I^{2\times 2}.$$

This initial trigonometric definition (171) with (165 - II) and (177) gives correct final result:

$$[\cos\tilde{\Phi}_{12}]^{2\times 2} = \cos\varphi_{12}\cdot\begin{bmatrix} +\cos(\varphi_1+\varphi_2) & \sin(\varphi_1+\varphi_2) \\ \sin(\varphi_1+\varphi_2) & -\cos(\varphi_1+\varphi_2)\end{bmatrix} = \cos\varphi_{12}\cdot\{\sqrt{I_{2\times 2}}\}_c =$$

$$= \cos\varphi_{12}\cdot[\cos\,(\tilde{\Phi}_1+\tilde{\Phi}_2) + \sin\,(\tilde{\Phi}_1+\tilde{\Phi}_2)] = \cos\varphi_{12}\cdot Ref_{\boxplus}\{+(\tilde{\Phi}_1+\tilde{\Phi}_2)\}. \qquad (400)$$

Here

$$\cos(\varphi_2-\varphi_1)\cdot\cos(\varphi_2+\varphi_1) = \cos^2\varphi_2+\cos^2\varphi_1-1,$$
$$\cos(\varphi_2-\varphi_1)\cdot\sin(\varphi_2+\varphi_1) = \cos\varphi_2\sin\varphi_2+\cos\varphi_1\sin\varphi_1.$$

Consider a 2×2-cell of the tensor sine. And according to definition (163) it is

$$[\sin\tilde{\Phi}_{12}]^{2\times 2} = \overleftarrow{\mathbf{e_2 e_2'}} - \overleftarrow{\mathbf{e_1 e_1'}} = \mathbf{e_2 e_2'} - \mathbf{e_1 e_1'}.$$

This initial trigonometric definition (163) with (165 - I) and (176) gives corrrect final result:

$$[\sin\tilde{\Phi}_{12}]^{2\times 2} = \sin\varphi_{12}\cdot\begin{bmatrix} -\sin(\varphi_1+\varphi_2) & \cos(\varphi_1+\varphi_2) \\ \cos(\varphi_1+\varphi_2) & +\sin(\varphi_1+\varphi_2)\end{bmatrix} = \sin\varphi_{12}\cdot\{\sqrt{I_{2\times 2}}\}_s =$$

$$= \sin\varphi_{12}\cdot[\cos\,(\tilde{\Phi}_1+\tilde{\Phi}_2) - \sin\,(\tilde{\Phi}_1+\tilde{\Phi}_2)] = \sin\varphi_{12}\cdot Ref_{\boxplus}\{-(\tilde{\Phi}_1+\tilde{\Phi}_2)\}. \qquad (401)$$

Here

$$\sin(\varphi_2-\varphi_1)\cdot\sin(\varphi_2+\varphi_1) = \sin^2\varphi_2-\sin^2\varphi_1,$$
$$\sin(\varphi_2-\varphi_1)\cdot\cos(\varphi_2+\varphi_1) = \cos\varphi_2\sin\varphi_2-\cos\varphi_1\sin\varphi_1.$$

Condition $\varphi_1+\varphi_2=0$ and its tensor form $\tilde{\Phi}_1+\tilde{\Phi}_2=\tilde{Z}$ determines the Cartesian base of the diagonal cosine, i. e., the trigonometric base for angles $\tilde{\Phi}$ and Φ. Under this condition all tensor angles and their trigonometric functions as well as all their eigenreflectors have canonical forms determined in Ch. 5. Secants and tangents of tensor angles have similar representations. The mirror of the mid-reflector (253) is the mid-subspase of a tensor angle, it is clearly seen in the 2×2-cells considered above.

8.4 The general Sine inequality

The sine ratio (135) defines the *sine trigonometric norm of a tensor angle*. It is nonzero if the two lineors are completely linearly independent. From (227), (228) the following *internal multiplication criterion for at least partial parallelism or linear dependence of two lineors of sizes $n\times r_1$ and $n\times r_2$* or planars $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ is derived:

$$det\ G_{1,2} = det\ [(A_1|A_2)'(A_1|A_2)] = 0 \ \Leftrightarrow\ |\{A_1A_2'\}|_{\sin} = 0. \qquad (402)$$

Similar to the cosine ratio, the sine ratio may be represented as direct product of sine ratio (124) in each eigen planes according to the lineors sine trigonometric spectrum.

If lineors A_1 and A_2 are linearly independent, the superposition matrix $(A_1|A_2)$ has rank $r_1+r_2\le n$. Its external homomultiplication $B_{1,2} = [(A_1|A_2)(A_1|A_2)']$ is a symmetric positive (semi-)definite $n\times n$-matrix. Due to (120) and (402) we have

$$k(B_{1,2}, r_1+r_2) = det\ G_{1,2} \ge 0. \qquad (403)$$

Then, by the analogy with (135), through the *external multiplication* $\{B_{1,2}, r_1 + r_2\}$ *(or internal multiplication* $\{G_{1,2}\}$*) of these two lineors of sine type*, we obtain

$$|\{A_1|A_2\}|^2_{\sin} = \frac{\mathcal{M}t^2(r_1+r_2)\{A_1|A_2\}}{\mathcal{M}t^2(r_1)A_1 \cdot \mathcal{M}t^2(r_2)A_2} = \frac{k(B_{1,2}, r_1+r_2)}{k(A_1A_1', r_1)k(A_2A_2', r_2)}. \qquad (404)$$

In addition, due to (62), (159), and (163), *for two completely linearly independent lineors* A_1 and A_2 ($\nu' = 0$), in the subspace of non-zero values of $\sin \Phi_{12}$, there holds

$$\overleftarrow{\sin \tilde{\Phi}_{12}} = \overleftarrow{B_{12}} = \frac{K_2(B_{1,2}, r_1+r_2)}{k(B_{1,2}, r_1+r_2)}, \quad (\nu' = 0 \to rank\{\sin \tilde{\Phi}_{12}\} = r_1 + r_2 \leq n). \qquad (405)$$

Consider the trigonometric spectrum of the coefficient $K_2(B_{1,2}, r_1 + r_2)$ and express it as the following algebraic sum with the use of the principle of binarity:

$$K_2(B_{1,2}, r_1+r_2) = \sum_{i=1}^{r_1-\nu''} \overrightarrow{S_i} \cdot K_2(B_{1,2}, r_1+r_2) \cdot \overrightarrow{S_i} + \overrightarrow{S_d} \cdot K_2(B_{1,2}, r_1+r_2) \cdot \overrightarrow{S_d}. \qquad (406)$$

Here $\overrightarrow{S_d}$ is the orthogonal projector into the defect subspace of intersections $\langle \mathcal{P}_d \rangle \equiv \langle \langle im\ A_2 \cap ker\ A_1' \rangle \cup \langle im\ A_1 \cap ker\ A_2' \rangle \rangle$ of dimension $(r_2 - r_1 + 2\nu'')$.

This coefficient may be represented also as the direct orthogonal sum

$$K_2(B_{1,2}, r_1+r_2) = \sum_{j=1}^{r_1-\nu''} \boxplus\ det\ [(A_1|A_2)(A_1|A_2)']_j^{2\times 2} \cdot I_j^{2\times 2} \boxplus det\ (A_1A_1')^{\nu''\times\nu''} \cdot\ I^{\nu''\times\nu''} \boxplus$$

$$\boxplus\ det\ (A_2A_2')^{(r_2 - r_1 + \nu'')\times(r_2 - r_1 + \nu'')} \cdot I^{(r_2 - r_1 + \nu'')\times(r_2 - r_1 + \nu'')} \boxplus$$

$$\boxplus\ Z^{(n - r_1 - r_2)\times(n - r_1 - r_2)}, \qquad (407)$$

where (as the illustration see Figure 2):
$[(A_1|A_2)(A_1|A_2)']_j^{2\times 2}$ is the nonsingular 2×2-matrix of rank 2, which corresponds to j-th trigonometric cell, its highest matrix coefficient is evaluated by (29), and the highest scalar coefficient is its determinant (their summary dimension here is $2(r_1 - \nu'')$;
$(A_1A_1')^{\nu''\times\nu''}$ and $(A_2A_2')^{\nu''\times\nu''}$ are the nonsingular matrices in the spectrum corresponding to the subspaces $\langle im\ A_1 \cap ker\ A_2' \rangle$ and $\langle im\ A_2 \cap ker\ A_1' \rangle$, their highest coefficients also are specified as determinants;
$Z^{(n - r_1 - r_2)\times(n - r_1 - r_2)}$ is the zero block; if $\nu' \neq 0$, the dimension rises by $2\nu'$.

In the direct sum, the orthoprojector onto the image of homomultiplication $B_{1,2}$ is

$$\overleftarrow{B_{1,2}} = \sum_{j=1}^{r_1-\nu''} \boxplus\ I_j^{2\times 2} \boxplus$$

$$\boxplus\ I^{(r_2 - r_1 + 2\nu'')\times(r_2 - r_1 + 2\nu'')} \boxplus Z^{(n - r_1 - r_2)\times(n - r_1 - r_2)}. \qquad (408)$$

With the use of the principle of binarity, from (407), (408) and (378), (379) we may infer relations between higher scalar coefficients and direct products over the trigonometric subspaces as in sect. 8.1. But the two latter for lineors A_1 and A_2 transform into analogous formulae (387) and (389). Suppose in the sequel $r_2 \geq r_1$ (see Figure 2). If lineors are completely linearly independent, then $r_1 + r_2 \leq n$ and $\nu' = 0$.

For the i-th trigonometric cell, due to (124) there holds

$$0 \le \sin^2\varphi_i = \frac{det\ [(A_1|A_2)(A_1|A_2)']_i^{2\times 2}}{tr\ (A_1A_1')_i^{2\times 2} tr\ (A_2A_2')_i^{2\times 2}} \le 1, \qquad (409)$$

where φ_i is the eigen angle between the planars $\langle im\ (A_1A_1')_i^{2\times 2}\rangle$ and $\langle im\ (A_2A_2')_i^{2\times 2}\rangle$ of rank 1 (similar to one in cosine variant (395)).

Further, evaluate the highest scalar coefficient of matrix $B_{1,2}$ with the use of (407)–(409).

$$k(B_{1,2}, r_1 + r_2) =$$

$$= \prod_{i=1}^{r_1-\nu''} det\ [(A_1|A_2)(A_1|A_2)']_i^{2\times 2} \cdot det\ (A_1A_1')_i^{\nu''\times\nu''} \cdot det\ (A_2A_2')_i^{(r_2-r_1+\nu'')\times(r_2-r_1+\nu'')} =$$

$$= \prod_{i=1}^{r_1-\nu''} \{\sin^2\varphi_i \cdot tr(A_1A_1')_i^{2\times 2} \cdot tr(A_2A_2')_i^{2\times 2}\} det(A_1A_1')^{\nu''\times\nu''} det(A_2A_2')^{(r_2-r_1+\nu'')\times(r_2-r_1+\nu'')} =$$

$$= \prod_{i=1}^{r_1-\nu''} \sin^2\varphi_i \cdot k(A_1A_!', r_1) \cdot k(A_2A_2', r_2) \qquad (410)$$

(here ν'' values of $\sin^2\varphi_i = 1$, for $i > r_1 - \nu''$, are omitted.

Finally, the *general sine inequality in the normalized form* for lineors A_1 and A_2 of size $n \times r_1$ and $n \times r_2$ follows from (404) and (410) (where $\varphi_i \in (0; \pi/2]$):

$$0 \le \prod_{i=1}^{r_1-\nu''} \sin^2\varphi_i = |\{A_1|A_2\}|^2_{\sin} = \frac{\mathcal{M}t^2(r_1+r_2)\{A_1|A_2'\}}{\mathcal{M}t^2(r_1)A_1\mathcal{M}t^2(r_2)A_2} =$$

$$= |\mathcal{D}l(r_1+r_2)\sin\tilde{\Phi}_{12}| \le 1. \qquad (411)$$

If $n > 2$, the inequality has only the normalized form. The extremal special cases are:
$|\{A_1|A_2\}| = 0$ if the lineors are at least partially parallel,
$|\{A_1|A_2\}| = 1$ if the lineors are completely orthogonal.
If lineors A_1, A_2 are equirank, then general inequalities (396) and (411) may be united:

$$0 \le \sqrt[r]{|\{A_1A_2\}|^2_{\cos}} + \sqrt[r]{|\{A_1A_2\}|^2_{\sin}} \le 1. \qquad (412)$$

This is derived with applying the algebraic Cauchy Inequality (sect. 1.2) for the arithmetic and geometric means to squared eigenvalues of the cosine and sine, and further summating both the results. The right equality in (412) holds iff $|\varphi_i| = \text{const}$, $i = 1, \dots r$.

If two planars have the same rank 1 (straight lines) or $n-1$ (hyperplanes), then the tensor angle between these planars has exactly one trigonometric cell, it corresponds to the unique trigonometric eigen plane. Then inequalities (412) are transformed into usual identity $\cos^2\varphi + \sin^2\varphi = 1$.

Consider a $n \times r$-matrix A of rank r and its arbitrary partition into j column blocks $A = \{A_1|A_2|\dots|A_j\}$. This form of the matrix corresponds to the polyhedral tensor angle, the sides of the angle are determined by the lineors $A_1, \dots, A_j$. If each block consists of exactly one column, then the polyhedral tensor angle is r-edges. Apply the general sine inequality j times sequentially to this block-matrix A, obtain

$$\mathcal{M}t(r)A \le \mathcal{M}t(r)A_1 \cdot \mathcal{M}t(r)A_2 \cdots \mathcal{M}t(r)A_j. \qquad (413)$$

Equality holds iff the lineors (the vectors) are mutually orthogonal. Inequality (413) is the most complete generalization of the Hadamard Inequality [25] of sine nature.

Chapter 9

Geometric norms of varied orders for matrix objects

9.1 Quadratic and hierarchical norms

Norms for matrices and matrix objects have as usually positive or non-negative values. The geometric norms must be invariant under admissible geometric transformations in the space containing the objects, including parallel translations. For example, homogeneous transformations in $\langle \mathcal{Q}^{n+q}\rangle$ are determined by a reflector tensor: they are trigonometrically compatible with pure rotations and reflections. In $\langle \mathcal{E}^n\rangle$ the reflector tensor is an unity matrix. As both these basis spaces have the same Euclidean metric (see in sect. 5.7), the geometric norms, defined in $\langle \mathcal{E}^n\rangle$, may be used in $\langle \mathcal{Q}^{n+q}\rangle$ too.

For objects of rank 1 (vectors) in arithmetic space $\langle \mathcal{E}^n\rangle$, the Euclidean norm of length is naturally used. However, for objects of rank r greater than 1, the Frobenius norm (i. e., a norm of the same order 1 similarly to Euclidean one) is only the first special norm from the *set of geometric norms of orders* t $(1 \le t \le r)$. That is why defining geometric norms of higher orders (up to r) for objects of rank r is the problem of great interest. In principle, there are two ways for defining a geometric norm of a $r \times n$-lineor A as the geometric object (or a $r \times n$-matrix A as the algebraic transformation).

Way 1. At first, an intermediate norm of homomultiplication $A'A$ is evaluated, it depends on eigenvalues $\sigma_i^2 > 0$ of this matrix. Then the norm of the original matrix A may be obtained as the positive square root of the intermediate norm for $A'A$.

Way 2. A norm is defined in terms of positive eigenvalues σ_i of the *arithmetic square root* $\sqrt{A'A}$. But evaluating this square root is a long and complicated process.

(If $A = S$ is a symmetric matrix, then the results of both ways are equivalent.)

Thus, in the book, we use only way 1. Norms constructed with this method are called *quadratic*, as they are based on the set of eigenvalues σ_i^2. For example, symmetric matrix functions $\cos\tilde{\Phi}$, $\sin\tilde{\Phi}$, $\tan\tilde{\Phi}$, $\sec\tilde{\Phi}$ are sign-indefinite. Their nonzero quadratic norms depend on squared eigenvalues of $\cos^2\varphi_i$, $\sin^2\varphi_i$, $\tan^2\varphi_i$, $\sec^2\varphi_i$. Consequently, they are the same for trigonometric functions of motive and projective tensor angles. (For tensor angles, the general cosine and sine norms were defined in previous chapter.)

Correct definition of *general and particular quadratic norms* will be given with the use of geometric analogies similar to (126), (127) in sect. 3.1 and of the general inequality of means, more precisely, its chain (11) for algebraic means expressed in terms of positive Viéte coefficients (sect. 1.2). Our analysis of (126), (127) in section 3.1 gave clear interpretation of the positive Viéte coefficients for matrices homomultiplication. Remember, that algebraic means (and other ones), inferred from the positive Viéte coefficients, form a hierarchical sequence. (As before, we use a bar to denote means.)

Let A be a $r \times n$-matrix A and $rankA = r$. Define its *parametric* and *hierarchical* geometric norms of order t and degree h as

$$||A||_t^h = [\sqrt[2t]{k(A'A,t)}]^h > 0, \qquad (414)$$

$$\overline{||A||_t^h} = [\sqrt[2t]{k(A'A,t)/C_r^t}]^h > 0. \qquad (415)$$

Formally all these norms are regarded to be zero if $t > r$ and unity if $t = 0$.

Parametric norm (414) with $h = t$ may be consider geometrically as t-dimensional *volume parameter* for the lineor A, and with $t = 1$ as its *length parameter* – see (127) in sect. 3.1.

Hierarchical norms (415) may be consider as hierarchical medians of order $t = 1, \ldots, r$ and degree h, according to original chain (11) for scalar coefficients of the matrix $B = AA'$ (see the general inequality of means in sect. 1.2). In particular, both these norms of highest orders are identical

$$||A||_r^r = \sqrt{det\ (A'A)} = \mathcal{M}t(r)A = \overline{||A||_r^r}.$$

Accordingly, for quadratic nonsingular and singular matrices B, there hold:

$$||B||_n^n = \sqrt{det\ (B'B)} = \sqrt{det\ (BB')} = |det\ B| = \overline{||B||_n^n},$$

$$||B||_r^r = \sqrt{k(B'B, r)} = \sqrt{k(BB', r)} = \mathcal{M}t(r)B = \overline{||B||_r^r}.$$

By the definition, *any general norms for a matrix* have maximal order t equal to its rank r. If in (414), (415) $h = r$, then the general norm of a matrix is its minorant.

In that number, this definition belongs to general norms for the tensor cosine and the tensor sine (projective and motive). For example, *general quadratic trigonometric norms of degree* $h = 1$ are defined similarly with maximal order, according their ranks:

$$0 \leq ||\cos\Phi_{12}||_n^1 = \sqrt[2n]{det\ \cos^2\Phi_{12}} = \sqrt[n]{\prod_{i=1}^{r-\nu'} \cos^2\varphi_i} = \sqrt[n]{(A_1|A_2)|_{\cos}^2} \leq 1, \qquad (416)$$

$$0 \leq ||\sin\Phi_{12}||_{r_1+r_2}^1 = \sqrt[2(r_1+r_2)]{\mathcal{D}_l(r_1+r_2)\sin^2\Phi_{12}} =$$

$$= \sqrt[r_1+r_2]{\prod_{i=1}^{r_1-\nu''} \sin^2\varphi_i} = \sqrt[r_1+r_2]{|(A_1|A_2)|_{\sin}^2} \leq 1. \qquad (417)$$

These norms characterize binary tensor angles $\tilde{\Phi}_{12}$ and Φ_{12} between the lineors A_1 and A_2 or between the planars $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ (the planars $\langle ker\ A_1'\rangle$ and $\langle ker\ A_2'\rangle$).

In its turn, the scalar characteristic

$$0 \leq ||\cos\Phi_B||_n^1 = \sqrt[n]{|\{B\}|_{\cos}^2} \leq 1 \qquad (418)$$

is the general trigonometric norm of degree 1 for the cosine of binary tensor angles $\tilde{\Phi}_B$ and Φ_B between the planars $\langle im\ B\rangle$ and $\langle im\ B'\rangle$ (the planars $\langle ker\ B\rangle$ and $\langle ker\ B'\rangle$).

According to the Le Verrier-Waring direct recurrent formula or the Newton system of equations (see in sect. 1.1), there exist only r independent geometric norms of each type. Just norms (414) and (415) completely determine scalar properties of a linear matrix object of rank r by these two set of its geometric invariants. The quadratic geometric norm of degree 1 and order 1 is the Frobenius norm, for example:

$$||A||_1^1 = \sqrt{tr\ (A'A)} = \sqrt{\sum_{k=1}^{n}\sum_{j=1}^{r} a_{jk}^2} = \sqrt{\sum_{i=1}^{r} \sigma_i^2} = ||A||_F > 0, \qquad (419)$$

where a_{jk} – elements of A, σ_i^2 – eigenvalues of $A'A$. The Euclidean norm $||\mathbf{a}||_E$ is similar. Note, that a power manner for norms defining (in terms of eigenvalues of $\sqrt{A'A}$) give the Euclidean and Frobenius norms as *degree norms of order* θ with $\theta = 2$:

$$\sqrt[\theta]{S_\theta(\sigma_i)} = \sqrt[\theta]{\sum_{i=1}^{r} \sigma_i^\theta}, \quad \theta = 1, \ldots, r.$$

On the other hand, both these ways (1 and 2) of norms defining (see above) are equivalent only for norms of the highest order, i. e., for general ones:

$$||\sqrt{A'A}||^1_r = \sqrt[2r]{s_r(\sigma_i)} = \sqrt[2r]{det\ (A'A)} = \sqrt[r]{\mathcal{M}t(r)A} = \sqrt[r]{s_r(\sigma_i)} = \sqrt[r]{det\sqrt{A'A}}.$$

(In particular, this holds, if $r = 1$.) Way 1 defines norms in terms of scalar characteristic coefficients of the same internal homomultiplication $A'A$ (i. e., not directly in terms of eigenvalues of $A'A$). Way 2 of norms defining demands computing a matrix arithmetic square root through eigenvalues of $A'A$. This is the essential difference between these two ways and *the main reason for choosing by us only the manner corresponding to way 1.*

The Frobenius norm of order 1 and degree 1 is the invariant of length. The general norm of order r and degree r (the minorant), is the invariant of r-dimensional volume. The characteristic $||A||^1_r = \overline{||A||^1_r}$ is the invariant of degree 1 of this volume (the general hierarchical norm). The geometric norms $\overline{||A||^1_t}$ (the small medians) form the hierarchy in order of t values ($1 \leq t \leq r$) corresponding to inequality chain (11) – see sect. 1.1.

The *hierarchical quadratic trigonometric norms of order* $t = 1$ are defined similarly:

$$\overline{||\cos\Phi||^1_1} = \sqrt{\frac{tr\ \cos^2\Phi}{n}}, \qquad \overline{||\sin\Phi||^1_1} = \sqrt{\frac{tr\ \sin^2\Phi}{n}}.$$

Taking into account (182) and (264), we obtain also with $t = 1$ the simplest invariant:

$$\overline{||\cos\Phi||^2_1} + \overline{||\sin\Phi||^2_1} = 1. \tag{420}$$

Quadratic trigonometric norms of the highest order are defined as (416) and (417). So, if chain (11) consists of mean invariants of a tensor trigonometric function, then (12) contains mean invariants of the inverse function (with respect to multiplication). The hierarchical invariants of the spherical cosine and sine range in $[0; 1]$, that of the spherical secant and the tangent range in $[1; \infty)$ and $[0; \infty)$.

9.2 Absolute and relative norms

Consider definitions and properties of various geometric norms for matrix objects. Let A be a complex-valued $n \times m$-matrix of rank r. It represents algebraically a certain geometric object such as either as an 1-valent tensor in $\langle\mathcal{A}^n\rangle$, $m < n$, or as a 2-valent tensor in $\langle\mathcal{A}^{n\times n}\rangle$, $m = n$.

For a complex-valued $n \times m$-matrix A of rank r, its *absolute geometric norm of order* t, $0 \leq t \leq r$, *and degree* h is the scalar characteristic $||A||^h_t$ with the following defining conditions:

(a) $||A||^h_t = [||A||^1_t]^h > 0$ if $1 \leq t \leq r$,

(a') $||A||^h_0 = 1$ if $t = 0$,

(a") $||A||^h_t = 0$ if $t > r$,

(b) $||c \cdot A||^h_t = |c|^h \cdot ||A||^h_t$,

(c) $||U_1 \cdot A \cdot U_2||^h_t = ||A||^h_t$,

(d) $||A^*||^h_t = ||A||^h_t$.

For example, (414)–(419) are *definite absolute geometric norms*. If the symbol " $>$" in defining condition (a) is replaced above by " $\geq$ " , then such norms are called *semi-definite absolute geometric norms* of order t and degree h. They are used only for square matrices B representing 2-valent tensors and denoted as $|\{B\}|_t^h$. Their examples are

$$|\{B\}|_t^t = |k(B,t)| \geq 0, \quad |\{B\}|_r^r = |k(B,r)| \geq 0, \quad |\{B\}|_1^1 = |tr\ B| \geq 0. \tag{421}$$

A *relative norm of order t and degree h* is the ratio of a semi-definite absolute norm and definite one. They are always dimensionless and have here *trigonometric nature*. Examples of relative norms of order $t = r$ are the cosine and sine ratios introduced in Ch. 3. These geometric norms are called *general* if $t = r$ and *particular* if $t < r$. General norms were interpreted before. Reveal the geometrical sense of particular ones.

9.3 Geometric interpretation of particular norms

Consider particular norms, using as clear model, the *particular cosine ratio* (i. e., under condition $t < r$). The general cosine inequalities (396), (398) and the cosine ratios corresponding to these inequalities may be further developed and their quasi-analogs for orders $t < r$ may be inferred.

Let A_1 and A_2 be $n \times r$-lineors. For each j-th subset of t columns, $j = 1, \ldots, C_r^t$, choose the pair of $n \times t$-submatrices $\{A_1\}_j$ and $\{A_2\}_j$ with the same subset of columns. Write down all the submatrices $\{A_1\}_j$ one under another and do the same with $\{A_2\}_j$. This operation transforms A_1 and A_2 into the pair of *ranged $nC_r^t \times t$-lineors* of rank t.

For each pair $\{A_1\}_j$ and $\{A_2\}_j$, the cosine inequalities similar to (396), (398) hold:

$$-1 \leq det\ \{A_1'A_2\}_j / \left(\sqrt{det\ (A_1'A_1)_j} \cdot \sqrt{det\ (A_2'A_2)_j}\right) \leq +1.$$

The numerator of the fraction is the j-th principal minor of order t of $\{A_1'A_2\}$, as the internal multiplication of $\{A_1\}_j$ and $\{A_2\}_j$. Summate separately j numerators and j denominators of these inequalities, we obtain from two sums a united inequality (that is generally a *Rule of summing homogeneous fraction inequalities, i. e., with constant left and right constraints and positive denominators, in a united fraction inequality*):

$$-1 \leq \frac{\sum_{j=1}^{C_r^t} det\ \{A_1'A_2\}_j}{\sum_{j=1}^{C_r^t} \sqrt{det\ \{A_1'A_1\}_j} \cdot \sqrt{det\ \{A_2'A_2\}_j}} \leq +1.$$

Further, apply to the denominator the geometric cosine Cauchy Inequality (sect. 3.3) for a paired set of positive numbers, obtain the following intermediate inequality:

$$-1 \leq \frac{\sum_{j=1}^{C_r^t} det\ \{A_1'A_2\}_j}{\sqrt{\sum_{j=1}^{C_r^t} det\ \{A_1'A_1\}_j \cdot \sum_{j=1}^{C_r^t} det\ \{A_2'A_2\}_j}} \leq +1.$$

Using (120) and (121), obtain the *particular quasi-cosine inequalities* in the sign form:

$$-1 \leq \frac{k(A_1'A_2,t)}{\sqrt{k(A_1'A_1,t)} \cdot \sqrt{k(A_2'A_2,t)}} = \frac{k(A_1A_2',t)}{\sqrt{k(A_1A_1',t)} \cdot \sqrt{k(A_2A_2',t)}} \leq +1. \tag{422}$$

The quasi-cosine inequalities in the signless form define the particular relative norms:

$$0 \leq \frac{|\{A_1A_2'\}|_t^1}{||A_1||_t^1 \cdot ||A_2||_t^1} \leq 1. \quad (1 \leq t < r) \tag{423}$$

Trigonometric sense of the quasi-cosine ratio as a norm of order $t < r$ is explained with its inference, it is connected with ranged lineors. If $t = 1$, then

$$-1 \leq \frac{tr\ (A_1'A_2)}{\sqrt{tr\ (A_1'A_1)} \cdot \sqrt{tr\ (A_2'A_2)}} = \frac{tr\ (A_1A_2')}{\sqrt{tr\ (A_1A_1')} \cdot \sqrt{tr\ (A_2A_2')}} \leq +1, \tag{424}$$

$$0 \leq \frac{|\{A_1A_2'\}|_1^1}{||A_1||_1^1 \cdot ||A_2||_1^1} \leq 1. \tag{425}$$

From these inequalities the classical triangle and parallelogram inequalities for the Frobenius norms ($t = 1$) of the original $n \times r$-lineors follow:

$$||A_1 + A_2||_1^1 \leq ||A_1||_1^1 + ||A_2||_1^1. \tag{426}$$

$$\big|\ ||A_1||_1^1 - ||A_2||_1^1\ \big| \leq ||A_1 \pm A_2||_1^1 \leq ||A_1||_1^1 + ||A_2||_1^1. \tag{427}$$

These particular inequalities are of linear nature. They define the Frobenius norm of lineors as an invariant of extent (or length for vectors). However, particular inequalities (422), (424) and (426), (427) characterize the lineors A_1 and A_2 if $r > 1$ not directly, but in terms of ranged $nC_r^t \times t$-lineors $\{A_1\}$ and $\{A_2\}$. For illustrations, get Frobenius norms: they describe these lineors in terms of ranged $nr \times 1$-vectors $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$||A_1||_1^1 = ||\mathbf{a}_1||_E, \quad ||A_2||_1^1 = ||\mathbf{a}_2||_E, \quad ||A_1 \pm A_2||_1^1 = ||\mathbf{a}_1 \pm \mathbf{a}_2||_E;$$

$$tr\ (A_1' \cdot A_2) = tr\ (A_1 \cdot A_2') = \mathbf{a}_1'\mathbf{a}_2.$$

Consequently, the *Pythagorean Theorem for the Frobenius norms of the lineors A_1 and A_2 holds iff ranged vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal*:

$$\mathbf{a}_1'\mathbf{a}_2 = 0 = tr\ (A_1'A_2) \ \leftrightarrow\ ||A_1 \pm A_2||_1^2 = ||A_1||_1^2 + ||A_2||_1^2. \tag{428}$$

Similarly, from the trigonometric point of view, particular quasi-cosine ratios (423) and (425) as relative norms characterize also tensor angles $\tilde{\Phi}_{12}$ and Φ_{12} between the lineors A_1 and A_2 not directly, but only in terms of ranged lineors $\{A_1\}$ and $\{A_2\}$.

9.4 Lineors of special kinds and some figures formed by lineors

In the *lineor Euclidean space* $\langle\mathcal{E}^n\rangle$, according to (130) in sect. 3.1, an $n \times r$-lineor *($of\ rank\ A = r$)* may be represented in the *unambiguous quasi-polar decomposition*:

$$A = \{A \cdot (\sqrt{A'A})^{-1}\} \cdot \sqrt{A'A} = Rq \cdot |A|,$$

where $|A| = \sqrt{A'A}$ is the $r \times r$-*matrix module* of the original $n \times r$-lineor A, and matrix $Rq = \{A \cdot (\sqrt{A'A})^{-1}\}$ is its *own quasi-orthogonal lineor*. This decomposition is similar to one for a vector: $\mathbf{a} = \mathbf{e} \cdot |\mathbf{a}|$, where $|\mathbf{a}| = \sqrt{\mathbf{a}'\mathbf{a}} = ||\mathbf{a}||_E$. The $r \times r$-matrix module of the lineor is similar to the scalar module of a vector, but with respect to the set of r basis unity vectors $\{\mathbf{e}_i\} = Rq$ in $\langle\mathcal{E}^n\rangle$. These vectors determine independent directional axes in $\langle\mathcal{E}^n\rangle$ of the given $n \times r$-lineor A. Consequently, there hold

$$\overleftarrow{AA'} = \overleftarrow{Rq \cdot Rq'} = Rq \cdot Rq', \quad Rq' \cdot Rq = Rq^+ \cdot Rq = I_{r\times r}, \quad (Rq' = Rq^+).$$

Each lineor formally belongs to its basis planar: $A \in \langle im\ A\rangle$ (as $\mathbf{a} \in \langle im\ \mathbf{a}\rangle$). The condition (154) determines the set of *coplanar lineors* with respect to the basis planar $\langle im\ A_1\rangle$ (for the vector $\mathbf{a}_2$ this condition is $\mathbf{a}_2 \in \langle im\ A_1\rangle$).

Equirank lineors ($r_A = r = \text{const}$) with the same basis planar $\langle im\ A\rangle$ form the complete set of *colplanar lineors* with respect to the basis planar $\langle im\ A\rangle$. If $r = 1$, they are *collinear vectors*. Two equirank lineors are colplanar iff they satisfy (153). The complete set of colplanar $n \times r$-lineors $\langle AC\rangle$ with respect to the basis planar $\langle im\ A\rangle$ is parametrically determined with a free nonsingular $r \times r$-matrix C by relation

$$\overleftarrow{AA'} = \overleftarrow{(AC)(AC)'} = \text{Const}. \tag{429}$$

Colplanar lineors A_k are defined by the following invariant relations:

$$\overleftarrow{A_k A_k'} = \overleftarrow{Rq_k Rq_k'} = Rq_k \cdot Rq_k' = \overleftarrow{AA'} = \text{Const}_{n\times n}, Rq_k' \cdot Rq_k = I_{r\times r} = \text{Const}_{r\times r}. \tag{430}$$

Further, in the set of colplanar lineors $\langle A\rangle$, separate the subset of *coaxial lineors*. They are defined stronger with additional condition $Rq_k = Rq = \text{Const} = \{\mathbf{e}_i\}$. Such lineors differ only by their matrix moduli $|A_k|$. If A_1, A_2 are coaxial lineors, then

$$|A_1 \pm A_2|^2 = \big|\ |A_1| \pm |A_2|\ \big|^2 = (|A_1| \pm |A_2|)^2, \quad A_1' A_2 = |A_1| \cdot |A_2|, \ A_2' \cdot A_1 = |A_2| \cdot |A_1|.$$

Let A_1 and A_2 be equirank lineors, may be linearly entirely independent or not, but under the same conditions (224) and (230), and lying in their own basis planars $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$. Then the oblique projector $\overleftarrow{A_1 A_2'}$ exists. Using formulae (186) and (187) from sect. 5.2, and (226) from sect. 5.4, we obtain:

$$\overleftarrow{A_1 A_2'} = \overleftarrow{Rq_1 Rq_2'} = Rq_1\ Rq_1' \cdot \sec\tilde{\Phi}_{12} = \sec\tilde{\Phi}_{12} \cdot Rq_2\ Rq_2'. \tag{431}$$

Expressions (430) and (431) may be useful in *QR-factorizations* of lineors with similar conditions – see (129), (130) in sect. 3.1. They can be illustrated easily and visually on the simplest unity lineors $\mathbf{e}_1$ and $\mathbf{e}_2$, as we have done earlier too.

In conclusion, define also *rotationally congruent lineors*:

$$A_2 = Rot\ \Phi_{12} \cdot A_1 \ \Rightarrow\ \{Rq_2 = Rot\ \Phi_{12} \cdot Rq_1,\ |A_1| = |A_2| = |A|\}. \tag{432}$$

Such lineors differ only by their quasi-orthogonal lineors Rq_k.

For these lineors A_1 and A_2 we have these symmetric matrix module expressions:

$$\left.\begin{array}{c} |A_1 \pm A_2|^2 = 4 \cdot |A|^2 \cdot [(I \pm \cos\Phi_{12})/2], \\ |A_1 + A_2|^2 = 4 \cdot |A|^2 \cdot \sin^2(\Phi_{12}/2), \\ |A_1 - A_2|^2 = 4 \cdot |A|^2 \cdot \cos^2(\Phi_{12}/2). \end{array}\right\} \tag{433}$$

With the use of parallel translations, rotationally congruent lineors A_1 and A_2 form a $2r$-dimensional rhombus. In particular, centered equimodule vectors are rotationally congruent. If $\Phi_{12} = \pi/2$, then these lineors form the following $2r$-dimensional square:

$$|A_1 + A_2| = \sqrt{2} \cdot |A|.$$

One may construct from such lineors corresponding triangles, parallelograms and so on. Thus lineors, as well as vectors, can form, but more complex, geometric figures with various geometric properties. Euclidean and quasi-Euclidean spaces of lineors (*lineor spaces*) have, as well as vector spaces, valency 1.

Chapter 10

Complexification of tensor trigonometry

10.1 Adequate complexification

Complex-valued projective and motive *spherical* angles are expressed adequately in terms of real-valued spherical and hyperbolic tensor angles with their binary eigen angles in the following forms

$$\tilde{\Psi} = \tilde{\Phi} + i\tilde{\Gamma},\ (\tilde{\Psi}' = \tilde{\Phi} - i\tilde{\Gamma});\quad \Psi = \Phi + i\Gamma,\ (\Psi' = -\Phi + i\Gamma);\quad \psi_j = \varphi_j + i\gamma_j, \tag{434}$$

where $\tilde{\Phi}' = \tilde{\Phi}$, $\tilde{\Gamma}' = -\tilde{\Gamma}$; $\Phi' = -\Phi$, $\Gamma' = \Gamma$ (see the angles in Chs. 5 and 6).

In the *adequate complex n-dimensional Euclidean space*, complex tensor trigonometry is realized in the complex Cartesian bases with the use of adequate complexification (sect. 4.2). Complex tensor angles have their transposed forms indicated above. All geometric notions and formulae except norms and inequalities stay valid and do not change. In particular, complex minorants and complex matrix modules are defined with the use of transposing.

Complex numbers $+c$ and $-c$ have the analogous adequate complex module, and it is evaluated also in terms of c^2 by Moivre formula:

$$\pm c = \pm\rho(\cos\alpha + i\sin\alpha),\quad 0 \le \alpha < \pi,$$

$$(\pm c)^2 = c^2 = \rho^2(\cos 2\alpha + i\sin 2\alpha) = \rho^2(\cos\beta + i\sin\beta),\ 0 \le \beta < 2\pi,$$

$$|\pm c| = |c| = \rho(\cos(\beta/2) + i\sin(\beta/2)) = \rho(\cos\alpha + i\sin\alpha). \tag{435}$$

It is seen that $|c^2| = c^2$.

The adequate matrix Euclidean module $|A| = \sqrt{A'A}$ of a complex matrix A (sect. 9.4) is evaluated with intermediate diagonalization of its interior multiplication and complex orthogonal modal transformation:

$$R' \cdot A'A \cdot R = D\{A'A\} = \{\sigma_j^2\},\quad \sigma_j^2 = \rho_j^2(cos\beta_j + i\sin\beta_j) = |\sigma_j|^2,\quad 0 \le \beta_j < 2\pi.$$

From this, by Moivre formula, we obtain

$$|\sigma_j| = \rho_j[\cos(\beta_j/2) + i\sin(\beta_j/2)],\qquad |A| = R\cdot\{|\sigma_j|\}\cdot R',\qquad |A|^2 = A'A.$$

In the adequate complexification variant, all geometric characteristics, as complex angles and their trigonometric functions, are decomposed into real and imaginary parts, though each whole characteristic may be represented in the most suitable adequate form. The adequate variant in its simplest form is used in complex-valued Euclidean plane geometry, in particular, in scalar complex Euclidean trigonometry. In general case, complex squared identity (142), in that number in its variant of the sine-cosine Lagrange Identity for two vectors, does not change. The scalar sine and cosine ratios in (124) and (141) may be used for evaluating of the complex angles between two vectors and their trigonometric functions. The general scalar ratios (135) and (140) have also their adequate complex-valued forms.

10.2 Hermitean complexification

In the *Hermitean space*, Hermitean complexification of real-valued Euclidean geometry with tensor trigonometry is used (sect. 4.3). A projective *spherical* tensor angle is an Hermitean matrix $\tilde{\mathcal{H}} = \tilde{\Phi} + i\tilde{\Gamma} = \tilde{\mathcal{H}}^*$, where $\tilde{\Phi}^* = \tilde{\Phi}$, $\tilde{\Gamma}^* = -\tilde{\Gamma}$. Its eigenvalues are real spherical angles $\pm\eta_j$ and zero. A motive *spherical* tensor angle is a skew-Hermitean matrix $\mathcal{K} = \Phi + i\Gamma = -\mathcal{K}^*$, where $\Phi^* = -\Phi$, $\Gamma^* = \Gamma$ (Chs. 5, 6). Its eigenvalues are imaginary pseudohyperbolic angles $\pm i\eta_j$ and zero. Hermitean modules of linear objects are positive definite. Normalized general inequalities (Ch. 8), geometric and trigonometric norms (Ch. 9) preserve their real positive forms in the Hermite's variant.

The principle of binarity also stays valid in complex adequate and Hermitean variants of tensor trigonometry, as all its preliminaries do hold.

Hermitean analogs of cell formulae (399) and (400) in sect. 8.3 are inferred with analogous complex-valued unity vectors. Here the two sides of the tensor angle $\tilde{\mathcal{H}}_{12}$ between planars $\langle im\ A_1\rangle$ and $\langle im\ A_2\rangle$ of rank r are represented at the level of elementary trigonometric 2×2-cells as unity eigenvectors:

$$\mathbf{u}_1 = \begin{vmatrix} \cos\alpha_1 \\ \sin\alpha_1 \end{vmatrix}, \qquad \mathbf{u}_2 = \begin{vmatrix} \cos\alpha_2 \\ \sin\alpha_2 \end{vmatrix},$$

where

$$\cos\alpha\cdot\overline{\cos\alpha} + \sin\alpha\cdot\overline{\sin\alpha} = 1,$$

$$\cos\alpha = \cos\eta\cdot\exp i\beta_c, \qquad \sin\alpha = \sin\eta\cdot\exp i\beta_s,$$

$$\cos\alpha\cdot\overline{\cos\alpha} = \cos^2\eta, \qquad \sin\alpha\cdot\overline{\sin\alpha} = \sin^2\eta;$$

$$[\cos\tilde{\mathcal{H}}_{12}]^{2\times 2} = \overleftarrow{\mathbf{u}_1\cdot\mathbf{u}_1^*} + \overleftarrow{\mathbf{u}_2\cdot\mathbf{u}_2^*} - I^{2\times 2} = \mathbf{u}_1\cdot\mathbf{u}_1^* + \mathbf{u}_2\cdot\mathbf{u}_2^* - I^{2\times 2} =$$

$$= \left[\begin{array}{c|c} \cos\alpha_1\cdot\overline{\cos\alpha_1} + \cos\alpha_2\cdot\overline{\cos\alpha_2} - 1 & \cos\alpha_1\cdot\overline{\sin\alpha_1} + \cos\alpha_2\cdot\overline{\sin\alpha_2} \\ \overline{\cos\alpha_1}\cdot\sin\alpha_1 + \overline{\cos\alpha_2}\cdot\sin\alpha_2 & \overline{\sin\alpha_1}\cdot\sin\alpha_1 + \overline{\sin\alpha_2}\cdot\sin\alpha_2 - 1 \end{array}\right] =$$

$$= \begin{bmatrix} +|c_1| & \overline{s_1} \\ s_1 & -|c_1| \end{bmatrix};$$

$$-det\ [\cos\tilde{\mathcal{H}}_{12}]^{2\times 2} = |c_1|^2 + s_1\cdot\overline{s_1} = \cos^2(\eta_2-\eta_1) - \Delta = \cos^2\eta_{12},$$

$$\Delta = (1/2)\cdot\sin(2\eta_1)\cdot\sin(2\eta_2)\cdot[1 - \cos(\beta_{c_1})\cos(\beta_{c_2})\cos(\beta_{s_1})\cos(\beta_{s_2})];$$

$$[\sin\tilde{\mathcal{H}}_{12}]^{2\times 2} = \overleftarrow{\mathbf{u}_2\cdot\mathbf{u}_2^*} - \overleftarrow{\mathbf{u}_1\cdot\mathbf{u}_1^*} = \mathbf{u}_2\cdot\mathbf{u}_2^* - \mathbf{u}_1\cdot\mathbf{u}_1^* =$$

$$= \left[\begin{array}{c|c} \cos\alpha_2\cdot\overline{\cos\alpha_2} - \cos\alpha_1\cdot\overline{\cos\alpha_1} & \cos\alpha_2\cdot\overline{\sin\alpha_2} - \cos\alpha_1\cdot\overline{\sin\alpha_1} \\ \overline{\cos\alpha_2}\cdot\sin\alpha_2 - \overline{\cos\alpha_1}\cdot\sin\alpha_1 & \overline{\sin\alpha_2}\cdot\sin\alpha_2 - \overline{\sin\alpha_1}\cdot\sin\alpha_1 \end{array}\right] = \begin{bmatrix} -|s_2| & \overline{c_2} \\ c_2 & +|s_2| \end{bmatrix};$$

$$-det\ [\sin\tilde{\mathcal{H}}_{12}]^{2\times 2} = |s_2|^2 + c_2\cdot\overline{c_2} = \sin^2(\eta_2-\eta_1) + \Delta = \sin^2\eta_{12}.$$

For the residue Δ we have $\Delta = 0 \ \Leftrightarrow\ \cos(\beta_{c_1})\cos(\beta_{c_2})\cos(\beta_{s_1})\cos(\beta_{s_2}) = 1 = |\cos\beta_k|$;

$$\Delta = 0 \ \Leftrightarrow\ \eta_{12} = \eta_2 - \eta_1, \quad \Delta \neq 0 \ \Leftrightarrow\ \eta_{12} \neq \eta_2 - \eta_1. \tag{436}$$

The cell forms with respect to the *trigonometric base* (see sect. 5.5) are

$$[\cos\tilde{\mathcal{H}}_{12}]^{2\times 2} = \cos\eta_{12}\cdot\begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}, \quad [\sin\tilde{\mathcal{H}}_{12}]^{2\times 2} = \sin\eta_{12}\cdot\begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix}. \qquad (437, 438)$$

In the Hermitean variant, all canonical W-forms of tensor trigonometric functions are real-valued and do not change. They are constructed with complex unitary modal matrices U_W. In an Hermitean plane and with respect to the trigonometric base (of the diagonal cosine), Hermitean shift of paired functions (cosine-sine, secant-tangent) at a phase angle β may take place – see respectively (179) and (259):

$$Exp\ (-i\beta/2) \cdot Ref\ \{B^*B\}_r \cdot Exp\ (+i\beta/2) = Ref\ \{B^*B\}_c, \tag{439}$$

$$Exp\ (-i\beta/2) \cdot Rot\ \{\mathcal{H}\}_r \cdot Exp\ (+i\beta/2) = Rot\ \{\varepsilon\}_c, \tag{440}$$

i. e.,

$$\begin{bmatrix} \ddots & & & \\ & \exp\left(\frac{-i\beta}{2}\right) & 0 & \\ & 0 & \exp\left(\frac{+i\beta}{2}\right) & \\ & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \ddots & & & \\ & +\cos\eta & \sin\eta & \\ & \sin\eta & -\cos\eta & \\ & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \ddots & & & \\ & \exp\left(\frac{+i\beta}{2}\right) & 0 & \\ & 0 & \exp\left(\frac{-i\beta}{2}\right) & \\ & & & \ddots \end{bmatrix} =$$

$$= \begin{bmatrix} \ddots & & & \\ & +\cos\eta & \sin\eta\cdot\exp(-i\beta) & \\ & \sin\eta\cdot\exp(+i\beta) & -\cos\eta & \\ & & & \ddots \end{bmatrix},$$

$$\begin{bmatrix} \ddots & & & \\ & \exp\left(\frac{-i\beta}{2}\right) & 0 & \\ & 0 & \exp\left(\frac{+i\beta}{2}\right) & \\ & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \ddots & & & \\ & \cos\eta & -\sin\eta & \\ & +\sin\eta & \cos\eta & \\ & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \ddots & & & \\ & \exp\left(\frac{+i\beta}{2}\right) & 0 & \\ & 0 & \exp\left(\frac{-i\beta}{2}\right) & \\ & & & \ddots \end{bmatrix} =$$

$$= \begin{bmatrix} \ddots & & & \\ & \cos\eta & -\sin\eta\cdot\exp(-i\beta) & \\ & +\sin\eta\cdot\exp(+i\beta) & \cos\eta & \\ & & & \ddots \end{bmatrix}.$$

That is why the Hermitean trigonometric base should lead to the diagonal cosine as before and also to real-valued W-forms. In each eigen Hermitean plane (at the level of each 2×2-cells), Hermitean shift at a phase angle β may be eliminated with the special unitary rotational modal transformation $Exp\ (i\beta/2)$, and as final result with reducing in real-valued canonical forms of tensor trigonometric functions.

Hermitean analogs of Cauchy and Hadamard Inequalities of cosine and sine nature (see in Ch. 3) and their cosine and sine tensor forms (see in Ch. 8)) with complex Lagrange Identity (142) for coordinates of two vectors or in general two lineors are inferred with the use of Hermitean transposing in their internal products. Hermitean spherical angle is a composite function of the linear objects coordinates. But in its trigonometric base, the tensor angle have the real-valued canonical form.

10.3 Pseudoization in binary complex spaces

Consider pseudoization as the important special case of adequate complexification of real-valued algebraic and geometric notions (see Ch. 4). Fix a binary complex affine space $\langle\mathcal{A}^{n+q}\rangle_c$ of index q. In any admissible binary affine base, this space may be considered as linear one. In particular, with respect to a certain pseudounity base $\tilde{E}_0$, the space $\langle\mathcal{A}^{n+q}\rangle_c$ is the direct sum of the following real and imaginary affine subspaces:

$$\langle\mathcal{A}^{n+q}\rangle_c \equiv \langle\mathcal{A}^n\rangle \oplus \langle i\mathcal{A}^q\rangle \equiv \text{CONST}. \tag{441}$$

Here the sum space and dimensions of summand subspaces are constant. In $\langle\mathcal{A}^{n+q}\rangle$, we admit linear transformations V preserving the binary structure:

$$\underset{V}{\begin{bmatrix} V_{11} & iV_{12} \\ iV_{21} & V_{22} \end{bmatrix}} \cdot \underset{\tilde{E}_0}{\begin{bmatrix} I^{n\times n} & Z^{n\times q} \\ Z^{q\times n} & \pm iI^{q\times q} \end{bmatrix}} = \underset{\tilde{E}}{\begin{bmatrix} V_{11} & \pm V_{12} \\ iV_{21} & \pm iV_{22} \end{bmatrix}}, \ det\ V_{jk} \neq 0. \tag{442}$$

First n columns of the base matrices generate $\langle\mathcal{A}^n\rangle$, other q columns generate $\langle i\mathcal{A}^q\rangle$. The modal matrix V^{-1} has the same structure, this matrix transforms an arbitrary binary base $\tilde{E}$ into simplest one, i. e., into diagonal (pseudounity) base $\tilde{E}_0$ and performs *passive* modal transformation of a linear element: $\mathbf{z}\{\tilde{E}\} = V \cdot \mathbf{z}\{\tilde{E}_0\}$.

The *binary local complex trigonometric bases* are expressed in the *left and right mutual forms* connected with the local real-valued trigonometric base $\tilde{E}_1 = \{I\}$ by pseudounity passive modal matrices (see initially in sect. 5.9 and sect. 6.1):

$$\tilde{E}_{01} = \begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & +i & \\ & & & \ddots \end{bmatrix} \cdot \tilde{E}_1 = (\sqrt{I^{\pm}})_D \cdot \{I\} = R_{c1} \cdot \{I\} = \{R_{c1}\}, \tag{443}$$

$$\tilde{E}_{02} = \begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & -i & \\ & & & \ddots \end{bmatrix} \cdot \tilde{E}_1 = (\sqrt{I^{\pm}})_D^{-1} \cdot \{I\} = R_{c2} \cdot \{I\} = \{R_{c2}\}. \tag{444}$$

With respect to an admissible binary complex base $\tilde{E}$, a linear element and the whole space are direct sums of their real and imaginary affine projections:

$$\mathbf{z} = \mathbf{x} \oplus i\mathbf{y} = \begin{bmatrix} \mathbf{x} \\ i\mathbf{y} \end{bmatrix}. \tag{445}$$

The space $\langle\mathcal{A}^{n+q}\rangle_c$ is affine, and hence the translations in it at linear elements (445) are admissible, and hence the space is homogeneous.

Right local base (443) is identical to one in (271) and used in canonical forms of *pseudo-hyperbolic* trigonometric matrices with angle eigenvalues $-i\varphi_j = \varphi_j/(+i)$ (see sect. 5.9). The sign "minus" at angles is due to the multiplier $+i$ at ordinates.

Left local base (444) represents canonical forms of trigonometric matrices in the *pseudospherical* variant of tensor trigonometry with binary eigen angles $\pm i\gamma_j = \pm\gamma_j/(-i)$ – primary and mutual. This base is identical to inverse (271), i. e., with the multiplier $-i$ at ordinates.

The modal transformation translates into base $\tilde{E}_{01}$ (443) similarly (322):

$$\overset{(\sqrt{I^{\pm}})^{-1}}{\begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & -i & \\ & & & \ddots \end{bmatrix}} \cdot \begin{bmatrix} \ddots & & & \\ & \cos\varphi_j & -\sin\varphi_j & \\ & +\sin\varphi_j & \cos\varphi_j & \\ & & & \ddots \end{bmatrix} \cdot \overset{\sqrt{I^{\pm}}}{\begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & +i & \\ & & & \ddots \end{bmatrix}} =$$

$$= \begin{bmatrix} \ddots & & & \\ & \cos\varphi_j & -i\sin\varphi_j & \\ & -i\sin\varphi_j & \cos\varphi_j & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & \cosh(-i\varphi_j) & \sinh(-i\varphi_j) & \\ & \sinh(-i\varphi_j) & \cosh(-i\varphi_j) & \\ & & & \ddots \end{bmatrix}.$$

And the modal transformation translates into base $\tilde{E}_{02}$ (444) similarly to (323):

$$\overset{\sqrt{I^{\pm}}}{\begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & +i & \\ & & & \ddots \end{bmatrix}} \cdot \begin{bmatrix} \ddots & & & \\ & \cosh\gamma_j & \sinh\gamma_j & \\ & \sinh\gamma_j & \cosh\gamma_j & \\ & & & \ddots \end{bmatrix} \cdot \overset{(\sqrt{I^{\pm}})^{-1}}{\begin{bmatrix} \ddots & & & \\ & 1 & 0 & \\ & 0 & -i & \\ & & & \ddots \end{bmatrix}} =$$

$$= \begin{bmatrix} \ddots & & & \\ & \cosh\gamma_j & -i\sinh\gamma_j & \\ & +i\sinh\gamma_j & \cosh\gamma_j & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & \cos(i\gamma_j) & -\sin(i\gamma_j) & \\ & +\sin(i\gamma_j) & \cos(i\gamma_j) & \\ & & & \ddots \end{bmatrix}.$$

Accordingly, in $\tilde{E}_{01}$ and $\tilde{E}_{02}$ of $\langle\mathcal{Q}^{n+q}\rangle_c$, we have the mixed pseudoized angles in two forms: $\gamma_k \boxplus (-i\varphi_j)$ and $\varphi_j \boxplus i\gamma_k$ (at the counter-clockwise angle φ) – see sect. 6.1 too.

Express coordinates of linear elements (445) in $\langle\mathcal{A}^{n+q}\rangle_c$ with respect to base (444). Define in $\tilde{E}_{02}$ the same and invariant under passive modal transformations scalar product for elements $\mathbf{z}$ (445) in $\langle\mathcal{Q}^{n+q}\rangle_c$ as in a usual Euclidean space:

$$\mathbf{z}_1'\{I^+\}\mathbf{z}_2 = \mathbf{z}_1'\mathbf{z}_2 = \mathbf{x}_1'\cdot\mathbf{x}_2 + \mathbf{i}\mathbf{y}_1'\cdot\mathbf{i}\mathbf{y}_2 = \mathbf{x}_1'\mathbf{x}_2 - \mathbf{y}_1'\mathbf{y}_2 = \mathrm{const}.$$

This is valid in the special *complex quasi-Euclidean space* with index q, its metric is as if *Euclidean-like*; but it is either real-valued or zero or imaginary-valued. First this construction (with $n = q = 1$) was made by H. Poincaré in his *group variant* of the relativistic theory [63]. The space is binary, it is the direct spherically orthogonal sum of the real-valued Euclidean subspace and the imaginary-valued *anti-Euclidean* one:

$$\langle\mathcal{Q}^{n+q}\rangle_c \equiv \langle\mathcal{E}^n\rangle \boxplus \langle i\mathcal{E}^q\rangle \equiv \mathrm{CONST} \Leftrightarrow \langle\mathcal{P}^{n+q}\rangle \equiv \langle\mathcal{E}^n\rangle \boxtimes \langle\mathcal{E}^q\rangle \equiv \mathrm{CONST}. \tag{446}$$

Here $\boxplus$ and $\boxtimes$ stand for direct *spherically and hyperbolically orthogonal summation*. Admissible transformations in these binary space are determined by the simplest reflector metric tensors $\{I^{\pm}\}$ and $\{I^{\mp}\}$ – see initially in Chs. 5 and 6. In particular, $\langle i\mathcal{E}^q\rangle$ degenerates into the axis iy or $\overrightarrow{iy}$ or into the time arrow $\overrightarrow{i\cdot ct}$ according to H. Poincaré.

Chapter 11

Tensor trigonometry of general pseudo-Euclidean spaces

11.1 Realification of complex quasi-Euclidean spaces

Return to *binary complex quasi-Euclidean space* (446). It is defined by the *unity metric tensor* $\{I^+\}$ and the reflector tensor $\{I^{\pm}\}$; its trigonometric base is $\tilde{E}_{02}$ (444). Further, apply to complex-valued space $\langle \mathcal{Q}^{n+q}\rangle_c$ realifying passive transformation (443) as $R_c = (\sqrt{I^{\pm}})_D$ (here passive transformation R_c has a geometric sense contrary to R_c (271) in sect. 5.9):

$$\tilde{E}_{02} = \{(\sqrt{I^{\pm}})_D^{-1}\} \;\rightarrow\; (\sqrt{I^{\pm}})_D \cdot \tilde{E}_{02} = \tilde{E}_1 = \{I\}, \;\; (\sqrt{I^{\pm}})_D^{-1} \cdot \mathbf{z}_{02} = \mathbf{u}. \tag{447}$$

The modal matrix is not admissible as $\sqrt{I^{\pm}}' \cdot \sqrt{I^{\pm}} \neq \{I^+\}$; it transfers into a *realificated pseudo-Euclidean* space $\langle \mathcal{P}^{n+q}\rangle$ of index q with the *metric and reflector tensor* $\{I^{\pm}\}$. Its quadratic metric is *pseudo-Euclidean*. The scalar product for the same element is

$$\mathbf{z}_{02}' \cdot \mathbf{z}_{02} = [(\sqrt{I^{\pm}})_D \cdot \mathbf{u}]' \cdot [(\sqrt{I^{\pm}})_D \cdot \mathbf{u}] = \mathbf{u}' \cdot \{I^{\pm}\} \cdot \mathbf{u} = \text{const}. \tag{448}$$

So, the spaces $\langle \mathcal{Q}^{n+q}\rangle_c$ and $\langle \mathcal{P}^{n+q}\rangle$ are isometric and expressed only in different forms! Now the same element is expressed in the base $\tilde{E}_1$, and it is denoted as $\mathbf{u}$. The new metric tensor $\{I^{\pm}\}$ in this *coaxially oriented* space $\langle \mathcal{P}^{n+q}\rangle$ is also its *reflector tensor*! Realification $\langle \mathcal{Q}^{3+1}\rangle_c \rightarrow \langle \mathcal{P}^{3+1}\rangle_r$ with introducing the metric tensor $\{I^{\pm}\}$ at $q = 1$ was suggested by Hermann Minkowski in 1909 [65], at the beginning into physical $4D$ space-time with $\langle \mathbf{x}, ct\rangle$.

Further, realize next and also isometric passive modal transformation in the similar binary space, but with an affine base $\tilde{E}$, connected with $\tilde{E}_1 = \{I\}$ by the constant binary real-valued modal matrix V. It is not compatible again with the former metric tensor $\{I^{\pm}\}$. In result we have the sequential transformations of the original base and element

$$\tilde{E} = V \cdot \tilde{E}_1, \;\; \tilde{E}_1 = (\sqrt{I^{\pm}})_D \cdot \tilde{E}_{02}, \tag{449}$$

$$\mathbf{w} = V^{-1} \cdot (\sqrt{I^{\pm}})_D^{-1} \cdot \mathbf{z}_{02} = [(\sqrt{I^{\pm}})_D \cdot V]^{-1} \cdot \mathbf{z}_{02}. \tag{450}$$

Now the original element $\mathbf{z}_{02}$ is expressed in the affine base $\tilde{E}$, it is denoted as $\mathbf{w}$. The inverse modal matrices of the passive transformations are written *in direct order* for sequential ones. The scalar product of this element, as its immanent characteristic at passive isometric base's transformations, does not change with respect to the new and now *affine base* $\tilde{E}$, and hence the metric reflector tensor (with the same reflector tensor) $\{I^{\pm}\}$ does change into the new certain symmetric metric reflector tensor:

$$\mathbf{z}_{02}' \cdot \mathbf{z}_{02} = [(\sqrt{I^{\pm}})_D \cdot V \cdot \mathbf{w}]' \cdot [(\sqrt{I^{\pm}})_D \cdot V \cdot \mathbf{w}] = \mathbf{w}' \cdot \{V' \cdot I^{\pm} \cdot V\} \cdot \mathbf{w} = \text{const}. \tag{451}$$

What is important, in fact, the binary *basis space* $\langle \mathcal{P}^{n+q}\rangle$) is preserved again, because we introduced in it only other (affine) base with one-valued linear transformation V.

Let $V \neq \text{Const}$ and respectively to its changing the metric reflector tensor does change too, because it is subjected to the permanent general congruent transformation

$$\{G^{\pm}\} = \{V' \cdot I^{\pm} \cdot V\} = \{V' \cdot I^{\pm} \cdot V\}' = \{G^{\pm}\}'. \tag{452}$$

Then the new metric tensor operates in *Special curvilinear coordinates* in the binary space with *Riemannian local metric* due to function $\{G^{\pm}\}(\mathbf{w})$. Its mutual tensor is

$$\{\hat{G^{\pm}}\} = \{G^{\pm}\}^{-1} = \{V^{-1} \cdot I^{\pm} \cdot V'^{-1}\}. \tag{453}$$

This binary space with variable local metric and zero Riemannian–Christoffelian curvature is isometric and topologically equivalent to $\langle \mathcal{P}^{n+q} \rangle$, where latter is the *basis space* by the definition. Curvilinear and pseudo-Cartesian coordinates act in fact in the same flat space. However, if the curvature is non-zero, we have the pseudo-Riemannian space. Both these binary spaces (flat and curve) will be used in Chapter 9A. The geometry, if $V = \text{Const}$, may be considered as linear mapping of pseudo-Euclidean one in admissible affine bases

$$\langle \tilde{E}_{af} \rangle \equiv \langle T_{af} \rangle \cdot \tilde{E} \tag{454}$$

with the constant metric reflector tensor $G^{\pm}$. There holds

$$T'_{af} \cdot \{V' \cdot I^{\pm} \cdot V\} \cdot T_{af} = T'_{af} \cdot \{G^{\pm}\} \cdot T_{af} = \{G^{\pm}\}. \tag{455}$$

Equalities $\boldsymbol{det}\ T_{af} = \pm 1$ follow from (455). We define the group of affine *continuous* trigonometric transformations $\langle T_{af} \rangle$ with respect to $G^{\pm}$ by more exact conditions:

$$T'_{af} \cdot \{G^{\pm}\} \cdot T_{af} = \{G^{\pm}\} = \text{Const}, \quad \boldsymbol{det}\ T_{af} = +1. \tag{456}$$

Due to (448), the metric tensor $\{I^{\pm}\}$ is identical to its mutual analog. This condition, generally, is $G = G' = G^{-1} \to \{G^{\pm}\} = \{\sqrt{I}\}_S$. Hence, in any metric spaces $\langle \mathcal{P}^{n+q} \rangle$ and only in them, contravariant and covariant coordinates are identical, in particular if $q = 0$ or $n = 0$. That is why pseudo-Cartesian bases are uniquely applicable in $\langle \mathcal{P}^{n+q} \rangle$. The metric reflector tensors $\{\sqrt{I}\}_S$ are the general variant of ones for pseudo-Euclidean spaces, when their metric is quadratic and has no distortions.

11.2 The general Lorentzian group of pseudo-Euclidean rotations

In (452) put $V = R$, this spherically orthogonal transformation is not compatible with the simplest metric reflector tensor $\{I^{\pm}\}$ too (sect. 6.3). Then we obtain the following metric reflector tensor in the general form, what is identical to its mutual analog:

$$\{R' \cdot I^{\pm} \cdot R\} = \{\sqrt{I}\}_S = \{\sqrt{I}\}'_S = \{\sqrt{I}\}_S^{-1}. \tag{457}$$

Here $\{\sqrt{I}\}_S$ is a symmetric and hence prime certain square root of I (see more about these in sect. 5.9). Formula (457) describes a metric reflector tensor of the *non-coaxially oriented* pseudo-Euclidean space $\langle \mathcal{P}^{n+q} \rangle$ as well as a reflector tensor of the similar quasi-Euclidean space (see their common definition in sect. 6.3).

The complete *group* $\langle T \rangle$ *of rotational trigonometric transformations* in $\langle \mathcal{P}^{n+q} \rangle$ is determined by conditions similar to (456) with the new metric reflector tensor $\{\sqrt{I}\}_S$ as follows:

$$T' \cdot \{\sqrt{I}\}_S \cdot T = \{\sqrt{I}\}_S = T \cdot \{\sqrt{I}\}_S \cdot T' = \text{Const}, \quad \boldsymbol{det}\ T = +1. \tag{458}$$

In space $\langle \mathcal{P}^{n+q} \rangle$, admissible transformations may be defined in terms of internal or external products, what is equivalent to the identity of contravariant and covariant coordinates:

$$\left.\begin{array}{c} T' \cdot \{\sqrt{I}\}_S \cdot T = \{\sqrt{I}\}_S \ \leftrightarrow T' \cdot \{\sqrt{I}\}_S \cdot T \cdot \{\sqrt{I}\}_S = I \ \leftrightarrow \\ \leftrightarrow T' \cdot \{\sqrt{I}\}_S \cdot T \cdot \{\sqrt{I}\}_S \cdot T' = T' \ \leftrightarrow \\ \leftrightarrow \{\sqrt{I}\}_S \cdot T \cdot \{\sqrt{I}\}_S \cdot T' = I \ \leftrightarrow T \cdot \{\sqrt{I}\}_S \cdot T' = \{\sqrt{I}\}_S. \end{array}\right\} \tag{459}$$

The relation $T' \cdot \{\sqrt{I}\}_S \cdot T = T \cdot \{\sqrt{I}\}_S \cdot T' = \{\sqrt{I}\}_S$ is pseudo-analog of Euclidean one $R' \cdot R = R \cdot R' = I$. But, if $V = I$ in (452), we have again the coaxially oriented space with the metric reflector tensor $\{I^{\pm}\}$ and admissible trigonometric transformations:

$$T' \cdot \{I^{\pm}\} \cdot T = \{I^{\pm}\} = T \cdot \{I^{\pm}\} \cdot T' = \text{Const}, \quad \det T = +1. \tag{460}$$

In (458)–(460) the set $\langle T \rangle$ is called the *Lorentz group of homogeneous transformations* in $\langle \mathcal{P}^{n+q} \rangle$ – in accordance with the initial definition of Poincaré [63]. (Its complex analog exists for the binary complex pseudo-Euclidean space $\langle \mathcal{P}^{n+q} \rangle_c$!) The groups $\langle T \rangle$ and $\langle T_{af} \rangle$ are isomorphic and homothetic:

$$(V^{-1} \cdot T \cdot V)' \cdot \{V' \cdot I^{\pm} \cdot V\} \cdot (V^{-1} \cdot T \cdot V) = \{V' \cdot I^{\pm} \cdot V\}, \quad \langle T_{af} \rangle = V^{-1} \cdot \langle T \rangle \cdot V. \tag{461}$$

An *absolute* pseudo-Euclidean space with respect to its metric reflector tensor $\{\sqrt{I}\}_S$ may be represented in any its pseudo-Cartesian base $\tilde{E}_k$ by the *hyperbolically orthogonal direct sum* of the two real-valued *relative* Euclidean subspaces:

$$\langle \mathcal{P}^{n+q} \rangle \equiv \langle \mathcal{E}^n \rangle^{(k)} \boxtimes \langle \mathcal{E}^q \rangle^{(k)} \equiv \text{CONST}. \tag{462}$$

Moreover, the real-valued subspace $\langle \mathcal{E}^q \rangle$ is obtained as result of realification (447) from the imaginary anti-Euclidean subspace $\langle i\mathcal{E}^q \rangle$. In original complex variant, the *absolute* quasi-Euclidean space $\langle \mathcal{Q}^{n+q} \rangle_c$ is represented in any its quasi-Cartesian base $\{\tilde{E}_k\}_c$ as a *spherically orthogonal direct sum* of the Euclidean and anti-Euclidean subspaces:

$$\langle \mathcal{Q}^{n+q} \rangle_c \equiv \langle \mathcal{E}^n \rangle^{(k)} \boxplus \langle i\mathcal{E}^q \rangle^{(k)} \equiv \text{CONST}. \tag{463}$$

Here and in the sequel, $\boxplus$ and $\boxtimes$ stand for spherically and hyperbolically orthogonal direct summation with respect to a metric reflector tensor. In the indicated both absolute spaces decompositions, these paired summands as the orthogonal complements of each other (in admissible bases $\tilde{E}_k$) are connected one-to-one rigorously functionally, as $\langle \mathcal{E}^q \rangle \equiv Y \langle \mathcal{E}^n \rangle$ and $\langle \mathcal{E}^n \rangle \equiv Y^{-1} \langle \mathcal{E}^q \rangle$. These subspaces are *relative*, but the whole space is *absolute* ! Here $Y(X)$ is some matrix function, connected one-to-one these two spaces. So, for example, we have $y(x) = a - x \leftrightarrow x(y) = a - y$, where a is an absolute.

Due to relation (462), the pseudo-Euclidean space has binary structures determined generally by the reflector metric tensor $\{\sqrt{I}\}_S$ and pseudo-Cartesian bases $\tilde{E}_k$. In this type space, an 1-valent tensor is decomposed in the two hyperbolically orthogonal projections into $\langle \mathcal{E}^n \rangle^{(k)}$ and $\langle \mathcal{E}^q \rangle^{(k)}$; a 2-valent tensor is decomposed in the homogeneous $n \times n$-biprojection into $\langle \mathcal{E}^n \rangle^{(k)}$ and $q \times q$-biprojection into $\langle \mathcal{E}^q \rangle^{(k)}$), and the mixed $n \times q$ and $q \times n$ projections into $\langle \mathcal{E}^n \rangle^{(k)}$ and $\langle \mathcal{E}^q \rangle^{(k)}$ transposed to each other.

For 1-valent tensor objects (under unified compatible binary structure with $\{I^{\pm}\}$ and T), the internal and external multiplications in the base $\tilde{E} = T \cdot \tilde{E}_1$) are determined as follow:

$$\left.\begin{array}{ll} \mathbf{a}_1' \cdot \{I^{\pm}\} \cdot \mathbf{a}_2 = c_{12}, & A_1' \cdot \{I^{\pm}\} \cdot A_2 = C_{12}; \\ \sqrt{I^{\mp}} \cdot T \cdot (\mathbf{a}_1 \mathbf{a}_2') \cdot T' \sqrt{I^{\mp}} = B_{12}, & \sqrt{I^{\mp}} \cdot T \cdot \{A_1 A_2'\} \cdot T' \sqrt{I^{\mp}} = B_{12}. \end{array}\right\} \tag{464}$$

These multiplications are translated into original complex quasi-Euclidean space (463). Thus they may be used in Euclidean geometry including its tensor trigonometry!
Hence, a *metric* reflector tensor in the space $\langle \mathcal{P}^{n+q} \rangle$ executes the following operations:

- it defines the space binary structure,
- it determines the admissible transformations,
- it translates internal and external products into the original space $\langle \mathcal{P}^{n+q} \rangle_c$.

In particular, by this way the following analogs of (120) and (121) are inferred:

$$\left.\begin{array}{l} c_{12} = tr\ B_{12}, \qquad k(C_{12}, t) = k(B_{12}, t); \\ \mathbf{a}' \cdot \{I^{\pm}\} \cdot \mathbf{a} = tr\ (\sqrt{I^{\pm}} \cdot T \cdot \mathbf{a}\mathbf{a}' \cdot T' \cdot \sqrt{I^{\pm}}); \\ k[(A' \cdot \{I^{\pm}\} \cdot A), t] = k[(\sqrt{I^{\pm}} \cdot T \cdot AA' \cdot T' \cdot \sqrt{I^{\pm}}), t]. \end{array}\right\} \qquad (465)$$

These scalar characteristics of admitted vector and lineor objects in a pseudo-Euclidean space are their *real-valued pseudonorms*, in addition to semi-definite norms of sect. 9.2.

For $t = r$, define the *pseudominorant* and the *pseudodianal*:

$$\left.\begin{array}{l} \mathcal{M}p^2(r)A = k[(\sqrt{I^{\pm}} \cdot T \cdot AA' \cdot T' \cdot \sqrt{I^{\pm}}), r] = det\ (A' \cdot \{I^{\pm}\} \cdot A), \\ \\ \mathcal{D}l(r)B_{12} = k(B_{12}, r) = det\ C_{12}. \end{array}\right\} \qquad (466)$$

Rotational matrices and reflectors compatible with a metric tensor do not change internal multiplications (464) and scalar angles in W-forms of projective trigonometric functions of tensor angles between linear objects (vectors, lineors, planars). Note, that in $\langle \mathcal{P}^{n+q} \rangle$, reflectors as well as projectors may be also spherically, hyperbolically, or, generally, pseudo-Euclidean orthogonal. The same relates to geometric objects too.

1-valent tensor objects are *pseudo-orthogonal* if $C_{12} = Z$, this is similar to (155); and they are *at least partially pseudo-orthogonal* if $detC_{12} = 0$, this is similar to (229).
If two objects are spherically orthogonal, then they both are either in $\langle \mathcal{E}^n \rangle$, or in $\langle \mathcal{E}^q \rangle$!
If two objects are hyperbolically orthogonal, then one of them is in $\langle \mathcal{E}^n \rangle$ and another one is in $\langle \mathcal{E}^q \rangle$! The latter is true for decompositions of $\langle \mathcal{P}^{n+q} \rangle$ into its relative subspaces.

Also hyperbolic and spherical analogs of eigenprojectors considered in Ch. 2 operate in this space as shown, for example, in sect. 6.3.

The set of *universal bases* is identical to the set of orthospherical rotational matrices compatible with $I^{\pm}$ with respect to the trigonometric base $\tilde{E}_1 = \{I\}$ – see (352):

$$\left.\begin{array}{l} \langle \tilde{E}_{1u} \rangle \equiv \langle Rot\ \Theta \rangle \cdot \{I\} \equiv \langle \{Rot\ \Theta\} \rangle, \\ Rot'\ \Theta \cdot \{I^{\pm}\} \cdot Rot\ \Theta = \{I^{\pm}\} = Rot\ \Theta \cdot \{I^{\pm}\} \cdot Rot'\ \Theta. \end{array}\right\} (det\ Rot\ \Theta = +1) \qquad (467)$$

The scalar angles in trigonometric rotations (460) and invariant scalar angles between linear objects (in W-forms) are real-valued numbers, they may be spherical (θ_k) or hyperbolic (γ_j) compatible separately with the constant-sign or alternating-sign parts of the metric reflector tensor $I^{\pm}$. In their W-forms, these structures correspond to exactly pure rotational trigonometric types considered in Ch. 5 and 6:

$$T = \{Rot\ (\pm\Theta)\}_{can} \qquad\qquad \{I^{\pm}\}$$

$$\begin{bmatrix} \ddots & & & \\ & \cos\theta_k & \mp\sin\theta_k & \\ & \pm\sin\theta_k & \cos\theta_k & \\ & & & \ddots \end{bmatrix} \Leftrightarrow \begin{bmatrix} \ddots & & & \\ & \pm 1 & 0 & \\ & 0 & \pm 1 & \\ & & & \ddots \end{bmatrix}, \qquad (468)$$

$$T = \{Roth(\pm\Gamma)\}_{can}$$

$$\begin{bmatrix} \ddots & & & \\ & \cosh\gamma_j & \pm\sinh\gamma_j & \\ & \pm\sinh\gamma_j & \cosh\gamma_j & \\ & & & \ddots \end{bmatrix} \Leftrightarrow \begin{bmatrix} \ddots & & & \\ & \pm 1 & 0 & \\ & 0 & \mp 1 & \\ & & & \ddots \end{bmatrix}. \qquad (469)$$

These structures generate with not admissible modal transformation R'_W two *pure types* of general rotational matrices determined with respect to reflector tensor (457) as in (458) and a certain new base. These types are orthospherical and hyperbolic:

$$\left.\begin{array}{l} R_W \cdot \{Rot\ \Theta\}_{can} \cdot R'_W = Rot\ \Theta = T_{(1)}, \quad (T'_{(1)} \cdot T_{(1)} = T_{(1)} \cdot T'_{(1)} = I), \\ T'_{(1)} \cdot \{I^{\pm}\} \cdot T_{(1)} = \{I^{\pm}\} = T_{(1)} \cdot \{I^{\pm}\} \cdot T'_{(1)}, \quad det\ T_{(1)} = +1; \end{array}\right\} \qquad (470)$$

$$\left.\begin{array}{l} R_W \cdot \{Roth\ \Gamma\}_{can} \cdot R'_W = Roth\ \Gamma = T_{(2)}, \quad (T_{(2)} = T'_{(2)}), \\ T_{(2)} \cdot \{I^{\pm}\} \cdot T_{(2)} = \{I^{\pm}\} = T_{(2)} \cdot \{I^{\pm}\} \cdot T_{(2)}, \quad det\ T_{(2)} = +1. \end{array}\right\} \qquad (471)$$

Modal matrices R'_W not compatible with $\{I^{\pm}\}$ change it as in (457) and condition (460) into (458). Thus the group $\langle T \rangle$ contains as pure types $Rot\ \Theta$ and $Roth\ \Gamma$ (Ch. 6).

Generally, an arbitrary transformation T may be a composition of them with respect to certain unity base $\tilde{E}_1$ of their definition:

$$T = \cdots Rot\ \Theta_{(t-1)t} \cdot Roth\ \Gamma_{(t-1)t} \cdots . \qquad (472)$$

Hyperbolic rotations in trigonometric cells, by (469), must correspond to two different blocks from the positive and negative unity parts of a reflector tensor. If $q = 1$, the elementary hyperbolic rotations with their frame axes are (363) and (364). Orthospherical rotations must be compatible with the positive and negative unity parts of a reflector tensor as below:

$$\begin{array}{cc} Rot\ \Theta & I^{\pm} \\ \begin{bmatrix} Rot\ \Theta^{n\times n} & Z^{n\times q} \\ Z^{q\times n} & Rot\ \Theta^{q\times q} \end{bmatrix}, & \begin{bmatrix} +I^{n\times n} & Z^{n\times q} \\ Z^{q\times n} & -I^{q\times q} \end{bmatrix}. \end{array} \qquad (473)$$

The zero quadratic Minkowski invariants $\rho^2(\mathbf{u}) = 0$, centralized with respect to any centered admissible bases $\tilde{E}$, partitions the pseudo-Euclidean space into three subspaces. For metric tensor $\{I^{\pm}\}$, the middle of them is the following dividing conic hypersurface of the 2-nd order:

$$\rho^2(\mathbf{u}) = \sum_{s=1}^{n} x_s^2 - \sum_{t=1}^{q} y_t^2 = \rho^2(\mathbf{x}) - \rho^2(\mathbf{y}) = 0, \quad or \quad \rho^2(\mathbf{u}) = \mathbf{u}' \cdot \{I^{\pm}\} \cdot \mathbf{u} = 0.$$

The hypersurface is invariant with respect to Lorentz passive bases transformations (460). According to this equation, the metric $\rho(\mathbf{u})$ is zero over all of the dividing conic hypersurface. Its generating lines are central middle straight rays. This hypersurface divides $\langle \mathcal{P}^{n+q} \rangle$ into its invariant conic internal and external cavities (if $n > q$) called the *internal and external isotropic cones*. The vertex of these two isotropic cones with this dividing hypersurface is the common origin of all the centralized admissible pseudo-Cartesian bases $\tilde{E}_k$.

For visuality and determinacy, we choose an universal base $\tilde{E}_1$ for tensor trigonometric descriptions with the use sometimes of the dividing hypersurface and two cones at $n > q$. The external isotropic cone ($\rho^2(\mathbf{u}) > 0$) is the open region outside the dividing conic hypersurface, it is also the union of the subspaces $\langle \mathcal{E}^n \rangle^{(k)}$ in decompositions (462). The internal isotropic cone ($\rho^2(\mathbf{u}) < 0$) is the open region inside the dividing conic hypersurface, it is also the union of the subspaces $\langle \mathcal{E}^q \rangle^{(k)}$ in decompositions (462).

The set of admissible rotations in the space $\langle \mathcal{P}^{n+q} \rangle$ with respect to any centralized pseudo-Cartesian base consists of the two connected subsets of Lorentz homogeneous transformations inside and outside the dividing conic hypersurface, what stipulates *isotropy* of these internal and external cones. In general, these motions of any tensor objects have hyperbolically orthogonal homogeneous and mixed projections into instantaneous $\langle \mathcal{E}^n \rangle^{(k)}$ and $\langle \mathcal{E}^q \rangle^{(k)}$, i. e., these motions realized in these two instantaneous isotropic cones.

Hence, $\langle\mathcal{P}^{n+q}\rangle$ *in the whole is isotropic* too for any admissible motions. On the other hand, the parallel translations into its any point are admissible too, and stipulates *homogeneity* of the space $\langle\mathcal{P}^{n+q}\rangle$.

If $q = 1$, then $\langle\mathcal{P}^{n+1}\rangle$ is the Minkowski space (see in Ch. 12) with its *internal double isotropic cone* $(\rho^2(\mathbf{u}) < 0)$ and *external circle isotropic cone* $(\rho^2(\mathbf{u}) > 0)$. In special theory of relativity (STR), the double internal isotropic cone, where $\mathbf{u}$ is *time-like*, is formed by the upper and lower conic parts as so called the *cone of the future* and the *cone of the past*, i. e., in accordance with the positive and negative directions of the ordinate $\overrightarrow{y}^{(k)}$-axis. These parts are situated inside the same dividing conic hypersurface, in STR called the *light cone*. They are the union of the ordinate $\overrightarrow{y}^{(k)}$-axes. In its turn, the external circle isotropic cone, where $\mathbf{u}$ is *space-like*, is the union of the spaces $\langle\mathcal{E}^n\rangle^{(k)}$.

11.3 Polar representation of general pseudo-Euclidean rotations

Any composite continuous transformation (460), for example (472), of geometric objects in internal and external cavities of an isotropic cone, with respect to an universal base $\tilde{E}_1$, may be reduced to the non-commutative product of hyperbolic and orthospherical modal matrices (and as *general measureless tensors of motions*) in the following *two polar forms*:

$$T = Roth\,\Gamma \cdot Rot\;\Theta = Rot\;\Theta \cdot Roth\;\overset{\angle}{\Gamma}, \qquad (474), (475)$$

where $\quad Roth\,\Gamma = \{\sqrt{TT'}\}_{S+} = \sqrt{Roth\;2\Gamma} = Roth'\;\Gamma = Roth^{-1}\;(-\Gamma),$

$$Roth\;\overset{\angle}{\Gamma} = \{\sqrt{T'T}\}_{S+} = \sqrt{Roth\;2\,\overset{\angle}{\Gamma}}$$

are one-valued symmetric arithmetic (and trigonometric) square roots (sect. 5.7, 6.2);

$$Rot\;\Theta = \sqrt{TT'}^{\,-1} \cdot T = Roth\;(-\Gamma) \cdot T = T \cdot \sqrt{T'T}^{\,-1} = T \cdot Roth\;(-\overset{\angle}{\Gamma}) = Rot'\;(-\Theta).$$

Note (!): the polar representations strictly correspond to definition (351) of $\langle\mathcal{P}^{n+q}\rangle$.

From (474), (475) the simple connection between these two principal rotations as well as their two motive hyperbolic tensor angles follows:

$$Roth\;\overset{\angle}{\Gamma} = Rot'\;\Theta \cdot Roth\,\Gamma \cdot Rot\;\Theta = Rot\;(-\Theta) \cdot Roth\,\Gamma \cdot Rot\;\Theta. \qquad (476)$$

Polar representation can be inferred with the use of arithmetic roots by the two ways:

1) $T = S^+ \cdot R \;\Rightarrow\; TT' = S^2,\;\; T'T = R' \cdot S^2 \cdot R \Rightarrow\; T'T = R' \cdot TT' \cdot R \;\Rightarrow$

$$\Rightarrow\; \sqrt{T'T} = R' \cdot \sqrt{TT'} \cdot R \;\Rightarrow\; T = \sqrt{TT'} \cdot R = R \cdot \sqrt{T'T};\;\; det\;T = +1 \;\Rightarrow\; R = Rot\;\Theta;$$

2) $(460), (267), (325) \;\Rightarrow\; (TT') \cdot I^{\pm} \cdot (TT') = I^{\pm} = (T'T) \cdot I^{\pm} \cdot (T'T) \;\Rightarrow\; (471) \;\Rightarrow$

$$\Rightarrow \begin{cases} TT' = Roth\;2\Gamma, & \sqrt{TT'} = Roth\;\Gamma \;\Rightarrow\; (474), \\ T'T = Roth\;2\,\overset{\angle}{\Gamma}, & \sqrt{T'T} = Roth\;\overset{\angle}{\Gamma} \;\Rightarrow\; (475); \end{cases} \quad det\;T = +1 \;\Rightarrow (476).$$

By (476), Γ and $\overset{\angle}{\Gamma}$ *have the same angles eigenvalues spectrum* $\langle\gamma_j\rangle$.

We shall use widely such polar representations of a general rotational transformation for simple description of polysteps hyperbolic or spherical principal rotations, for example, of the relativistic motions in STR, and motions in spherical and hyperbolic geometries.

Further consider the polar representation of trigonometric modal transformations:

$$\left.\begin{array}{l} T = \sqrt{TT'} \cdot R = S_1 \cdot R = (S_1 \cdot R \cdot S_1^{-1}) \cdot S_1 = \\ = R \cdot \sqrt{T'T} = R \cdot S_2 = (R \cdot S_2 \cdot R') \cdot R. \end{array}\right\} (R = Rot\ \Theta) \qquad (477, 478)$$

The symmetric matrices of principal rotations $S_1 = Roth\ \Gamma$ and $S_2 = Roth\ \overset{\angle}{\Gamma}$ are expressed in (474), (475) in canonical form (324) in the unity base $\tilde{E}_1 = \{I\}$. But the latter acts in the base $\tilde{E}_{1u} = Rot\ \Theta \cdot \tilde{E}_1$ and then is transformed in it by the rotation R.
The orthospherical rotation $Rot\ \Theta$ is expressed initially in $\tilde{E}_1 = \{I\}$ too. But $Rot\ \Theta$ acts really in the base $\tilde{E}_{1h} = Roth\ \Gamma \cdot \tilde{E}_1$ and then is transformed in it by rotation S_1.

According to (477) the matrix S_1 acts in the base $\tilde{E}_1$ and realizes the base rotation at the angle Γ, and then the orthospherical matrix R acts in this hyperbolically rotated base $\tilde{E}_{1h}$ and realizes the base rotation at the angle Θ. According to (478) the matrix R acts in the base $\tilde{E}_1$ and realizes the base rotation at the angle Θ, and then the matrix S_2 acts in this spherically rotated base $\tilde{E}_{1u}$ and realizes the base rotation at the angle Γ. Both these modal transformations of the base $\tilde{E}_1$ are *formally* equivalent.

Similar sense of these two variants of multiplications S and R appears in a *passive transformation* of an element $\mathbf{u}^{(1)}$ coordinates:

$$\mathbf{u}^{(2)} = (S_1 \cdot R)^{-1} \cdot \mathbf{u}^{(1)} = R^{-1} \cdot S_1^{-1} \cdot \mathbf{u}^{(1)} = \{R' \cdot S_1 \cdot R\}^{-1} \cdot R^{-1} \cdot \mathbf{u}^{(1)} =$$

$$= (R \cdot S_2)^{-1} \cdot \mathbf{u}^{(1)} = S_2^{-1} \cdot R^{-1} \cdot \mathbf{u}^{(1)} = \{S_2^{-1} \cdot R \cdot S_2\}^{-1} \cdot S_2^{-1} \cdot \mathbf{u}^{(1)}. \qquad (479)$$

In a linear pseudo-Euclidean space, separate the full set of right pseudo-Cartesian bases $\langle T \cdot \tilde{E}_1 \rangle$. All these bases are rotationally connected as $det\ T = +1$. Transition from $\tilde{E}_1$ to a new base $\tilde{E}$ may be represented, by (474) and (475), in the following two polar forms – straight and inverse:

$$\tilde{E} = T \cdot \tilde{E}_1 = Roth\ \Gamma \cdot Rot\ \Theta \cdot \tilde{E}_1 = (Roth\ \Gamma \cdot Rot\ \Theta \cdot Roth^{-1}\ \Gamma) \cdot Roth\ \Gamma \cdot \tilde{E}_1, \qquad (480)$$

$$\tilde{E} = T \cdot \tilde{E}_1 = Rot\ \Theta \cdot Roth\ \overset{\angle}{\Gamma} \cdot \tilde{E}_1 = (Rot\ \Theta \cdot Roth\ \overset{\angle}{\Gamma} \cdot Rot'\ \Theta) \cdot Rot\ \Theta \cdot \tilde{E}_1. \qquad (481)$$

These two forms give the two possible sequences of these hyperbolic and orthospherical rotations execution. For both these variants: in the left multiplications these matrices are expressed in the base $\{I\}$ of their definitions; in the right multiplications these matrices are expressed in the bases of their actions! Hence, these two polar forms realize the principal hyperbolic rotation in different bases: straight polar form (480) in the base $\tilde{E}_1$ and inverse polar form (481) in the other universal base $\tilde{E}_{1u} = Rot\ \Theta \cdot \tilde{E}_1$.

For any pseudo-Cartesian base $\tilde{E}_k$, first n columns of its matrix determine the subspace $\langle \mathcal{E}^n \rangle^{(k)}$, other q columns determine $\langle \mathcal{E}^q \rangle^{(k)}$ in hyperbolically orthogonal sum (462). The matrix $Rot\ \Theta$ has structure (473), that is why only hyperbolic rotations of any pseudo-Cartesian base $\tilde{E}_k$ give new subspaces $\langle \mathcal{E}^n \rangle^{(j)}$ and $\langle \mathcal{E}^q \rangle^{(j)}$ determined by the columns of the new base $\tilde{E}_j$ matrix. If the new base $\tilde{E}$ connected with $\tilde{E}_1 = \{I\}$ by a modal matrix T or $Roth\ \Gamma$, then in the base we have the following identities:

$$\left.\begin{array}{l} \langle \mathcal{E}^n \rangle \equiv im\ [\tilde{E}]^{(n+q)\times n} \equiv im\ [T]^{(n+q)\times n} \equiv im\ [Roth\ \Gamma]^{(n+q)\times n}, \\ \langle \mathcal{E}^q \rangle \equiv im\ [\tilde{E}]^{(n+q)\times q} \equiv im\ [T]^{(n+q)\times q} \equiv im\ [Roth\ \Gamma]^{(n+q)\times q}. \end{array}\right\} \qquad (482)$$

This means that all trigonometric rotations (460) applied to the Euclidean subspaces $\langle\mathcal{E}^n\rangle$ and $\langle\mathcal{E}^q\rangle$ *in the whole* as sets of point elements are reduced to their pure hyperbolic rotation from (474). In particular, for a Minkowski space $\langle\mathcal{P}^{n+1}\rangle$, the n and 1 columns of the matrices $\tilde{E}$, T, $roth\ \Gamma$ determine the space $\langle\mathcal{E}^n\rangle$ and the axis $\overrightarrow{y}$ as the relative subspaces in the base $\tilde{E}$ after the base $\tilde{E}_1$ rotation by the matrix T or $roth\ \Gamma$.

Hence, the polar formula (474) reduces any admissible transformation T of the two relative subspaces in the whole from the original base $\tilde{E}_1 = \{I\}$ into any admissible pseudo-Cartesian base $\tilde{E}$ till their pure hyperbolic rotation $Roth\ \Gamma = \sqrt{TT'}$.

The polar representation of a general trigonometric transformation of the relative subspaces in the whole as hyperbolic rotation does not hold for subsets of these subspaces, in particular, the base coordinate axes. This can be seen in (481): the coordinate axes are subjected to orthospherical rotation and then hyperbolic rotation.

The matrix of a transformation T, due to (460), is a bivalent pseudo-Euclidean quasi-biorthogonal tensor. This is true for the matrix of the base $\tilde{E} = T \cdot \{I\}$ too. The tensor is splitted projectively into the pair of symmetric homogeneous ($n \times n$ and $q \times q$) and the pair of mutually transposed mixed ($n \times q$ and $q \times n$) tensor projections:
$[\tilde{E}]^{n\times n}$ is orthoprojection of space-like unity basis vectors into the subspace $\langle\mathcal{E}^n\rangle^{(1)}$;
$[\tilde{E}]^{q\times q}$ is orthoprojection of time-like unity base vectors into the subspace $\langle\mathcal{E}^q\rangle^{(1)}$;
$[\tilde{E}]^{n\times q}$ and $[\tilde{E}]^{q\times n}$ are mutually transposed oblique projections into $\langle\mathcal{E}^n\rangle^{(1)}$ and $\langle\mathcal{E}^q\rangle^{(1)}$.
If the base matrix is transposed, then these projections are reflected with respect to the matrix main diagonal. This takes place, in particular, under changing the direction of a multistep hyperbolic rotation sequence (see in next sect.).

If $q = 1$, then the matrix $Rot\ \Theta^{q\times q}$ in (473) degenerates into I. Then in the Minkowski space $\langle\mathcal{P}^{n+1}\rangle$, an 1-valent tensor is decomposed in two hyperbolically orthogonal projections into $\langle\mathcal{E}^n\rangle^{(k)}$ and onto $\overrightarrow{y}^{(k)}$; a 2-valent tensor is decomposed in an homogeneous projection $n \times n$-tensor into $\langle\mathcal{E}^n\rangle^{(k)}$, an invariant scalar onto $\overrightarrow{y}^{(k)}$-axis, and two mixed projections – $n \times 1$-vectors into $\langle\mathcal{E}^n\rangle^{(k)}$ and onto $\overrightarrow{y}^{(k)}$. World events in STR are described here from the view-point of a relatively immobile Observer with respect to an universal base. Among them, $\tilde{E}_1 = \{I\}$ is the simplest original one. Any concrete spherical-hyperbolic analogy (from sect. 6.2) is realized with respect to this base!

In this Minkowski space, Lorentz transformation (460) of a point element on the $\overrightarrow{y}^{(1)}$-axis is reduced by polar representation up to either it hyperbolic rotation together with the ordinate axis (under passive transformation), or it hyperbolic rotation off the ordinate axis in the direction given by the orthospherical tensor angle (under active transformation). Consider two examples with elementary matrices useful in STR.

Example 1.

$$\mathbf{u}^{(\jmath)} = \{rot'\ \Theta \cdot roth\ \Gamma \cdot rot\ \Theta\}^{-1} \cdot rot'\ \Theta \cdot \mathbf{u}^{(1)} = \{rot'\ \Theta \cdot roth\ \Gamma \cdot rot\ \Theta\}^{-1} \cdot \mathbf{u}^{(1)}, \quad (483)$$

where $\mathbf{u}^{(1)} \in \langle\overrightarrow{y}^{(1)}\rangle$ is a point object with respect to $\tilde{E}_1$, and $\mathbf{u}^{(\jmath)}$ is the same object with respect to $\tilde{E}_\jmath = T_{1\jmath} \cdot \tilde{E}_1$. However, its pure hyperbolic passive transformation (in brackets) was realized here from the base $\tilde{E}_{1u} = rot\ \Theta \cdot \tilde{E}_1$ into the final base $\tilde{E}_\jmath$!

Example 2.

$$\mathbf{u}_\jmath = T_{1\jmath} \cdot \mathbf{u}_1 = \{rot\ \Theta \cdot roth\ \overset{\angle}{\Gamma} \cdot rot'\ \Theta\} \cdot rot\ \Theta \cdot \mathbf{u}_1 = \{rot\ \Theta \cdot roth\ \overset{\angle}{\Gamma} \cdot rot'\ \Theta\} \cdot \mathbf{u}_1, \quad (484)$$

where $\mathbf{u}_1 \in \langle\overrightarrow{y}^{(1)}\rangle$ is a point element, it generated in $\tilde{E}_1$ the element $\mathbf{u}_\jmath = T_{1\jmath} \cdot \mathbf{u}_1$. Here the pure hyperbolic active rotation was realized off $\overrightarrow{y}^{(1)}$ under the angle Θ!

11.4 Polysteps hyperbolic rotations with polar decomposition

The summarized polysteps hyperbolic rotation is pure hyperbolic if its particular rotations are trigonometrically compatible with each other (see Ch. 6), i. e., they can be reduced to cell-forms (324) in common base. In particular, vectors of directional cosines for elementary hyperbolic rotations in (363) are equal to each other up to coefficients ± 1. If particular rotational matrices are not trigonometrically compatible (though each of them is compatible with the given metric reflector tensor), then a composite formula of non-symmetric (in general) polysteps hyperbolic rotations can be reduced always till polar forms (474), (475).

Specify the sequence of particular hyperbolic rotations as measureless hyperbolic tensors of motions, realizing geodesic motions on hyperboloid hypersurfaces, at $\rho^2(\mathbf{u}) = \text{const}$, and expressed in the original unity base $\tilde{E}_1 = \{I\}$: $Roth\ \Gamma_{12}$, $Roth\ \Gamma_{23}, \ldots,$ $Roth\ \Gamma_{(t-1)t}$. – with their canonical form (324) in $\langle \mathcal{P}^{n+q} \rangle$, either elementary one (363) in $\langle \mathcal{P}^{n+1} \rangle$.

For descriptive analysis in $\tilde{E}_1 = \{I\}$, the matrices in the own bases $\tilde{E}_k$ have the new forms, because the following matrices realize hyperbolic rotations in $\tilde{E}_k$. All these forms correspond to an adopted tensor $\{I^{\pm}\}$. These sequential bases are transformed as follows:

$$\tilde{E}_1 = \{I\}, \tilde{E}_2 = \{Roth\ \Gamma_{12}\}_{(\tilde{E}_1)} \cdot \tilde{E}_1, \ldots, \tilde{E}_t = \{Roth\ \Gamma_{(t-1)t}\}_{(\tilde{E}_{t-1})} \cdot \tilde{E}_{t-1}.$$

Translate the matrix $\tilde{E}_t$ from the base of its action into the original base $\tilde{E}_1 = \{I\}$ for rotations analysis, obtain the dual formula for resulting multisteps transformation:

$$\tilde{E}_t = T_{1t} \cdot \tilde{E}_1 = \{Roth\ \Gamma_{(t-1)t}\}_{(\tilde{E}_{t-1})} \cdots \{Roth\ \Gamma_{23}\}_{(\tilde{E}_2)} \cdot \{Roth\ \Gamma_{12}\}_{(\tilde{E}_1)} \cdot \tilde{E}_1 =$$

$$= T_{1t} \cdot \tilde{E}_1 = Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23} \cdots Roth\ \Gamma_{(t-1)t} \cdot \tilde{E}_1. \tag{485}$$

It is the **Rule** of executing multisteps transformations (proved by induction on $t \geq 3$).

$$\tilde{E}_3 = \{Roth\ \Gamma_{23}\}_{(\tilde{E}_2)} \cdot \tilde{E}_2 = \{Roth\ \Gamma_{23}\}_{(\tilde{E}_2)} \cdot \{Roth\ \Gamma_{12}\}_{(\tilde{E}_1)} \cdot \tilde{E}_1 =$$

$$= \{Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23} \cdot Roth^{-1}\ \Gamma_{12}\} \cdot \{Roth\ \Gamma_{12}\} \cdot \tilde{E}_1 = Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23} \cdot \tilde{E}_1. \tag{486}$$

The sequence of the canonical matrices in (485) is inversed (see, for example, [21, p. 428]).

Coordinates of linear objects are transformed *passively*, but the sequence of the inverse rotational matrices in their canonical form is direct:

$$\mathbf{u}^{(t)} = Roth\ (-\Gamma_{(t-1)t}) \cdots Roth\ (-\Gamma_{23}) \cdot Roth\ (-\Gamma_{12}) \cdot \mathbf{u}^{(1)} =$$

$$= \{Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23} \cdots Roth\ \Gamma_{(t-1)t}\}^{-1} \cdot \mathbf{u}^{(1)}, \tag{487}$$

$$\mathbf{u}^{(3)} = Roth\ (-\Gamma_{23}) \cdot \mathbf{u}^{(2)} = Roth\ (-\Gamma_{23}) \cdot Roth\ (-\Gamma_{12}) \cdot \mathbf{u}^{(1)} =$$
$$= \{Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23}\}^{-1} \cdot \mathbf{u}^{(1)}. \tag{488}$$

Active polysteps hyperbolic rotational transformations of generating element $\mathbf{u}$, for example, in $\tilde{E}_1 = \{I\}$, are realized similarly to analogous polysteps hyperbolic transformations of the base, when particular rotational matrices are ordered inversely (as in (485), because they are determined and act sequentially with respect to $\tilde{E}_1$:

$$\mathbf{u}_t = Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23} \cdots Roth\ \Gamma_{(t-1)t} \cdot \mathbf{u}_1, \tag{489}$$

$$\mathbf{u}_3 = \{Roth\ \Gamma_{12} \cdot (Roth\ \Gamma_{23})_{\tilde{E}_1} \cdot Roth^{-1}\Gamma_{12}\}_{\tilde{E}_2} \cdot Roth\ \Gamma_{12} \cdot \cdot \mathbf{u}_1 =$$
$$= Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23} \cdot \mathbf{u}_1 = \{Roth\ \Gamma_{23}\}_{\tilde{E}_2} \cdot \mathbf{u}_2. \tag{490}$$

Formulae (485)–(490) are special cases of the *General rule of polysteps transformations*. Other special cases of the rule relate to similar sequences of principal spherical rotations – motions in a quasi-Euclidean binary space $\langle Q^{n+q}\rangle$ (Ch. 8A).

In pseudo-Euclidean geometry, matrices of pure hyperbolic (principal) rotations can be or not be symmetric, but they are always prime. This depends on the bases of their definition and action. A matrix is symmetric in canonical forms (324), (362), (363) with respect to any unity base of its definition. The matrix $T \cdot Roth\ \Gamma \cdot T^{-1}$ represents the hyperbolic rotation with respect to the universal base $\tilde{E}_1$ and acting in the pseudo-Cartesian base $\tilde{E} = T \cdot \tilde{E}_1$. Prime matrices of hyperbolic rotations also belong to the Lorentz group with the metric tensor $I^{\pm}$. A prime hyperbolic matrix may be represented in $\tilde{E}_1$ in polar forms (474), (475) for its analysis. The analogous statements hold for orthospherical rotations $Rot\ \Theta$ and $T \cdot Rot\ \Theta \cdot T^{-1}$ too. They may be expressed with respect to either the original base $\tilde{E}_1$, or the base $\tilde{E} = T \cdot \tilde{E}_1$ of their action. *All pure orthospherical rotations form their complete continuous subgroup* of the Lorentz group of homogeneous (or continuous) transformations.

For a generating or transforming element $\mathbf{u}$, *its continuous Lorentz transformations do not change the value of the invariant* $\rho^2(\mathbf{u}) = [T \cdot \mathbf{u}]' \cdot \{I^{\pm}\} \cdot [T \cdot \mathbf{u}] = \mathbf{u}' \cdot \{I^{\pm}\} \cdot \mathbf{u}$ *similar to continuous motions on the hyperboloid surface with invariant* $\rho^2(\mathbf{u}) = const$!

Further, in order to analyze and reduce the expressions for two-steps and polysteps hyperbolic rotations, we use again polar representations (474), (475). There hold:

$$\tilde{E}_t = T_{1t} \cdot \tilde{E}_1 = Roth\ \Gamma_{1t} \cdot Rot\ \Theta_{1t} \cdot \tilde{E}_1 = Rot\ \Theta_{1t} \cdot Roth\ \overset{\angle}{\Gamma}_{1t} \cdot \tilde{E}_1, \tag{491}$$

$$\left.\begin{aligned} &Roth\ \Gamma_{13} = \sqrt{TT'} = \sqrt{Roth\ \Gamma_{12} \cdot Roth\ (2\Gamma_{23}) \cdot Roth\ \Gamma_{12}} = \\ &= \sqrt{Roth\ (2\Gamma_{13})}, \\ &Rot\ \Theta_{13} = Roth\ \Gamma_{31} \cdot Roth\ \Gamma_{12} \cdot Roth\ \Gamma_{23} = Rot'\ (-\Theta_{13}) = \\ &= Rot^{-1}\ (-\Theta_{13}) = Rot'\ \Theta_{31} = Rot\ (-\Theta_{31}), \end{aligned}\right\}(t=3) \tag{492}$$

$$\left.\begin{aligned} &\mathbf{u}^{(t)} = (Rot'\ \Theta_{1t} \cdot Roth\ \Gamma_{1t} \cdot Rot\ \Theta_{1t})^{-1} \cdot Rot'\ \Theta_{1t} \cdot \mathbf{u}^{(1)} = \\ &= \{Roth\ \Gamma_{1t}\}^{-1}_{\tilde{E}_{1u}} \cdot \mathbf{u}^{(1u)}, \\ &A^{(f)} = (Rot'\ \Theta_{1t} \cdot Roth\ \Gamma_{1t} \cdot Rot\ \Theta_{1t})^{-1} \cdot Rot'\ \Theta_{1t} \cdot A^{(1)} = \\ &= \{Roth\ \Gamma_{1t}\}^{-1}_{\tilde{E}_{1u}} \cdot A^{(1u)}. \end{aligned}\right\}(t \geq 3) \tag{493}$$

The rotation $Rot\ \Theta_{13}$ is executed separately in the bases of particular rotations actions in the sequence 31, 12, 23 along of legs of the orthospherical triangle 123 in Euclidean subspace.

So, polysteps hyperbolic geodesic motions of a point element, when $\rho^2(\mathbf{u}) = \rho^2 = \text{const}$, sequentially produce apices of a certain geometric figure, for examples, triangle or polygon. A necessary condition for such entire construction be a geometric figure is that the sequential hyperbolic rotations form a closed circuit with summarized hyperbolic angle annihilation: $\prod_{k\geq 3} Roth\ \Gamma_{(k)}\mathbf{u}_1 = Rot\ \Theta_{1t} \cdot \mathbf{u}_1$.

Geometry of such figures from geodesic hyperbolic segments is realized, for example, on two *invariant hyperboloid hypersurfaces*, i. e., generally of maximal dimension, with their given quadratic centralized Minkowski invariants $\rho^2(\mathbf{u}) = \pm R^2$ (see above in sect. 11.2):

$$\mathbf{u}' \cdot \{I^{\pm}\} \cdot \mathbf{u} = \sum_{s=1}^{n} x_s^2 - \sum_{t=1}^{q} y_t^2 = \rho^2(\mathbf{x}) - \rho^2(\mathbf{y}) = \rho^2(\mathbf{u}) = \pm R^2, \quad (R = \text{const}). \tag{494}$$

If $R = 0$, then, in any admissible to $\{I^{\pm}$ pseudo-Cartesian bases with the same origin, we have a centralized invariant conic surface dividing the pseudo-Euclidean space into its internal and external cavities. For pure hyperbolic geometric figures, their segments are continuous, that is why, this constructed figure is contained in exactly one cavity of this conic surface: either inside the internal cone with $\rho^2(\mathbf{u}) = -R^2$ ($\rho^2(\mathbf{y}) > \rho^2(\mathbf{x})$), or inside the external cone with $\rho^2(\mathbf{u}) = +R^2$ ($\rho^2(\mathbf{x}) > \rho^2(\mathbf{y})$).

However from (494) we may get else, as trivial cases, real-valued n- and q-dimensional spheres with their equations: $\sum_{s=1}^{n} x_s^2 = \rho^2(\mathbf{x}) = +R^2$ if we put $y_t = 0 \to \rho^2(\mathbf{y}) = 0$ and $-\sum_{t=1}^{q} y_t^2 = -\rho^2(\mathbf{y}) = -R^2$ if we put $x_t = 0 \to \rho^2(\mathbf{x}) = 0$. They have the usual spherical geometry for a sphere in Euclidean space. Here the geometry may have place on the spheres with the radius R in two Euclidean subspaces $\langle\mathcal{E}^n\rangle_{(x)}$ and $\langle\mathcal{E}^q\rangle_{(y)}$ in any admissible bases of the pseudo-Euclidean space $\langle\mathcal{P}^{n+q}\rangle$.

The active homogeneous Lorentzian transformations perform motions of the generating element $\mathbf{u} = T \cdot \mathbf{u}_1$ on this hyperboloid with Minkowski invariant $\rho^2(\mathbf{u}_1) = \rho^2(\mathbf{u}) = const.$ If this circuit of hyperbolic motions is complete and closed at $t = 3$ or $t \geq 3$ in (485), i. e., these principal hyperbolic motions form on it a closed geometric figure (hyperbolic triangle or hyperbolic polygon) with the quadratic Minkowski invariant $\rho^2(\mathbf{u})$, then here as the result is the appearance of the induced secondary orthospherical precession $Rot\ \Theta$. In Appendix we'll prove that its orthospherical angle θ is equal to the angular deviation of Gauss–Bonnet in such a closed figure in non-Euclidean geometries of constant radius-parameter R, and the precession is the deviation algebraic cause explained by tensor trigonometry!

In the Minkowski space-time $\langle\mathcal{P}^{3+1}\rangle$ of STR, the orthospherical rotation in (491) is the result of summing motions (velocities) with different directions $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$. In STR, this is a *secondary rotation*. Its well-known case is a *Thomas precession*. The principal hyperbolic motion is called a *boost*. The feature of velocities summation law is explained by hyperbolic nature of principal motions in this space.

In conclusion of this Chapter note, that the sum of motions is invariant under a choice either passive or active transformations of coordinates. We choose T for the original base $\tilde{E}_1$ transformation as a more descriptive variant, and we shall use this in Appendix.

Chapter 12

Tensor trigonometry of Minkowski pseudo-Euclidean space with geometries of two embedded hyperboloids

12.1 Trigonometric models of bi-associated hyperbolic geometries

Now consider more in details the *coaxially oriented* pseudo-Euclidean space $\langle\mathcal{P}^{n+1}\rangle$, i. e., as *geometric Minkowski space* and as *Minkowski space-time* of STR at $n=3$ [65]. Due to (462) at $q=1$, it is expressed in the base $\tilde{E}_k$ as such a *hyperbolically orthogonal direct sum*

$$\langle\mathcal{P}^{n+1}\rangle \equiv \langle\mathcal{E}^n\rangle^{(k)} \boxtimes \overrightarrow{y}^{(k)} \equiv \langle\mathcal{E}^n\rangle^{(k)} \boxtimes \overrightarrow{ct}^{(k)} \equiv \text{CONST } \{\text{under acting } I^{\pm} \text{ or } I^{\mp} \ (17A)\}.$$

Tensor trigonometry in this pseudo-Euclidean space are realized in *elementary forms* of the tensor angle Γ and its trigonometric functions (362)-(365) with the frame ordinate axis $\overrightarrow{y}$, as $q=1$, – see this in sect. 6.5. Note, in any pseudo-Euclidean and quasi-Euclidean spaces – see in sect. 6.3., the tensor trigonometry in its different kinds is realizable and applicable due to homogeneity and isotropy of these spaces! First homogeneity and isotropy to the space-time of events were stated by H. Poincaré [63, 64]. We use two quadric relations for definitions in the base $\tilde{E}_1$ of $\langle\mathcal{P}^{n+1}\rangle$ of two *perfect* hyperboloidal hypersurfaces (Ch. 6A) with different topologies and Minkowski invariants $\rho^2(\mathbf{v}) = +R^2$ and $\rho^2(\mathbf{u}) = -R^2$ at $R = \text{const}$:

$$\mathbf{v}'\cdot\{I^{\pm}\}\cdot\mathbf{v} = \sum_{s=1}^{n} x_s^2 - y^2 = \rho^2(\mathbf{x}) - y^2 = \rho^2(\mathbf{v}) = +R^2 = (\pm R)^2, \ \ (||\mathbf{x}||_E > |y|_P), \quad (495-I)$$

$$\mathbf{u}'\cdot\{I^{\pm}\}\cdot\mathbf{u} = \sum_{s=1}^{n} x_s^2 - y^2 = \rho^2(\mathbf{x}) - y^2 = \rho^2(\mathbf{u}) = -R^2 = (iR)^2, \ \ (||\mathbf{x}||_E < |y|_P). \quad (495-II)$$

Here $\mathbf{u}$ and $\mathbf{v}$ are the radius-vectors of points on these hyperboloidal hypersurfaces, $\mathbf{x}$ is their vector projection into $\langle\mathcal{E}^n\rangle$, y is their scalar projection onto $\overrightarrow{y}$ (and $||\mathbf{u}||_P = ||\mathbf{v}||_P = R$). As *invariant geometric objects* in $\tilde{E}_1 = \{I\}$, they are Minkowski *hyperboloids I and II* [65]. They have own two non-Euclidean geometries: two-sheets hyperbolic (II) and one-sheet hyperbolic–elliptical (I) – see further. Their internal geometries are equivalent to their external tensor trigonometries, with exactness up to coefficient of similarity "R". (Such a property relates to all perfect surfaces – see their definition in the Introduction.)

Due to equation (I), for any value of the ordinate y it is possible on a hyperboloid I to realize spherical figures (till circles) of radius $r = +\sqrt{y^2+R^2}$. Due to equation (II), for the values of the ordinate $|y| > R$ it is possible on a hyperboloid II to realize also spherical figures of radius $r = +\sqrt{y^2-R^2}$. Their equations are $\sum_{s=1}^{k} x_s^2 = r^2, (k \le n)$. At $\rho^2 = 0$, these equations give an asymptotic invariant *isotropic or light cone*, dividing and placing these geometric objects in external and internal cavities of the cone (Figure 4). Our measureless hyperbolic tensor of motion (363) as $roth\ (\pm\Gamma) \equiv F(\pm\gamma, \mathbf{e}_\alpha)$ determines homogeneous motions of the points $\mathbf{u}$ and $\mathbf{v}$ on both these Minkowski hyperboloids, – as well as our measureless spherical tensor of motion (314) as $rot\ (\pm\Phi) \equiv F(\pm\varphi, \mathbf{e}_\alpha)$ determines homogeneous motions of the point $\mathbf{u}$ on the Special *hyperspheroid*. All these main geometric objects are constructed in the universal base $\tilde{E}_1 = \{I\}$, in order to study descriptively on them various types of motions and equivalent rotations in their enveloping binary spaces, using various spherical–hyperbolic analogies. (They are trigonometric at parameter $R=1$.) Hyperboloids are invariant to Lorentz transformations (sects. 6.3, 6,5, 11.2). Hyperspheroid is invariant to new Special transformations, introduced by us in sects. 5.7, 5.12, 6.3. Both hyperboloids with $R=1$ may be seen in cut at $n=2$ on Figure 4 with hyperbolic motions on them. Their initial points (the unique $\mathbf{u}_1$ for II and, for example, $\mathbf{v}_1$ for I) are rotated hyperbolically into other ones $\mathbf{u}_k$ and $\mathbf{v}_k$ along the pure hyperbolic meridians of II and I.

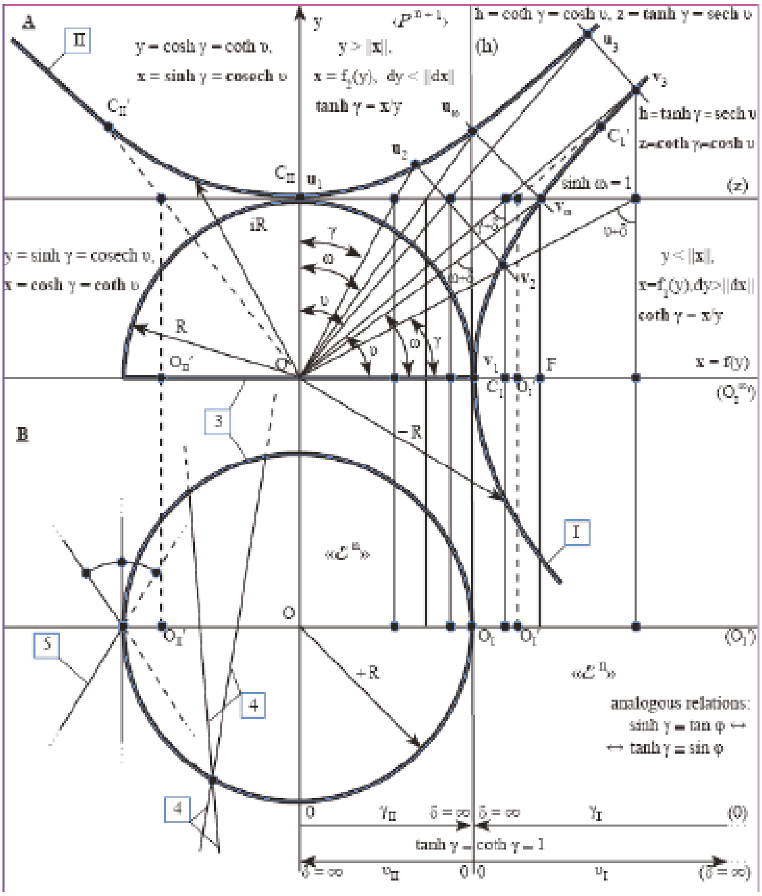

Figure 4. Trigonometric models of the hyperboloids II (tangent-secant) and I (cotangent-cosecant), and of the hyperspheroid (sine-cosine) – in their upper parts.

A. Trigonometric correspondence between points of the Minkowskian hyperboloids I and II (at R=1) in the cut by centered pseudoplane in $\langle \mathcal{P}^{n+1} \rangle$. The hyperbolic angles are presented: acute principal γ, complementary υ, especial ω ($\sinh\ \omega = 1$), right infinite δ, obtuse $\upsilon \uplus \delta$.
B. Trigonometric models in the universal trigonometric base $\bar{E}_1 = \{I\}$, or projective models with respect to the Cayley absolute oval into the projective hyperplane $\langle\langle \mathcal{E}^n \rangle\rangle$.

(I) one-sheet hyperboloid I of $\rho = \pm R$ with cylindrical topology and cotangent model, where $\coth\ (\gamma, \upsilon) = \cosh(\upsilon, \gamma)$, identical to projective model outside the Cayley oval,
(II) a two-sheets hyperboloid II of radii $\rho = +iR$ (upper) and $\rho = -iR$ (lower) with flat topology and tangent model identical to projective Klein model inside the Cayley oval,
(3) tangent-cotangent projections with Klein disk and Cayley oval as an isotropic cone,
(4) conjugate parallel straight lines (geodesic) inside and outside the Cayley oval,
(5) correspondences between straight lines (geodesic) inside and outside the Cayley oval.

Rotation of a time-like hyperbola generates the one sheet hyperboloid I (it is seeming as *"horn shaped"*). Rotation of two space-like hyperbolae generates the hyperboloid II from two coupled sheets (they are seeming as *two symmetric cups*) inside the external double cone. One sheet hyperboloid I has radii $+R$ ($y \geq 0$) and $-R$ ($y < 0$). Two sheets hyperboloid II has 1-st sheet radius $+iR$ ($y \geq +R$) and 2-nd sheet radius $-iR$ ($y \leq -R$). This stipulates their constant radius-parameter R at constant negative Gaussian curvature K_G in $\langle \mathcal{P}^{n+1} \rangle$. Rotation of a large circle around the ordinate axis $\overrightarrow{y}$ generates a hyperspheroid with radius R in the base $\bar{E}_1$. This stipulates its constant radius-parameter R at constant positive Gaussian curvature K_G. These rotations are executed with $(n-1)$ degrees of freedom.

Centered (with the center O) pure hyperbolic motions in $\tilde{E}_1$ of radius-vectors $\mathbf{u_1}$ (space-like) and $\mathbf{v_1}$ (time-like) at Figure 4 on these hyperboloids are expressed by rotational matrix functions $roth\ \Gamma = F(\gamma, \mathbf{e}_\alpha)$ due to formulae (362), (363). Non-centered in $\tilde{E}_1$ *hyperboloidal motions* of elements $\mathbf{u_2}$ and $\mathbf{v_2}$ are presented as purely hyperbolic only as centered in $\tilde{E}_2$ as $T_{12}\{roth\ \Gamma_{23}\}_{\tilde{E}_1} T_{12}^{-1}$ – see (490) in sect. 11.4. In both these cases, it is possible to have own orthospherical rotations $d\alpha_{(1)}$ and $d\alpha_{(2)}$ as independent or secondary induced ones in polar decomposition (111A). (The angle γ ranges in $[0; +\infty)$ if $dy > 0$ and in $[0; -\infty)$ if $dy < 0$.) (The angle γ ranges in $[0; +\infty)$ if $dy > 0$ and in $[0; -\infty)$ if $dy < 0$.)

On the hyperboloid II in $\langle \mathcal{P}^{n+1} \rangle$, the extent of space-like geodesic hyperboloidal arcs is the pseudo-Euclidean external natural measure of a length λ in the base $\tilde{E}_1$, expressed in the centered cutting k-th pseudo-Euclidean plane. On the hyperboloid I in $\langle \mathcal{P}^{n+1} \rangle$, the extent of time-like geodesic hyperboloidal arcs is the pseudo-Euclidean external natural measure of a length λ in the base $\tilde{E}_1$, expressed in the same centered cutting k-th pseudo-Euclidean plane; but the extent of space-like extremal ellipsoidal arcs is the pseudo-Euclidean external natural measure of a length λ in $\tilde{E}_1$, expressed in the centered cutting j-th pseudo-Euclidean plane. In all these cases, these cutting planes include the point O and such lines arcs. Since these hyperboloids are *perfect hypersurfaces* (see in Introduction and Chs. 6, 6A, 7A, 8A), their natural λ and angular λ/R Lambert measures expressed proportionally with radius-parameter R. Their external pseudo-Euclidean geometries are isometric to their internal non-Euclidean geometries. Hence, both hyperboloids have own isometric external pseudo-Euclidean and internal non-Euclidean geometries on their perfect superfaces of constant radius-parameter R with affine and cylindrical topologies. They may be simplest descriptive isometric representations in $\langle \mathcal{P}^{n+1} \rangle$ of two certain real-valued n-dimensional non-Euclidean geometries with natural l and angular l/R measures of a length of segments and arcs.

The two-sheets hyperboloid II with space-like hyperbolae as principal geodesics has natural measures λ_γ with its angular γ. The upper and lower parts of a hyperboloid II are reduced by tangent *cross projecting* to the isomorphic finite tangent model as the Klein *open disk* (ball at $n > 2$), with its affine topology of usual $\langle \mathcal{E}^n \rangle$, on the projective hyperplane $\langle\langle \mathcal{E}^n \rangle\rangle$ inside the Cayley oval of radius R (trigonometric circle at $R = 1$), when $\gamma \to \tanh\ \gamma$, $\cos\alpha_k = \text{const}_k$, $k = 1, \ldots, n$ (Figure 4). In the Klein disk, hyperbolic and hyperboloidal arcs are mapped as straight segments in own *cross bases* (Chs. 4A, 7A). The tangent projections of two *limit circumferences of radius* R from two upper and lower parts of a hyperboloid II are asymptotes inside to Cayley oval (with orthospherical arcs $r \leq R$). External geometry of II is isometric and hence homeomorphic to the internal Lobachevsky–Bolyai geometry [40–42], with their identical natural measures and the same parameters n and R [52, 53]. Indeed, tangent or Klein model is also *projective map* of the Lobachevsky–Bolyai plane or space into the homogeneous coordinates onto the same projective hyperplane as n-dimensional disk (ball) of radius R without its border inside the Cayley oval in $\langle\langle \mathcal{E}^n \rangle\rangle$. (This first finite model of the Lobachevsky–Bolyai plane was anticipated by Eugenio Beltrami in 1868 [44, 45]!)

The geometries on both sheets of a hyperboloid II are different only in the signs of the hyperbolic angle and of its directional vector in trigonometric matrices for mirror-symmetric motions with respect to $\langle \mathcal{E}^n \rangle$. Latter statements must true also for *two antipodal parts of the two-sheets Lobachevsky-Bolyai space and geometry*. If $R = c$, then at $n = 3$ this radius-vector, as time-like 4-velocity $\mathbf{c}$ of Poincaré [63], gives proper $\mathbf{v}^*$ and coordinate $\mathbf{v}$ velocities as its hyperbolic sine and tangent orthoprojections into $\langle\langle \mathcal{E}^3 \rangle\rangle$. In the Kleinian model, the natural measures of a length (for both geometries) are transformed into projective *tangent measure* $R\tanh(\lambda/R) \equiv R\tanh(l/R)$, identifiable in the projective hyperplane $\langle\langle \mathcal{E}^n \rangle\rangle$ with Euclidean measure inside the Cayley oval. This projective measure is limited by R. Note, if $R \to \infty$, then two sides of Klein disk together with two sheets hyperboloid II are transforming into two infinite antipodal Euclidean spaces $\langle \mathcal{E}^n \rangle$. On the n-$D$ hyperboloid II in $\langle \mathcal{P}^{n+1} \rangle$, the hyperbolic n-$D$ space with the Lobachevsky–Bolyai geometry maps without problems for any $n \geq 2$. But this has not place in its real-valued isomorphism – see below.

On the hyperboloid II (top sheet), diametrical hyperbolic lines inside the Cayley oval has the center O, which is the center of projecting in $\tilde{E}_1$ and the origin for counting the Euclidean tangent measure inside the Caylay oval at $R = 1$ (Figure 4). If $\gamma \to 0$ at any R, the natural measure $\lambda = R\gamma$ and the projective Euclidean *tangent measure* $R \tanh \lambda/R$ in the projective Euclidean plane $\langle\langle \mathcal{E}^n \rangle\rangle$ are became identical as $R\gamma \to R \tanh \gamma$ with the Infinitesimal Pythagorean theorem on the hyperboloid II (see in detail in Chs. 7A and 10A). If $R = c$, then it is the hyperboloid of 4- and 3-velocities.

For the hyperboloid II, the countervariant *visual spherical* Lobachevsky parallel angle $\Pi(a) = \xi$, correct in universal base $\tilde{E}_1$, is produced from the chain: $\tanh \gamma = \mathrm{sech} v \equiv \cos \xi$, where v and γ are countervariant and covariant hyperbolic parallell angles *in hyperbolic geometries*, correct in any admitted pseudo-Cartesian bases – see in detail to the end of Ch. 6.

Corollary 1. *The Minkowski hyperboloid II of constant radii $+iR$ and $-iR$ in $\langle \mathcal{P}^{n+1} \rangle$ has interal n-D hyperbolic non-Euclidean geometry, with affine topology, isometric to the two separated antipodal Lobachevsky-Bolyai geometries. Its internal geometry is equivalent to its external tensor trigonometry in $\langle \mathcal{P}^{n+1} \rangle$ with the exactness up to the similarity coefficient R.*

The one-sheet hyperboloid I with time-like hyperboloidal geodesics and space-like ellipsoidal extremals is mapped by cotangent *cross projecting* into the isomorphic *infinite cotangent model* on the *projective hyperplane $\langle\langle \mathcal{E}^n \rangle\rangle$ outside the double Cayley oval as the double ring with two its external radii R* and without external borders, when $\gamma \to \coth \gamma$, $\cos \alpha_k = \mathrm{const}_k$, $k = 1, \dots, n$ (Figure 4). Its first measure λ_γ with γ associates this cotangent model of I with the tangent model of II through the double Cayley oval. Indeed, the four conjugate pure hyperbolic lines on I and II may be interpreted as the quadrohyperbola in the common cutting pseudoplane on the trigonometric hyperbolic diagram (Figures 3 and 4). Such a pseudoplane is determined by two coupled eigenvectors along isotropic diagonals of $roth\, \Gamma$. In result of such projecting, the centered pseudoplane with the quadrohyperbola cuts the *projective two-sided hyperplane* $[\langle\langle \mathcal{E}^n \rangle\rangle]$ along four straight lines as projective maps of the quadrohyperbola in their *United tangent-cotangent projective flat model* inside and outside double Cayley oval. The cotangent projections of two *limit circumferences of radius R* from the upper and lower parts of a hyperboloid I are asymptotes outside to Cayley oval, but for the two sheets hyperboloid II its both *limit circumferences of radius R* from two upper and lower parts are asymptotes inside double Cayley oval contrary each to another in tangent projection of II and cotangent projection of I at $R \tanh \gamma = R \coth \gamma = R$ if $\gamma \to \infty$.

Express simultaneously the equivalent connections of the natural length λ_{23} of segments between two points on the hyperbolic lines: on the hyperboloid II with Euclidean tangent projective length $R \tanh \lambda/R$ and on the accompanied to it hyperboloid I with Euclidean cotangent projective length $R \coth \gamma = R \coth(\lambda/R)$ as difference $[-R(\coth \gamma_{13} - \coth \gamma_{12})]$ outside the Cayley oval for the cotangent projective model. Then, in a collinear case, a length of the segment between two points on the hyperbolic geodesic with natural hyperbolic and Euclidean tangent–cotangent projective lengths are calculated in the tvs-forms as follows:

$$\mathbf{u}_2 = \{roth\, \Gamma_{12}\} \cdot \mathbf{u}_1,\ \mathbf{u}_3 = \{roth\, \Gamma_{12}\} \cdot \{roth\, \Gamma_{23}\} \cdot \mathbf{u}_1 \to\ \mathbf{u}_{23} = \mathbf{u}_3 - \mathbf{u}_2,\ \Gamma_{23} = \Gamma_{13} - \Gamma_{12}.$$

$$\mathbf{v}_2 = \{roth\, \Gamma_{12}\} \cdot \mathbf{v}_1,\ \mathbf{v}_3 = \{roth\, \Gamma_{12}\} \cdot \{roth\, \Gamma_{23}\} \cdot \mathbf{v}_1 \to\ \mathbf{v}_{23} = \mathbf{v}_3 - \mathbf{v}_2,\ \Gamma_{23} = \Gamma_{13} - \Gamma_{12}.$$

$$\lambda_{23} = R \cdot \ln \sqrt{\frac{(1 + \tanh \gamma_{13})(1 - \tanh \gamma_{12})}{(1 - \tanh \gamma_{13})(1 + \tanh \gamma_{12})}} \equiv R \cdot \ln \sqrt{\frac{(\coth\ \gamma_{13} + 1)(\coth\ \gamma_{12} - 1)}{(\coth\ \gamma_{13} - 1)(\coth\ \gamma_{12} + 1}}.$$

The identical formulae for a distance between points correspond to the Rule of tangent-cotangent summing collinear hyperbolic motions in Appendix (time-like and space-like). They are interpreted through Caylay oval the unity of the flat tangent-cotangent common model with conjugated straight projections in $\langle\langle \mathcal{E}^n \rangle\rangle$, as it is demonstrated at Figure 4. But all this applies so far only to the hyperbolic part of the hyperboloid I non-Euclidean geometry. Its internal geometry includes pseudo-normally directed time-like hyperboloidal geodesics and space-like ellipsoidal extremals, separated by the gorolines with zero metric. The closed ellipsoidal geodesics lead to the cylindrical topology of the hyperboloid I.

On the hyperboloid I, these three types of extremal curves exist together: hyperboloidal with time-like slope, ellipsoidal with space-like slope and gorolines with infinity slope. In each point hyperboloidal and ellipsoidal curves are intersected. See more in Chs. 7A and 10A.

The hyperboloid I is also *perfect regular hypersurface*, but as hyperbolic–elliptical one; and, maybe, there is corresponding to it a certain real-valued hyperbolic–elliptical perfect regular surface, with the same hyperbolic–elliptical internal non-Euclidean geometry. This hyperbolic–elliptical non-Euclidean geometry is 3-rd one and it is additional to the well-known classic elliptical and hyperbolic non-Euclidean geometries. However it has a feature as the limited freedom of motions of the geometric figures due to its cylindrical topology. Generally, it is clarity seen, how two points on the hyperboloid I can be connected uniquely with two manners: either by hyperbolic or hyperboloidal segments, and either by circular or ellipsoidal extremals in two direction, i. e., without restrictions in the base $\tilde{E}_1$. Of course, such simple solutions for two points do not relate to motions of figures.

The projective open ring between two upper and lower Cayley ovals (without them) in the closed whole two-sides projective hyperplane $[\langle\langle\mathcal{E}^n\rangle\rangle]$, as the flat cotangent map of the entire hyperboloid I, is equivalent topologically to cylindrical space $\langle\mathcal{C}^n\rangle$. It is produced continuously through the conventional infinitely far border between two sides of the projective hyperplane (in its upper and lower halves). This border is projected into the infinite cotangent map – as if an *equator* of the hyperboloid I at Figure 4, when $\gamma = \infty$ ($\coth\gamma = 1$). If $R \to \infty$, the hyperboloid I is transforming into the infinite cylindrical pseudo-Euclidean space (but its cotangent projection is transforming into infinite Euclidean double ring). If' $R = c$, it is a hyperboloid of cotangent supervelocities (chs. 4A, 6A). Note, that this hyperbolic-elliptical non-Euclidean geometry is presented in both cavities of isotropic cone.

Corollary 2. *The Minkowski hyperboloid I of constant pseudo-normal radii $\pm R$ in $\langle\mathcal{P}^{n+1}\rangle$ has internal n-D hyperbolic–elliptical non-Euclidean geometry, with cylindrical topology, and infinitesimally pseudo-Euclidean. Its internal geometry is equivalent to its external tensor trigonometry in $\langle\mathcal{P}^{n+1}\rangle$ with the exactness up to the similarity coefficient R.*

The super descriptive, whole and finite *tangent model* of **the hyperboloid I** is realized as its tangent projection onto the *projective cylindrical pseudo-Euclidean hypersurface* $[\langle\langle\mathcal{C}^n\rangle\rangle]$ *with the same radius* $R = 1$ *and with the heights* $\pm R = \pm 1$, centered in $\tilde{E}_1$ along the axis $\overrightarrow{y}$ upper anf lower of the center O. Its lateral cylindrical surface is bounded from above and below by two Cayley ovals (trigonometric circles) without them. From the trigonometric point of view, this model is a *tangent map* under $\gamma \to \tanh\gamma$, $\cos\alpha_k = \mathrm{const}_k$, $k = 1, \ldots, n$. Such map is the descriptive *Special cylindrical tangent model* of the hyperboloid I, realized on the lateral cylindrical pseudo-Euclidean hypersurface $[\langle\langle\mathcal{C}^n\rangle\rangle]$, where the original hyperbolic and hyperboloidal geodesics are mapped as if straight lines on this cylinder under their visual inclination $\varphi_R = |\pi/2|$ and $\varphi_R > |\pi/4|$, the original circular and ellipsoidal extremals are mapped as if elliptical curves on this cylinder under their visual inclination $\varphi_R = 0$ and $\varphi_R < |\pi/4|$ (with horocycles between them at $\varphi_R \to |\pi/4|$). The time-like hyperbolic angular Lambert measure γ of a length is transformed into the *tangent projective measure* $R\tanh a/R$. The space-like spherical angular Lambert measure φ of a length works here for the elliptical and circular maps. This model is identical topologically also to open cylindrical region outside and between two Cayley ovals without them. It includes the centered circular conventional border between its upper and lower parts as if the spherical n-equator of the hyperboloid I. *This cylinder tangent model is ideal for geometric projective summation of time-like finite segments of the hyperbolic geodesics and of space-like finite arcs of the elliptical extremals.*

Both ring and cylindrical models of the hyperboloid I are conventionally two-sided, as they are divided *not topologically* into halves, with positive and negative values of y. Passage from one side to another of the models corresponds to passage through the equator of I.

Metric forms of hyperboloids and hyperspheroid are considered in details in Appendix (Chs. 6A, 7A, 10A) in connection with the theory of regular curves in $\langle\mathcal{P}^{n+1}\rangle$ and $\langle\mathcal{Q}^{n+1}\rangle$.

Main Inference. *The United internal non-Euclidean geometry of both conjugated Minkowski hyperboloids – I with cylindrical topology and II with affine topology of its upper and lower parts, all with the same radius–parameter* R, *and separated asymptotically by the isotropic cone, so in the original base* $\tilde{E}_1$, *is equivalent completely to the Tensor Trigonometry of the enveloping Minkowski space* $\langle \mathcal{P}^{n+1} \rangle$ *with exactness up to the constant scale parameter* R.

Note, that analogous Main Inference is inferred for the Special hyperspheroid, presented at Figure 4 and having the frame axis $\overrightarrow{y}$ (introduced us preliminary in Chs. 5 and 6), its internal non-Euclidean geometry with spherical topology is equivalent completely to the Tensor Trigonometry of the enveloping quasi-Euclidean space $\langle \mathcal{Q}^{n+1} \rangle$ with this exactness.

The tangent *Whole United Cylinder-model of hyperboloids I and II* consists of two parts: the Special cylindrical tangent model of I as lateral surface of the cylinder with radius R and, on the heights $\pm R$ of this cylinder, the two Kleinian disks of radius R of the flat tangent model of II, as upper and lower bases of this cylinder. For these concomitant hyperboloids and their trigonometric models, the dividing asymptotically hypersurface (isotropic cone) and its finite tangent-cotangent projection ($(n-1)$-dimensional Cayley oval or trigonometric circle at $R = 1$) are *automorphisms*. In the base $\tilde{E}_1$, this oval is determined by the equation:

$$x_1^2 + \cdots + x_n^2 = R^2.$$

Let us dissect at $n = 2$ this finite tangent projective *Whole United Cylinder-model of hyperboloids I and II* by the *centered* cutting plane under a certain angle $\varphi_R(\gamma)$ to plane $\langle \mathcal{E}^2 \rangle$. If this angle is zero, we have an equivalent map as the real equator of the hyperboloid I. If this angle less $\pi/4$, we get on the cylindrical hypersurface one (at $n = 2$) closed elliptical curve as a map of the space-like ellipsoidal extremal on the hyperboloid I. If this angle is $\pi/4$, we get on the cylindrical hypersurface two isotropic segments till the two Cayley ovals as a map of two horocyrcles on the hyperboloid I with zero metric. If this angle more $\pi/4$, we get four one-to-one connected straight segments: two ones on the cylindrical hypersurface as a map of two time-like hyperboloidal curves on the hyperboloid I and two ones on the two Kleinian disks as a map of two space-like hyperboloidal curves on the hyperboloid II. Thus, on this model, they form an united closed projective quadrangle cycle from two pairs of the connected infinite parallel lines. Its four apexes lie formally on the two Cayley ovals. Such a geometric tangent projective sum in the Minkowski space $\langle \mathcal{P}^{n+1} \rangle$ of both conjugated complex-valued n-dimensional hyperboloids, dividing asymptotically by isotropic cone, as the *united three sheets non-Euclidean hyperplane* with its complete Lorentz group, can be mapped entirely into the whole two-sided projective n-dimensional hypersurface $[\langle\langle \mathcal{E}^n \rangle\rangle]$ with topology of n-sphere.

This closed construction maps the *United non-Euclidean hypersurface of three sheets* in $\langle \mathcal{P}^{n+1} \rangle$ is as if "the world in a water drop". In sect. 6.4 we considered so the hyperbolic tensor trigonometry on a pseudoplane with solving interior and exterior right triangles, where time-like and space-like hyperbolae at Figure 3 were as the future prototypes of both Minkowski hyperboloids in this Chapter.

Note (!!!), that continuing the cylindrical tangent model of I, we'll transfer only to the non-descriptive and infinite semi-closed cylindrical cotangent model of II.

Our external tensor-trigonometric approach to analysis of the Minkowski hyperboloids in pseudo-Euclidean space ($q = 1$), with vector and scalar projections from the introduced and used widely tensor trigonometric functions, represents descriptively and correctly these two objects initially in the unity universal base $\tilde{E}_1 = \{I\}$. At the same time, this universal base is the initial one for similar representation of the hyperspheroid in the quasi-Euclidean space. See on Figure 4 (at ($q = 1$). As a result, we can apply or notice the abstract and specific spherical-hyperbolic analogies from Ch. 6 in their connected and understandable tensor trigonometric considerations - see later in Chs. 6A, 7A, 8A, 10A of the Appendix.

The opportunities of our Tensor Trigonometry in these binary perfect spaces are more widely! So, we'll see this in Appendix at simplest constructions of various screwed motions!

Thus, the hyperboloid I and the Beltrami pseudosphere are itself-homothetic objects of common similarity coefficient R, i. e., similar to trigonometric variants at $R = 1$. Besides, they have the same Gaussian curvature $K_G = -1/R^2$ and are homeomorphic. Due to the Minding Theorem, both these geometric objects must be as if isometric in the large each to another. But, despite on these properties, there is one essential difference between them. Namely, the Minkowski hyperboloid I is a perfect hypersurface in $\langle\mathcal{P}^{n+1}\rangle$, but the Beltrami pseudosphere does not relate to the set of perfect surfaces, as it is embedded correctly in the Especial quasi-Euclidean binary space with one step principal spherical rotations and polysteps orthospherical ones, isometric to motions on the pseudosphere. Therefore, both these geometric objects are only one step isometric in the large! See in detail about the pseudosphere in Ch. 6A with its construction from the hyperboloid I together with generating tractrix and their pure trigonometric equations and metric forms also in one parameter R. Note, that local isometry of the Beltrami pseudosphere with the hyperboloid II, due to the Beltrami interpretation [44], is based on as if Lobachevsky–Bolyai geometry, as it is realized in the region of only hyperbolic geodesics motions – see in Ch. 6A. The pseudosphere was discovered by Ferdinand Minding in 1838 [43] as a surface of constant negative Gaussian curvature. Its area and volume were occurred by finite in contrast to these hyperboloids!

The fix idea about a possibility of the rigorous geometry in which the Fifth Euclidean Postulate may be not hold and the Hypothesis of the acute Saccheri angle [35) can be valid on "some imaginary sphere" was expressed first by Johann Lambert in 1766 [36]. Later it became more precise: the first property is a feature of *geometry in the large*, the second property is a feature of geometry *in the small*. They are bound in geometry with the free motions of figures. Carl Gauss made some drafts in this direction [39]. Ferdinand Schweikart introduced the factor parameter R of this geometry [37]. Franz Taurinus (his nephew) suggested a model of such geometry on a *hypothetic sphere of imaginary radius*, revealed that the sum of angles in its hyperbolic triangle is less π [38] and proved internal consistency of its planimetry at $n = 2$. Intuitive Lambert–Taurinus geometry anticipated the completely developed non-Euclidean geometry by Lobachevsky–Bolyai [40-42] presented in the certain hyperbolic plane and space. Many later, in XX century, this hyperbolic geometry as its complex-valued analog was presented by H. Jansen on the Minkowski hyperboloid II in 1909 [52]. In 1868 Eugenio Beltrami realized it in 1868 [44, 45] locally, *but as time-like*, on real-valued pseudosphere as a peculiar surface (Ch. 6A), which was discovered and analyzed earlier by Ferdinand Minding in 1840 [43] as with constant negative Gaussian curvature. The Kleinian projective model [48] reduced the problem of its non-contradiction on the whole to that of Euclidean geometry. David Hilbert proved that 2-dimensional Lobachevsky–Bolyai geometry can not be realized on the whole on some *non-peculiar* Riemann surface embedded into the 3-dimensional Euclidean space, as the Gaussian internal geometry [48]. But it does not mean that this geometry can not be realized on a saddle Riemann surface in a $(3+k)$-dimensional Euclidean space. Such surface must have constant negative curvature. If its embedding into an Euclidean space of minimal dimension is possible, then this should mean solvability of the *Beltrami problem*. The first results in this direction was obtained for $\langle\mathcal{E}^6\rangle$ and more for $\langle\mathcal{E}^{6n-5}\rangle$ by D. Blanusha in 1955 [50]. Later also other authors made a lot of contributions, particularly, E. Rosendorn in 1960 for $\langle\mathcal{E}^5\rangle$. The Beltrami problem was solved peculiarly by the original manner as the same embeddability, but into $\langle\mathcal{P}^{n+1}\rangle$, – see above.

Definition of an n-dimensional Riemannian surface and its geometry is not interrupted of an enveloping Euclidean superspace, but it is interrupted only of its dimension, which a priori may be in $[(n+1), \infty)$. A posteriori the dimension may be quite definite. Dimension of a Riemannian surface is the same for all its homeomorphisms, it is equal to dimension of a tangent Euclidean space in all its points. The latter generalized an one-dimensional *tangent line to a curve*, but dimension of its embedding may be in $[2, \infty)$. So, an infinite regular curve of *constant spherical curvature* can not be realized on a plane, however, it is realizable in the 3-dimensional Euclidean space as a screw line.

A similar curve of *constant hyperbolic curvature* is realizable in a pseudoplane as the hyperbola. Ulisse Dini, else in XIX century, with solving a problem posed by Beltrami of representing one surface on a second surface in such a way that geodesic lines in the first correspond to geodesic lines in the second [46] (in our Ch. 6A as hyperbola $\leftrightarrow$ tractrix), opened the helical twisting of the Beltrami pseudosphere into pseudospherical helicoid with constant negative curvature. He eliminated for pseudosphere irregularity in enveloping space, turning its circular equator in screw transforming in infinitely far horocycle. Interestingly, his teacher was Eugenio Beltrami, his favorite student was Ricci Curbastro – 3 geniuses!!!

Isometric images of the hyperbolic non-Euclidean geometry with completely free motions of figures in different surfaces (a hyperboloid II upper, a Lobachevsky–Bolyai hyperbolic space, a real-valued Riemannian surface of constant negative curvature) differ very much in visuality and complexity. The cylindrical hyperbolic-elliptical geometry may be clearly realized isometrically both on the hyperboloid I in $\langle \mathcal{P}^{3+1} \rangle$, and as one step one on the Beltrami pseudosphere in the real-valued Especial quasi-Euclidean space (see in Ch. 6A).

12.2 Rotations and deformations in their elementary tensor forms

There is isomorphism of the admissible rotations in the enveloping pseudo-Euclidean or quasi-Euclidean spaces and motions on the embedded into them hyperbolic or spherical hypersurfaces with radius R, what is inferred by their proportionality with respect to R.

The point elements $\mathbf{v}$ and $\mathbf{u}$ on the Minkowski hyperboloids I and II in $\langle \mathcal{P}^{n+1} \rangle$ are determined by their pseudo-Cartesian coordinates (x_k, y), $k = 1, \ldots, n$, as a rule, in the base $\tilde{E}_1$ (Figure 4). Any elements on these hyperboloids with radius-parameter R may be uniquely determined by "n" especial parameters in the base $\tilde{E}_1$ as follows: $\mathbf{u} = R\mathbf{i}$ for the hyperboloid II (with $\rho = +iR$) and $\mathbf{v} = R\mathbf{p}$ for the hyperboloid I (with $\rho = +R$), where $\mathbf{i} = (\sinh\gamma \cdot \mathbf{e}_\alpha,\ \cosh\gamma)'$ and $\mathbf{p} = (\cosh\gamma \cdot \mathbf{e}_\alpha,\ \sinh\gamma)'$ are the unity time-like and space-like 4-vectors in $\langle \mathcal{P}^{n+1} \rangle$; $\mathbf{e}_\alpha = \langle \cos\alpha_k \rangle$ is their Euclidean vector of directional cosines $\cos\alpha_k$, $k = 1, \ldots, n$ (i. e., for vector sine on II or cosine on I). In brackets, the orthoprojections in $\tilde{E}_1$ of these unity vectors are given. For the point $\mathbf{u}$ on the hyperboloid II on its upper part, $\overrightarrow{y}$ is the frame axis for counting absolute value of hyperbolic angle γ formed with its radius-vector $R\mathbf{i}$. Therefore n independent coordinates are sufficient, because $\sum_{k=1}^{n} \cos^2\alpha_k = 1$. For the point $\mathbf{v}$ on the one-sheet hyperboloid I, its frame axis lies in $\langle \mathcal{E}^n \rangle$, and it is always symmetrical with the axis $\overrightarrow{y}$ with respect to the dividing isotropic conic hypersurface. It forms the same hyperbolic angle γ with its radius-vector $R\mathbf{p}$. In the tensor trigonometry we use scalar, vectorial and most general tensor angles of motive and projective types – see their initial definitions and connections in Chs. 5 and 6.

Tensor functions of the motive complementary angles Γ and Υ, including rotation and deformation, can be reduced in $\langle \mathcal{P}^{n+1} \rangle$ with $\{I^{\pm}\}$ to their canonical hyperbolic forms (31A). With decompositions (324)-(327) and formulae (360), (361), we obtain the useful relations:

$$\left.\begin{aligned} roth\ \Gamma &= \cosh\Gamma + \sinh\Gamma = \coth\ (\pm\Upsilon) + \operatorname{csch}\ \Upsilon = \overline{roth}\ \Upsilon, \\ defh\ \Gamma &= \operatorname{sech}\ \Gamma + \tanh\Gamma = \tanh(\pm\Upsilon) + \operatorname{sech}\ \Upsilon = \overline{defh}\ \Upsilon. \end{aligned}\right\} \text{(where } \mathbf{e}_\alpha\mathbf{e}'_\alpha = \overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha})$$

$$roth\,\Gamma \qquad = \qquad \overline{roth}\ \Upsilon$$

$$\left| \begin{array}{c|c|c|c|c} \dfrac{\cosh\gamma\cdot\overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}'}{\sinh\gamma\cdot\mathbf{e}'_\alpha} & \dfrac{\sinh\gamma\cdot\mathbf{e}_\alpha}{\cosh\gamma} & \begin{array}{c}\cdots\\ \cdots\end{array} & \dfrac{\coth\upsilon\cdot\overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}'}{\operatorname{csch}\ \upsilon\cdot\mathbf{e}'_\alpha} & \dfrac{\operatorname{csch}\ \upsilon\cdot\mathbf{e}_\alpha}{\coth\upsilon} \end{array} \right|. \qquad (496-I)$$

$$defh\,\Gamma \qquad = \qquad \overline{defh}\ \Upsilon$$

$$\left| \begin{array}{c|c|c|c|c} \dfrac{\operatorname{sech}\ \gamma\cdot\overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}'}{+\tanh\gamma\cdot\mathbf{e}'_\alpha} & \dfrac{-\tanh\gamma\cdot\mathbf{e}_\alpha}{\operatorname{sech}\ \gamma} & \begin{array}{c}\cdots\\ \cdots\end{array} & \dfrac{\tanh\upsilon\cdot\overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}'}{+\operatorname{sech}\ \upsilon\cdot\mathbf{e}'_\alpha} & \dfrac{-\operatorname{sech}\ \upsilon\cdot\mathbf{e}_\alpha}{\tanh\upsilon} \end{array} \right|. \qquad (496-II)$$

$$\left.\begin{array}{l} \mathbf{sinh}(\Gamma,\Upsilon) = \mathrm{csch}\ (\Upsilon,\Gamma), \quad \mathbf{cosh}(\Gamma,\Upsilon) = \mathbf{coth}\ (\Upsilon,\Gamma). \ \mathbf{tanh}(\Gamma,\Upsilon) = \mathrm{sech}(\Upsilon,\Gamma). \\ \mathbf{cosh}^2(\Gamma,\Upsilon) - \mathbf{sinh}^2(\Gamma,\Upsilon) = I = \mathbf{coth}^2(\Upsilon,\Gamma) - \mathrm{csch}^2(\Upsilon,\Gamma) - \text{two invariants.} \\ \mathbf{tanh}^2(\Gamma,\Upsilon) + \mathrm{sech}^2(\Gamma,\Upsilon) = I = \mathrm{sech}^2(\Upsilon,\Gamma) + \mathbf{cosh}^2(\Upsilon,\Gamma) - \text{quasi-invariant.} \end{array}\right\} \quad (496-III)$$

Corollary 3. *In right triangles in* $\langle\mathcal{P}^{n+1}\rangle$ *with angles* γ, υ *and infinite angle* δ *there hold:* $\gamma+\upsilon < \delta = \infty$ and in addition $\gamma = \upsilon \leftrightarrow \gamma = \omega \leftrightarrow \upsilon = \omega$, $\mathbf{\Gamma} = \mathbf{\Upsilon} \Leftrightarrow \mathbf{\Gamma} = \mathbf{\Omega} \Leftrightarrow \mathbf{\Upsilon} = \mathbf{\Omega}$ – see original trigonometric relations (359)-(361) between these complementary angles (Ch. 6).

We used **Rules 4 and 5** (Ch. 5) for the tensor functions of motive angles (rotations and one-step deformations) with expansion in Ch. 6 and here to hyperbolic tensor analogues.

So, after exchanges in the cotangent-cosecant rotational function as in (496-1) of the angle $\mathbf{\Gamma}$ by its complementary angle $\mathbf{\Upsilon}$ as the operation $\mathbf{\Gamma} \to \mathbf{\Upsilon}$, according to these Rules, the new matrix function in $\mathbf{\Upsilon}$ gives rotation again at $\mathbf{\Gamma}$. This is spread into $\langle\mathcal{P}^{n+1}\rangle$. Analogous peculiarity acts for principal spherical rotations and one-step deformations in Ch. 5A.

These one-to-one functional connections of $\mathbf{\Gamma}$ and $\mathbf{\Upsilon}$ in tensor variant of relations (360), Ch. 6, in tensor pseudo-Euclidean right triangles in $\langle\mathcal{P}^{n+1}\rangle$ rise geometrically thanks to the fact that the usual hyperbolic cosine-sine orthogonal projecting with the angle $\mathbf{\Gamma}$ is equivalent to the hyperbolic cotangent–cosecant cross projecting with the complementary angle $\mathbf{\Upsilon}$, what is shown descriptively at Figure 4.

Factually in the sine-cosine pair the hyperbolic angle γ plays role of an acute angle in the hyperbolic right triangles (sect. 6.4), but in the cotangent-cosecant pair this hyperbolic angle plays role of the same, but of complementary hyperbolic angle inside the hyperbolically obtuse angle as intrinsic one (with infinite angle $+\delta$) at the contrary vertex of the right triangle! See this peculiarity in triangles on two sides of Cover to our book. It manifests itself in an unusual way in several places in the Appendix, explaining the dark places.

As we have seen, tensor trigonometry in its vector projective forms gives descriptive isometric models of both concomitant hyperbolic non-Euclidean geometries on corresponding to them the Minkowski hyperboloids I and II of the radius-parameters R or 1. Tangent and cotangent models display geodesic motions on them in their geometries into rectilinear mappings onto the projective plane or the projective cylinder. They display a linear part of motions as 1-st metric forms on the local tangent plane or pseudoplane at these hyperboloids.

Besides, invariants, quasi-invariants or modules for paired vector trigonometric functions of the same kinds of hyperbolic angles are similar to ones for the scalar functions of angles, because their valency is equal to $\mathbf{1}$. Thus, modules of these functions are bound by scalar relations (359), (360). For instance, all they form the invariants above in the pairs of sine-cosine and cotangent-cosecant (the latter is true only in hyperbolic geometries).

The vectorial nature of such linear mappings allows us in external geometries to impart this nature for two- and more steps metric forms of the 1-st order with the preservation of their scalar characteristics as the module values of the vectors. But their unity vectors determine the directions of motions in these forms. As we have seen above, such metric forms of absolute motions or segments on these hyperboloids in their geometries are mapped either into the tangent Euclidean and pseudo-Euclidean plane, or into the tangent Euclidean and pseudo-Euclidean cylindrical surface.

The unity hyperboloids I and II are ideal models for displaying metric forms of relativistic motions – the most varied! Even the cylindrical enveloping surfaces for swirling motions also figuratively fits into the vector trigonometric model, in that number, as partial fragments of its complete model. Examples of such two-step, multisteps and integral motions, with simplest important types of motions, are studied in Chs. 5A, 6A, 7A, and more generally in $\mathbf{3}D$ and $\mathbf{4}D$ tensor forms in last Ch. 10A.

In pseudo-Euclidean Minkowskian spaces $\langle \mathcal{P}^{n+1} \rangle$ admissible hyperbolic deformations, as one step transformations *with respect to the universal base* $\tilde{E}_1$ (where they are commutative), are of interest too. They have tangent-secant form (496) and canonical structure (364) in the universal base. Deformations are made in the pseudoplane at the same tensor angle Γ. With respect to the base of the diagonal cosine Γ, these matrices and the metric reflector tensor have the following *binary-cell structure* in the eigen pseudoplane:

$$\{defh\ \Gamma\}_{can} \qquad\qquad \{roth\ \Gamma\}_{can} \qquad\qquad I^{\pm} \quad (q=1)$$

$$\begin{bmatrix} \ddots & & & \\ & \operatorname{sech}\gamma & -\tanh\gamma & \\ & +\tanh\gamma & \operatorname{sech}\gamma & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & \cosh\gamma & \sinh\gamma & \\ & \sinh\gamma & \cosh\gamma & \\ & & & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & & & \\ & +1 & 0 & \\ & 0 & -1 & \\ & & & \ddots \end{bmatrix}.$$

This structure generates, similarly to (471), the *pure* type of the elementary (as $q=1$) hyperbolic measureless deformational tensors (one step) in the original base $\tilde{E} = R'_W \cdot \tilde{E}_1$:

$$\left. \begin{aligned} & R_W \cdot \{defh\ \Gamma\}_{can} \cdot R'_W = defh\ \Gamma, \\ & defh'\ \Gamma \cdot defh\ \Gamma = I = defh\ \Gamma \cdot defh'\ \Gamma, \end{aligned} \right\} (det\ defh\ \Gamma = +1.)$$

Modal matrices R'_W are not compatible with $\{I^{\pm}\}$ and change it as in (457). And the deformation do not belong to the Lorentz group as they do not satisfy (458) or (460). Note, that matrices $defh\ \Gamma$ act in sub-pseudoplanes, but matrices $rot\ \Theta$ act in sub-planes.

Recall also the following distinction of tensor deformations: **Rule 2** (sects. 5.7, 6.2) for summing trigonometrically compatible angles-arguments does not hold for deformations, though any deformational matrices with their compatible angles commute with each other! However such tensors may be used widely for *cross* non-Cartesian *projecting* in $\langle \mathcal{P}^{n+1} \rangle$. So, cross projecting in the space $\langle \mathcal{P}^{3+1} \rangle$ gives the mathematical model for Lorentz contraction of a moving object extents coaxially to the direction of physical motion – see in Ch. 4A.

Spherical–hyperbolic analogy of the two types (abstract in any $\tilde{E}$ and specific in $\tilde{E}_1$) generates *quart-circle* (341), Ch. 6 from elementary motive matrix transformation functions:

$$\begin{array}{ccccccc} rot\ (i\Gamma) & \equiv & defh\ (-i\Phi) & \Leftrightarrow & roth\ \Gamma & \equiv & def\ \Phi \\ & \Updownarrow & & & & \Updownarrow & \\ rot\ \Phi & \equiv & defh\ \Gamma & \Leftrightarrow & roth\ (-i\Phi) & \equiv & def\ (i\Gamma). \end{array}$$

All the matrices compatible with the metric reflector tensor act here in the same planes and pseudoplanes in the universal base $\tilde{E}_1$. Thus, if the initial conditions act, there hold:

$$defh\ \Gamma \cdot I^{\pm} \cdot defh\ \Gamma = rot\ \Phi \cdot I^{\pm} \cdot rot\ \Phi = I^{\pm} = roth\ \Gamma \cdot I^{\pm} \cdot roth\ \Gamma \ = def\ \Phi \cdot I^{\pm} \cdot def\ \Phi.$$

And four relations in the circle with respect to the universal base $\tilde{E}_1$ hold in hyperbolic as well as spherical geometry. That is why they are represented with angles Γ and Φ of rotation, and their middle reflector tensor is $I^{\pm} \equiv Ref\ \{\cos\tilde{\Phi}\}^{\ominus} \equiv Ref\ \{\cosh\tilde{\Gamma}\}^{\ominus}$.

In the pseudo-Euclidean trigonometry in $\langle \mathcal{P}^{n+1} \rangle$ and external hyperbolic geometry on hyperboloids, with respect to admissible pseudo-Cartesian bases, defining relations (348), (349) hold; in quasi-Euclidean trigonometry in $\langle \mathcal{Q}^{n+1} \rangle$ and external spherical geometry on hyperspheroid, with respect to admissible quasi-Cartesian bases, defining relations (257), (258) hold. Between them, the simple trigonometric relations act with the use of functions $\varphi(\gamma)$ and $\gamma(\varphi)$ introduced by us in Ch. 6 *with respect to the universal base* $\tilde{E}_1$.

In process of non-collinear polysteps or integral hyperbolic motions in $\langle \mathcal{P}^{n+1} \rangle$ and on both these hyperboloids, we'll deal with the secondary orthospherical rotations of these non-point geometric objects moving in them. There is a deep distinction between matrix representations of *rot* Θ and *roth* Γ. For *roth* Γ, the angle γ is counted from the current time-like frame axis $\overrightarrow{y}$ and space-like frame axis x_k. Structures (362, 363) and pseudoplane of rotation γ are determined by directional cosines with respect to Cartesian sub-base $\tilde{E}_1^{(3)}$.

Representation of *rot* Θ is defined by its general structure (473). So, in $\langle \mathcal{P}^{2+1} \rangle$, the structure of *rot* Θ includes the 2×2-block as its elementary spherical cell of the rotation in the Euclidean plane. In $\langle \mathcal{P}^{3+1} \rangle$, the structure of *rot* Θ includes the 3×3-block as its elementary spherical cell of the rotation in the Euclidean plane inside sub-space $\langle \mathcal{E}^3 \rangle$. It represents the orthospherical rotation with *fixed normal axis* $\mathbf{r}_N$ [21, p. 447]. This plane of the rotation, normal to the axis, are determined by the directional cosines of the *normal axis of rotation* $\mathbf{r}_N \in \langle \mathcal{E}^3 \rangle$ with respect to the Cartesian part of the universal base $\tilde{E}_1 = \{I\}$:

$$rot\ \Theta$$

$$\begin{array}{|c|c|c|c|} \hline \cos\theta + \frac{r_1^2}{1+\cos\theta} & -r_3 + \frac{r_1 r_2}{1+\cos\theta} & +r_2 + \frac{r_1 r_3}{1+\cos\theta} & 0 \\ \hline +r_3 + \frac{r_1 r_2}{1+\cos\theta} & \cos\theta + \frac{r_2^2}{1+\cos\theta} & -r_1 + \frac{r_2 r_3}{1+\cos\theta} & 0 \\ \hline -r_2 + \frac{r_1 r_3}{1+\cos\theta} & +r_1 + \frac{r_2 r_3}{1+\cos\theta} & \cos\theta + \frac{r_3^2}{1+\cos\theta} & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline \end{array} . \qquad (497)$$

Consider the angles Γ and $\overset{\angle}{\Gamma}$ in polar representations (474–476), Ch. 11 for the cases of direct and inverse orders of two-step pure hyperbolic motions γ_{12}, γ_{23} (γ_{23}, γ_{12}) with their tensor structures (362, 363) and their directional cosines $\cos\sigma_k$, $\cos\overset{\angle}{\sigma}_k$, $k = 1, 2, 3$: $\mathbf{e}_\sigma = \{\cos\sigma_k\}$, $\mathbf{e}_{\overset{\angle}{\sigma}} = \{\cos\overset{\angle}{\sigma}_k\}$. Applying structures (362, 363) with formula (476) we obtain:

$$\left.\begin{array}{l} rot'\ \Theta_{3\times3} \cdot \{\mathbf{e}_\sigma \cdot \mathbf{e}'_\sigma\} \cdot rot\ \Theta_{3\times3} = \mathbf{e}_{\overset{\angle}{\sigma}} \cdot \mathbf{e}_{\overset{\angle}{\sigma}}{}', \\ \mathbf{e}_{\overset{\angle}{\sigma}} = rot'\ \Theta_{3\times3} \cdot \mathbf{e}_\sigma = \{\mathbf{e}_{\overset{\angle}{\sigma}} \cdot \mathbf{e}'_\sigma\} \cdot \mathbf{e}_\sigma = \cos\theta \cdot \overleftarrow{\{\mathbf{e}_{\overset{\angle}{\sigma}} \cdot \mathbf{e}'_\sigma\}} \cdot \mathbf{e}_\sigma, \\ \mathbf{e}'_\sigma \cdot \mathbf{e}_{\overset{\angle}{\sigma}} = \mathbf{e}'_{\overset{\angle}{\sigma}} \cdot \mathbf{e}_\sigma = \cos\theta = tr\ [rot\ \Theta]_{3\times3}/2 - 1. \end{array}\right\} \mathbf{e}_2 \cdot \mathbf{e}'_1 = \frac{\overleftarrow{\mathbf{e}_2 \cdot \mathbf{e}'_1}}{\cos\theta_{12}} \qquad (498)$$

In $\langle \mathcal{E}^3 \rangle \in \langle \mathcal{P}^{3+1} \rangle$, the unity vectors $\mathbf{e}_\sigma$ and $\mathbf{e}_{\overset{\angle}{\sigma}}$, by (498), uniquely determine the vector of spherically normal axis of rotation *rot* $\Theta_{3\times3}$ as the following vectorial sine product:

$$\overrightarrow{\mathbf{r}_N}(\theta) = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \mathbf{e}_{\overset{\angle}{\sigma}} \times \mathbf{e}_\sigma = \begin{bmatrix} \cos\overset{\angle}{\sigma}_2 \cos\sigma_3 - \cos\overset{\angle}{\sigma}_3 \cos\sigma_2 \\ \cos\overset{\angle}{\sigma}_3 \cos\sigma_1 - \cos\overset{\angle}{\sigma}_1 \cos\sigma_3 \\ \cos\overset{\angle}{\sigma}_1 \cos\sigma_2 - \cos\overset{\angle}{\sigma}_2 \cos\sigma_1 \end{bmatrix} = -\sin\theta \cdot \overrightarrow{\mathbf{e}_N}. \qquad (499)$$

$\{|\sin\theta| = ||\mathbf{r}_N|| = \sqrt{r_1^2 + r_2^2 + r_3^2}\ ,\quad tr\ rot\ \theta = 2(\cos\theta + 1).\}$

We have $(det\{\mathbf{e}_{\overset{\angle}{\sigma}}, \mathbf{e}_\sigma, \overrightarrow{\mathbf{r}_N}\} > 0 \rightarrow \theta < 0)$, i. e., as the triple $(\mathbf{e}_{\overset{\angle}{\sigma}}, \mathbf{e}_\sigma, \overrightarrow{\mathbf{e}_N})$ is *left-handed*. The orthospherical shift angle θ in (498), (499) is also counterclockwise as the orthospherical angle ε between unity vectors $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$ of 1-st and 2-nd hyperbolic motions. But in $\langle \mathcal{E}^3 \rangle \in \langle \mathcal{P}^{3+1} \rangle$ and in non-Euclidean hyperbolic geometry (Ch. 7A), they are contrary as in (499). A cause of this fact is explained clear by our tensor trigonometry, as in it scalar and tensor hyperbolic angles with hyperbolic increments are imaginary-valued due to their nature – see in detail to the end of Ch. 10A. Therefore, we have here the **Rule** $\boxed{sgn\ \theta_{13} = -sgn\ \varepsilon\ !}$ (For two-step spherical principal rotations in $\langle \mathcal{Q}^{2+1} \rangle$ and motions in the spherical geometry with real-valued angles (Ch. 8A), under the same condition, we get the **Rule** $\boxed{sgn\ \theta_{13} = +sgn\ \varepsilon\ !}$.)

12.3 The Mathematical principle of relativity

All statements concerning ⟨Euclidean, quasi-Euclidean, pseudo-Euclidean⟩ geometry without its affine contents have covariant forms in any ⟨Cartesian, quasi-Cartesian, pseudo-Cartesian⟩ base of ⟨Euclidean, quasi-Euclidean, pseudo-Euclidean⟩ space. So, any geometry with the simplest quadratic invariant as a set of its own theorems does not depend in part of these theorems on a choice of its admissible base. In other words, ⟨Euclidean, quasi-Euclidean, pseudo-Euclidean⟩ geometries conserve covariant forms under their admissible transformations as ⟨orthogonal, quasi-orthogonal, pseudo-orthogonal⟩ and translations.

The *mathematical principle of relativity* takes place in any flat geometry with quadratic-type metric – thus, in the Minkowski geometry. For instance, in STR space-time, it is a mathematical source for the physical Postulate of Relativity by Galilei-Poincaré (1636, 1904), that all physical laws have covariant forms in any uniformly rectilinearly moving frames of reference up to nearly light velocity, i. e., under Lorentz transformations. *The physical-mathematical isomorphism* unites two Principles. Lorentzian transformations do not change the **absolute** Minkowskian space-time with dividing asymptotic hypersurface as *light cone*:

$$\langle \mathcal{P}^{3+1}\rangle \equiv \langle \mathcal{E}^3\rangle^{(k)} \boxtimes \overrightarrow{ct}^{(k)} \equiv \mathrm{CONST}, \quad (n=3,\ q=1);\ \Delta ct > 0\ ! \qquad (500)$$

Contrary, k-th $\langle \mathcal{E}^3\rangle$ and $\overrightarrow{ct}$ are relative, change under the Lorentzian transformations of the bases, but always complementary! Though they with their coordinate axes stay in own external and internal cavities of the cone. Although space $\langle \mathcal{E}^3\rangle^{(k)}$ and time-arrow $\overrightarrow{ct}^{(k)}$ are relative, but mutually dependent as direct hyperbolically orthogonal complements in $\langle \mathcal{P}^{3+1}\rangle$. Due to identity (500), there exists an one-to-one correspondence between them. Therefore, for STR this formula is the mathematical expression of the Poincaré–Minkowski inference about relativity, mutual dependence and unity of the space and the time! *Pay especial attention here to the fact that the Nature's Euclidean subspace is just as relative as the time!*

Let $\tilde{E}_m = roth\ \Gamma(\mathbf{v}) \cdot \tilde{E}_1$, where $\mathbf{v}$ is the velocity. In the pseudoplane $\langle \mathcal{P}^{1+1}\rangle$ of this rotation-motion, time and space coordinates axes in $\tilde{E}_m$ are seeming in $\tilde{E}_1$ as if dilated in the direction of $\mathbf{v}$ with coefficient $\mathbf{cosh}\,\gamma \equiv \mathbf{sec}\,\varphi(\gamma)$ *in the Euclidean metric* in $\tilde{E}_1$. Though they conserve in $\langle \mathcal{P}^{1+1}\rangle$ pseudo-Euclidean metric of length as in the base $\tilde{E}_1$ too – see at Figure 4. By this graphic reason, Hermann Minkowski in 1908 [66]) introduced for his new coordinates of relativistic space-time on such a pseudoplane the terms "dilation" for its time and space coordinates axes in moving system $\tilde{E}_m$. However on the pseudoplane $\langle \mathcal{P}^{1+1}\rangle$, in $\tilde{E}_m$, we have relativistic decreasing time and space interval of the given event with the coefficient $\mathbf{cosh}^{-1}\gamma(\mathbf{v})$ (Ch.5A) compared with ones in $\tilde{E}_1$. Such identical decreasing is caused by constancy of light velocity in any $\tilde{E}_m$, according to the Einsteinian physical Postulate [67].

Such polysteps decreasing of a space coordinate is not a one step Lorentzian contraction, Though both are gotten by *cosine projection*. Lorentzian contraction of space objects is gotten by *cross projecting* as a consequence of their seeming hyperbolic deformation (Ch. 4A).

In the 4*D* Lagrangian space-time $\langle \mathcal{L}^{3+1}\rangle \equiv \langle \mathcal{E}^3 \oplus \overrightarrow{t}\rangle \equiv \mathrm{CONST};\ \Delta t > 0,\ \mathcal{E}^3 \equiv \mathrm{CONST}'$, the Laws of the classical mechanics are form-invariant with respect to a choice of Galilean inertial frames of reference, or under Galilean transformations. It is the physical-mathematical form of the Galilean Principle of Relativity. The Lagrangian space and time-arrow form an absolute unity, as their sum is direct, but they are not orthogonal and, hence, not mutually dependent as in (500). From the mathematical point of view, the Lagrangian space-time is a simple case (at $n = 3$, $q = 1$) of the general affine-Euclidean space $\langle \mathcal{E}^n \oplus \mathcal{A}^q\rangle$ with the affine-Euclidean geometry and the Galilean group of affine-Euclidean transformations. The latter do not change the Euclidean subspace $\langle \mathcal{E}^3\rangle$ and the scalar time t: here they are absolute in Newtonian sense. Time-arrow $\overrightarrow{t}$ under slope $\mathbf{tan}\ \mathrm{v}$ is not constant as a directed world line in $\langle \mathcal{L}^{3+1}\rangle$. It is subjected to so-called "middle rotations" – between spherical and hyperbolic ones with respect to $\langle \mathcal{E}^3\rangle$ (see more further in Ch. 1A of Appendix).

From the other side, continuous transformations in Minkowskian space-time $\langle \mathcal{P}^{3+1} \rangle$ carry out relativistic elementary hyperbolic principal rotations with also elementary orthospherical induced ones in accordance with its reflector tensor $\{I^{\pm}\}$. Moreover, the space-time fixations of any geometric objects are subjected to relativistic hyperbolic deformations, which are described completely in the cross base $\tilde{E}_{i,j}$ with immobile Observer. Relativistic nature of the Lorentz transformations takes place according to hyperbolic nature of principal rotations and deformations. With Einsteinian physical approach [67], STR was based, in that number, with the as if axiomatic definition of events simultaneity. Factually this definition corresponds to the theorem in $\langle \mathcal{P}^{3+1} \rangle$, that the median and height in the pseudo-Euclidean right triangle are identical, which motivated the quadratic metric in the space-time of STR.

The abstract and specific spherical–hyperbolic analogies (the latter with respect to the universal base) connect initially quasi-Euclidean and pseudo-Euclidean geometries, and also as a consequence the spherical and hyperbolic types of non-Euclidean geometries of the same radius-parameter R. This enables one to describe them sometimes in the enveloping binary spaces $\langle \mathcal{Q}^{n+1} \rangle$ and $\langle \mathcal{P}^{n+1} \rangle$ by similar clear approaches based on the Tensor Trigonometry.

In the Lobachevsky–Bolyai geometry, a magnitude R is called *the Gauss–Schweikart Constant* ($1/R = K$ characterizes the distortion with respect to the flat Euclidean space); iR is the radius of a "hypothetical Lambert imaginary hyperbolic sphere" , realized in 1909 by H. Minkowski as the upper sheet of his hyperboloid II. This J. Lambert's original idea and its development by F. Taurinus pointed out the simplest and natural way for realization of the whole hyperbolic non-Euclidean geometry on the hypothetical sphere of imaginary radius iR. This way became quite possible after introducing pseudo-Euclidean space of $q = 1$ by H. Poincaré in 1905 [63] and later in 1909 by H. Minkowski [65] as space-time of STR.

A. Sommerfeld in 1909 established hyperbolic nature of the Poincaré – Einstein Law of relativistic velocities summation [86], considered its acting as if on the sphere with the imaginary radius ic. V. Varičak in 1910 conjectured that this Law of velocities summing is identical to the segments' summing in Lobachevsky–Bolyai geometry [87]. Later F. Klein constructed the theoretical basis for this Law, when he proved that the Lorentzian group in STR is equivalent to the group of motions in the Lobachevsky–Bolyai space. Before he interpreted this geometry in the large (1871) in the purely projective model inside the Cayley oval on the projective plane, which was anticipated by E. Beltrami in 1868 [44]. In 1928 F. Klein added this projective model on the projective plane, using the hyperboloid II of radius-parameter R with the sames hyperbolic geometry enveloped in the pseudo-Euclidean space of Minkowski [48].

The scenario for the further development of events in this area of geometric and physical researches was predetermined. The decisive role in understanding that different ways of constructing the same non-Euclidean geometry lead to identical final results was played by the projective models of Klein and Poincaré. And the choice of the simplest and most visual way of displaying and analytical study of non-Euclidean geometries with their applications in physical theories comes to the fore, what, for example, the tensor trigonometry gives by clear tools. Due to this all, as important applications, the tensor trigonometry interpretations of various motions in non-Euclidean geometries and in the Theory of Relativity with its kinematics and dynamics are exposed in Appendix, in addition to its fundamentals in Part II.

For your better understanding the author's presentation of Appendix with comments in its physical part, which concerns applications of Tensor Trigonometry in Theory of Relativity, we decided that it will be very useful to refer a little and maximum objectively, i. e. only on the basis of reliable facts given in the literature sources without some PR, to the history of the origin of this relativistic theory in the early XX century, in which three extraordinary personalities participated, each with own contribution in it, as the great scientific revolution. The author firmly adheres to the Rule that True Science should not be influenced by national and political lobbyists, as was in the twentieth century and, unfortunately, this crown of thorns of the fundamental Sciences has not up to now been eliminated completely.

Thus, it is appropriate to cite, thanks to French source [103], a very revealing and useful historically absentee dialogue between the greatest and most honest scientists of the early 20th century.
Henri Poincaré from his pioneer and well-known article "Sur la dynamique de l'électron." // Comptes Rendus de l'Académie des Sciences, Paris, v. 140, 5 juin 1905 [63]:

"Le point essential, établi par Lorentz, c'est que les équations du champ électromagnétique ne sont pas alterées par une certaine transformation, que j'appellerai du nom de Lorentz et qui est de la forme suivante: ...."
Hendrik Lorentz from his reaction on this pioneer article by Poincaré:

"Ce furent les considérations publiées par moi en 1904 qui donnèrent lieu à Poincaré d'écrire son article, dans lequel il a attaché mon nom à la transformation dont je n'ai pas tiré tout le parti possible ... J'ai pu voir plus tard dans le mémoire de Poincaré que j'aurais pu obtenir une plus grande simplification encore. Ne l'ayant pas remarqué, je n'ai pas établi le principe de relativité comme rigoureusement et universellement vrai. Poincaré, au contraire, a obtenu une invariance parfait ... et a formulé le Postulat de relativité, terme qu'il a été le premier à employer."
Against present situation in the Exact Sciences area, both must be ranked as Saints!
The pseudo-Euclidean space-time with **group** *of Lorentz transformations, introduced and named so by Henri Poincaré, and his Postulate of Relativity from 1904 with fundamental relation* $E = mc^2$ *discovered by him yet in 1900 [62], are the true foundation of the Theory of Relativity in 1904–1905, what is more, in modern understanding. All other following attributes are became derived concepts. The name of this new theory was later given by Max Planck as "Theory of Relativity".*

Some physicists are proud that Einstein in his article from 30 June 1905 [67] with derivation of space and time coordinates transformations, but well-known then and without reference to Lorentz, managed with, as it seems to them, two purely physical Postulates without the serious mathematics, as Poincaré did in his works. But what is the Einstein's proof from the point of view of mathematics.

According to the Einsteinian definition of simultaneity with two contrary light beams ways for fixing of the simultaneity of two events in the moving and resting frames $\tilde{E}$, they must meet at the middle point of the path, that is, at the median of the right triangle formed with these light beams. But then this median must also be the height of this right triangle, since the relative time and space must be orthogonally additive each other. This fact of identity of the median and the height in the external and internal right triangles is a Theorem of pseudo-Euclidean geometry (introduced by Poincaré initially on the pseudoplane), which was accepted implicitly. Although from such definition it would be possible to substantiate long-known Lorentzian contraction. Inference in similar difficult cases as always: "The Devil is in details". It is *Minkowski, Poincaré's friend and Einstein's teacher, on advised Einstein to study Poincaré's theory, which Einstein did without citing Poincaré* [110]. Carried away from youth by Dostoevsky's novels with their extreme heros and philosophy ("If there is no God, then everything is allowed!"), for a long time Einstein did not attach importance to the need to refer to previous authors and had be subjected to well-known ostracism from some eminent German scientists with Nobel laureates, and from England by eminent Edmund Whittaker [106].

In its turn, at the beginning of the 20th century, Poincaré's writings were very popular and much larger than those of Ernst Mach with his positivism. In 1904, he was even invited in the USA to give lecture for American physicists and mathematicians, that popularized his relativistic ideas.

In 1900, in the article "La Théorie de Lorentz et le Principe de réaction" [48], Poincaré, with publication of formula $m = E/c^2$, gives relativistic interpretation to the "temps local de Lorentz": "C'est le temps d'observateurs mobiles qui réglent leurs horloges par des signaux optiques en ignorant le mouvement de translation dont ils sont animés." In 1902, in own popular book "La Science et l' Hypothèse" he writes: "Il n'y pas d'espace absolu, et nous ne concevons que des mouvements relatifs ... Il n'y pas de temps absolu: dire que deux durées sont égales, c'est une assertion qui n'y pas de sens par elle-même et qui ne peut en acquérir un que par convention ... Nous n'avons pas l'intuition directe de la simultanéité de deux événements qui se produisant sur deux théâtres differents ... Nous pourrions énoncer les fait mécaniques en les rapportant à un espace non euclidien ..." The essence of new relativistic theory was published by him yet before his academic publication [63].
Nevertheless, the contribution of the very young at that time and recent student Albert Einstein (of 26 years), consisted in the facts that he began to operate realistically with time scales in the different Galilean frames near the light velocity under his physical concept of events simultaneity, in inferring his Law of summing two relativistic velocities, which seemed fantastic for that era!

APPENDIX

Trigonometric models of motions in STR and non-Euclidean Geometries

Preface

In Appendix we consider a lot of general or specific applications of tensor trigonometry in geometries and physics. For this we use our tensor trigonometric functions in the so-called *elementary form* with its single principal eigen angle and at it the unity vector of directional cosines $\mathbf{e}_\alpha$. This angle determines intensity and direction of geometric or physical motions. It is accompanied with the orthospherical angle either θ or α caused by rotation of the directed vector $\mathbf{e}_\alpha$. All they are used for complete tensor trigonometric descriptive analysis of these motions in affine-Euclidean, pseudo-Euclidean and quasi-Euclidean spaces with the index $q = 1$ and in embedded into them metric spaces of constant radius (and, therefore, of constant Gaussian curvature) with their non-Euclidean geometries. The main idea of such approach consists in that tensor trigonometry of these pseudo- and quasi-Euclidean spaces exist in one-to-one correspondence with non-Euclidean geometries of parameters n and R! All their common results are represented in the simplest and clear trigonometric forms. So, the widely used in STR so-called *relativistic factors* β and γ correspond in our notations to functions $\mathbf{tanh}\,\gamma$ and $\mathbf{cosh}\,\gamma$ of the angle γ with expansion till vector and tensor analogues.

In Chapter 1A, for initial illustration and use of these opportunities, the main Postulates and notions of the Special Theory of Relativity (STR) in the Minkowski space-time are represented in hyperbolic forms according to the original group approach of Poincaré in June 1905 [47] and then by Minkowski in 1909 [49]. Stated in the Theory of Relativity, according to our tensor trigonometric approach, *isotropy* and *homogeneity* of the space-time of events allow us to use the trigonometry in most wide aspects, than in its scalar form on the pseudo- and quasi- planes. This was impossible in the non-isotropic Lagrange space-time.

In the frame of the trigonometric aspects, we give renewed and universal conception of the *parallel angle* for both types non-Euclidean geometries in the hyperspaces of constant radius-parameter R, embedded respectively into quasi-Euclidean and pseudo-Euclidean spaces. Due to this conception, initial definitions of both types non-Euclidean geometries can be realized through a choice of the parallel angle type, whether spherical or hyperbolic, with corresponding to their nature two variants of the *global behavior* of parallel lines. As it was demonstrated in the Chapter's end, the Lobachevskian parallel angle is strictly correct only in the case of the spherical type geometry, because it has a spherical nature. The *universal parallel angle* is defined in the universal base $\tilde{E}_1$ of the enveloping or tangent space. In STR the hyperbolic parallel angle is defined also in $\tilde{E}_1$, which corresponds to the immovable Observer N_1 in the Minkowski space-time. And it is covariant, i. e. identical to the *hyperbolic motion angle* γ, defined initially in scalar form by velocity as $\gamma = \operatorname{artanh}\, v/c$. (Chapter 1A.)

The basic parameters of motions in the tensor trigonometric versions of non-Euclidean geometries, quasi- and pseudo-Euclidean geometries and also of STR are the tensor angles of hyperbolic and orthospherical rotations as in (259), (313, 314), (324), (362, 363), (497). The principal tensor angles Φ, Γ and Θ are arguments of their rotational matrix-functions $\boldsymbol{rot}\ \Phi$, $\boldsymbol{roth}\ \Gamma$ and $\boldsymbol{rot}\ \Theta$ as *measureless* tensors of motion. So, in STR the tensor hyperbolic interpretations of *Einsteinian dilation of time* and *Lorentzian contraction of extent* with concomitant to them relativistic effects are expressed very easy through hyperbolic rotation and deformation of coordinates. The Einsteinian physical Postulates with his definition of events' simultaneity are simplest theorems in the pseudo-Euclidean space-time, introduced in first by Henri Poincaré in June of 1905 (Chapters 2A÷4A.)

One-to-one correspondence between kinematic characteristics of relativistic inertial and uninertial motions of material objects or particles in Minkowski space-time with their tensor trigonometric models are established and used. For beginning, we constructed trigonometric descriptive models of various collinear motions (at $\mathbf{e}_\alpha = \text{const}$) relating to the rectilinear physical movements. Thus, we exposed the *hyperbolic motion* at $\boldsymbol{g} = \text{const}$ on a time-like hyperbola with coordinate velocity $\boldsymbol{v}$, on a kinematic catenary with proper velocity $\boldsymbol{v}^*$, and, with our original method, on a kinematic tractrix with supervelocities $\boldsymbol{s}$ and $\boldsymbol{s}^*$ under translation from pseudo-Euclidean space into two Special quasi-Euclidean spaces with primary hyperbolic and real spherical equations for a catenary and a tractrix of two kinds (of Huygens and of Minding). (In Ch. 10A we'll realize the tensor trigonometric model of *pseudoscrewed motion* as the 2-nd type of uniform motion at $\boldsymbol{g} = \text{const}$.) Such Minding tractrix equations was used for presentation of the Beltrami pseudosphere, realized in the Especial quasi-Euclidean space $\langle \mathcal{Q}_T^{2+1} \rangle^\ddagger$ *with one-step admitted principal spherical motions and polysteps admitted orthospherical motions*. We stated that two catenaries with two catenoids and two tractrices with two tractricoids are not mapping correctly in the usual Euclidean space, but only in their four Special quasi-Euclidean binary spaces! We added these surfaces by their four metric forms in the vector-scalar trigonometric presentations. The result is proved: *The Minkowski hyperboloid* I *in* $\langle \mathcal{P}^{n+1} \rangle$ *is one-step isometric with the Beltrami pseudosphere in 4-th* $\langle \mathcal{Q}_T^{n+1} \rangle^\ddagger$ *at* $n \geq 2$ *and common* R, *i. e., only for their one step principal motions and polysteps orthospherical ones.* By passing way, the *hyperbolic relativistic analog* of the Ziolkovsky cosmic formula is gotten. We calculated the cosmic travel on the "photon rocket" with an ideal reversible regime to the nearest Star "Proxima Centauri". However its disappointing conclusion: similar even optimal travels, but with acceleration $\boldsymbol{g}$, for the contemporary people (non-robots) are unreal in reasonable times! (Chapters 5A, 6A)

The general laws of summing two-steps, polysteps and integral non-collinear principal 3D rotations in $\langle \mathcal{P}^{3+1} \rangle$ and $\langle \mathcal{Q}^{3+1} \rangle$ around their frame axis, limiting by *constant* radius $R = 1$ due to the rotational Tensor Trigonometry, were inferred in their scalar, vector, tensor ("tvs") forms with their polar representation and revealing secondary orthospherical shift. And general tensor trigonometric formulae for the continuous Lorentzian (as a group) and Special quasi-Euclidean (as also a group!) transformations were inferred. The general laws of summing $3D$ two-steps, polysteps, integral angular motions on curvilinear hypersurfaces in $\langle \mathcal{P}^{3+1} \rangle$ or $\langle \mathcal{Q}^{3+1} \rangle$ under their non-Euclidean geometries of constant radius-parameter R, with motions (velocities, including superlight) in STR with Looking Glass, are isometric with the general laws of rotations above with exactness till factor R. Therefore both these rotations and motions have isomorphic own groups. For two-steps motions, we represented these laws in non-commutative sine and tangent biorthogonal forms with the Big and Small Euclidean Relative Pythagorean theorems reduced them to the initial Euclidean subspace. But in the case of the second differential principal motion in two-steps ones, the induced differential secondary orthospherical angular shift is revealed as the Thomas precession. In our tensor trigonometric analysis in the quasi-Euclidean, pseudo-Euclidean, non-Euclidean geometries and in the Theory of Relativity, we connected also the secondary and *induced* orthospherical angular shift by its common nature: (1) with the Harriot–Lambert angular deviations (excess or defect) in convex figures on the non-Euclidean *hypersurfaces of the radius-parameter* R; (2) with the relativistic Thomas precession in the STR, and (3) with the Coriolis acceleration in result of motions in the pseudo-Euclidean space along a curvilinear trajectory. Our tensor trigonometric compared descriptions in the base $\tilde{E}_1$ and the base $\tilde{E}_m$ have revealed the most universal and simplest formula for these induced angular shifts, including in time, as "the difference between real local rotation and its cosine orthoprojection into the original Cartesian subbase". These difference and shift are negative for hyperbolic cosine and positive for spherical cosine. We constructed the tensor trigonometric isomorphic models for kinematics and dynamics of a material body at integral non-collinear motions with the induced and oscillating Thomas precession. (Chapters 7A, 8A)

The main *measureless* concept of the tensor trigonometry in STR is the *hyperbolic tensor of motion roth* $\Gamma^{(m)} = F(\gamma, \mathbf{e}_\alpha)$, generated proportionally with constant coefficients m_0c the relativistic dynamic tensor of momentum and energy. It produces the pseudo-Euclidean interior right triangle of three momenta $\mathbf{P}_0 = m_0\mathbf{c}$, $P = mc$ and $\mathbf{p} = m\mathbf{v} = m_0\mathbf{v}^*$ with the *Absolute pseudo-Euclidean Pythagorean Theorem* in $\langle\mathcal{P}^{3+1}\rangle$. The own 4-momentum $\mathbf{P}_0$ as the hypotenuse has own scalar invariant of the Lorentzian transformations. An additional important concept is the *hyperbolic tensor of deformation defh* $\Gamma = D(\gamma, \mathbf{e}_\alpha)$ decreasing sizes of any moving object in the original Cartesian subbase $\tilde{E}_1^{(3)}$ in the direction of velocity from its own sizes in moving $\tilde{E}_2^{(3)}$ with the *Lorentzian seeming contraction.* (Chapter 4A, 5A, 7A)

With the use of *abstract and specific spherical-hyperbolic analogies*, a number of similar notions, formulae and theorems are given and inferred in their spherical kinds in the so-called *quasi-Euclidean space* with index $q = 1$ and on the embedded into it *Special hyperspheroid* of the constant radius R with its non-Euclidean spherical geometry. In addition to all these, we proposed the simple tensor trigonometric model of the geographic globe. (Chapter 8A)

In Chapter 9A, under the enough logical having and new arguments, we adopt that novel opportunities exist for correct studying and description of various relativistic motions in the presence of gravitation, with simplest and correct interpretations of all well-known GR-effects, in the same Minkowski space-time, using our tensor trigonometric approach in its tensor-vector-scalar (tvs) forms and, in addition, of the differential tensor trigonometry. The historical merit that inertia of any massive object is created by the mass of the Universe as a whole belongs to Ernst Mach [55] – eminent physicist and philosopher of science. True, the mechanism of action of this fantastic hypothesis remained unclear for a long time. Even Albert Einstein in his GTR refused it. This unique Mach system, associated with the center of Mass of the Universe, specified a priori the unique inertial system of Galileo, as Newtonian too, for example, for space-time, and relative to it all other Galilean systems. In 1964, the necessary theory was created by Peter Higgs [82], which explained, that during development of the Universe with formation of its Mass, the latter produces the specific Higgs field. It creates the Galileo's inertia of matter as a specific force of the Nature. Moreover, just like in space-time by Poincaré – Minkowski, the inertia at any point and in any direction of this field in the Universe depends only on the mass of any object, in accordance with the Galileo's Law! That is, this new material field of the Universe is homogeneous and isotropic, and, therefore, it combines with the space-time by Poincaré – Minkowski. Furthermore, due to the Newton's classical Equivalence Principle, inertial and gravitational mass are identical, and this fact has been repeatedly and accurately confirmed, starting with Newton's own experience. The term "uniform rectilinear motion" in the Higgs Theory has also been revived in the relativistic space-time! His material field is, as it were, a reincarnation of the rejected by Einstein world ether. I hope that this brief explanation commented to readers why the author develops, since first publication of this book in 2004 [15], various applications of his Tensor Trigonometry in the Theory of Relativity with the Poincaré – Minkowski space-time.

In Chapter 10A, we developed the differential tensor trigonometry of world lines in the flat Poincaré – Minkowski space-time $\langle\mathcal{P}^{3+1}\rangle$ with the unity *trigonometric* accompanied Minkowski hyperboloids I and II, and of regular curves in the $3D$ and $4D$ quasi-Euclidean spaces $\langle\mathcal{Q}^{n+1}\rangle$ with the unity accompanied *hyperspheroid.* The pseudo-Euclidean motions correspond to plane and spatial physical relativistic movements in $3D$ Euclidean space. We have identified that the tensor-vector-scalar metric forms of $3D$ Minkowski hyperboloids relate one-to-one to the full 3-steps metric form of any world lines. This relates also to the both connected metric forms of $3D$ hyperspheroid. These forms were expressed in the $4D$ Absolute Euclidean and pseudo-Euclidean Pythagorean theorems and in the $3D$ Relative Euclidean Pythagorean theorems. We given tensor trigonometric models of pseudoscrewed world lines and all quasiscrewed curves. The former correspond to the planetary movements. In addition to the Frenet–Serret theory in $\langle\mathcal{E}^3\rangle$, we created the theory of world lines in $\langle\mathcal{P}^{3+1}\rangle$ and regular curves in $\langle\mathcal{Q}^{2+1}\rangle$ and $\langle\mathcal{Q}^{3+1}\rangle$ with movable tetrahedron and two trihedrons.

Additional notations

$\{I^{\pm}\}$ or $\{R'_W\ I^{\pm} R_W\} = \{\sqrt{I}\ \}_S$ and $\{I^{\mp}\}$ or $\{R'_W\ I^{\mp} R_W\} = \{\sqrt{I}\ \}_S^{-1}$ are metric reflector tensors of the pseudo-Euclidean space $\langle\mathcal{P}^{n+1}\rangle$ by Minkowski and of its Looking Glass, or as only a reflector tensor of the quasi-Euclidean space $\langle\mathcal{Q}^{n+1}\rangle$,

ζ – dimension of embedding of the given regular curve into binary space $\langle\mathcal{P}^{n+1}\rangle$ or $\langle\mathcal{Q}^{n+1}\rangle$,

$\tilde{E}_1$ – the base for canonical trigonometric matrix forms, the initial unity base, in that number, as the universal unity base for realization of specific spherical-hyperbolic analogy (in STR, it is the base of relatively immovable Observer); $\tilde{E}_k^{(n)} \subset \tilde{E}_k$ is the Cartesian subbase of $\tilde{E}_k$,

l – natural Euclidean measure of length, λ – natural pseudo-Euclidean measure of length,

$\overrightarrow{ct}^{(1)}$ and $\mathcal{E}^{3(1)}$ – the time arrow and frame axis with the Euclidean subspace in the initial pseudo-Cartesian base $\tilde{E}_1$ of $\langle\mathcal{P}^{3+1}\rangle$,

$y^{(k)}$ or $ct^{(k)}$ at $n = 3$ and $\mathbf{x}^{(k)} \in \mathcal{E}^{n(k)}$ – two projections of element $\mathbf{u}$ in $\langle\mathcal{P}^{n+1}\rangle$ or $\langle\mathcal{Q}^{n+1}\rangle$,

$\overrightarrow{ct}^{(k)} = \overrightarrow{ct}$ – the k-th time arrow and frame axis as current relativistic time directed to future in the base $\tilde{E}_k$ of $\langle\mathcal{P}^{3+1}\rangle$ under hyperbolic inclination Γ to the initial time arrow $\overrightarrow{ct}^{(1)}$,

$\tau = t^{(k)}$ – the *proper time* along a world line,

$x_j^{(k)}$ – the j-th space coordinate in the base $\tilde{E}_k$ of $\langle\mathcal{P}^{n+1}\rangle$ or $\langle\mathcal{Q}^{n+1}\rangle$,

Note. Greek symbols as τ and χ are used here for the *proper time* and *proper extent*.

$\Phi, \varphi, d\varphi$ are the principal angles of rotation in the quasi-Euclidean space or identical motion in the spherical geometry on the embedded hyperspheroid; it is also the angle of latitude by Lambert's angular measure in the *tensor trigonometric model of the Earth globe* along spherical meridians as big circles from the Poles or from the Equator in $[0, \pm\pi/2]$,

Ξ, ξ are complementary to motion's angle above spherical angle in the osculating quasiplane between the tangent or the quasinormal to a regular curve and the χ axis; or defined by simplest formula $\xi = \pi/2 - \varphi$ (Ch. 5),

$\Gamma, \gamma, d\gamma$ are the *realificated* principal angles of rotation in the pseudo-Euclidean space or identical motion in the hyperbolic geometry on the embedded one sheet an two sheets Minkowskian hyperboloids; it is also the angle of latitude by Lambert's angular measure along hyperbolic meridians from the Poles (for II) or from the Equator (for I) in $[0, \pm\infty)$,

Υ, υ are complementary to motion angle above hyperbolic angle (in the osculating pseudo-plane between the tangent or the pseudonormal to a world line and the isotropic cone or the isotropic diagonal; or defined by formulae $\sinh\upsilon \cdot \sinh\gamma = 1 \sim \cosh\upsilon = \coth\gamma$ (Ch. 6),

A – is internal geometric orthospherical angle at tops of geometric figures on non-Euclidean surfaces, in particular, as A_{123} at the top 2 of the non-Euclidean triangle 123,
α, $d\alpha$ – are external angles of orthospherical motions or identical rotations,
Θ, θ, $d\theta$ are external independent or induced orthospherical angles of motions, or as of the non-Euclidean angular shift, or as the angle of Thomas induced relativistic precession,
ε and ϵ – external orthospherical angles between motions on non-Euclidean surfaces or identical rotations in enveloping spaces ($\varepsilon = \pi - A \rightarrow \cos\varepsilon = -\cos A,\ \sin\varepsilon = \sin A$),

$w^*_\varphi(\tau)$ and $\eta^*_\gamma(\tau)$ – spherical and hyperbolic angular proper velocities of rotations of a curve,
$w^*_\alpha(\tau)$ and $w_\alpha(t)$ – orthospherical angular proper and coordinate velocity of $\mathbf{e}_\alpha$,
$w_\theta(t)$ – induced Thomas orthospherical precession in sine normal plane $\langle\mathbf{e}_\alpha, \mathbf{e}_\nu\rangle$ around $\mathbf{e}_\mu$,

$roth\ \Phi = F_h(\varphi, \mathbf{e}_\alpha)$ – the trigonometric measureless tensor of motion in $\langle\mathcal{Q}^{n+1}\rangle$,

$roth\ \Gamma = F_s(\gamma, \mathbf{e}_\alpha)$ – the trigonometric measureless tensor of motion in $\langle\mathcal{P}^{n+1}\rangle$,

$\mathbf{e}_\nu = (\mathbf{e}_\beta - \cos\varepsilon \cdot \mathbf{e}_\alpha)/\sin\varepsilon$ – unity vector of the sine orthogonal increment of motion,

$\mathbf{e}_\mu = (\mathbf{e}_\kappa - \cos\epsilon \cdot \mathbf{e}_\alpha)/\sin\epsilon$ – unity vector of the cosine orthogonal increment of motion,

$\mathbf{e}_\sigma$, $\mathbf{e}_{\overset{\measuredangle}{\sigma}}$ – unity vectors of summing two- and multisteps motions for direct and inverse orders of partial motions along a world line, at the hyperboloids II and I, at the hyperspheroid.

$\mathbf{r}$ – $(n+1)\times 1$-radius-vector of some object in $\langle\mathcal{P}^{n+1}\rangle$ or $\langle\mathcal{Q}^{n+1}\rangle$ in the universal base $\tilde{E}_1$,

$\mathbf{i}$ and $\mathbf{p}$ – $(n+1)\times 1$ time-like and space-like vectors in $\langle\mathcal{P}^{n+1}\rangle$, including for a world line, $\mathbf{t}$ and $\mathbf{n}$ – $(n+1)\times 1$ analogous vectors in $\langle\mathcal{Q}^{n+1}\rangle$, including for a regular curve,

$\mathbf{c} = c\cdot\mathbf{i}_\alpha$ – vector of 4-velocity or supervelocity by Poincaré of absolute motion in $\langle\mathcal{P}^{3+1}\rangle$,

$\mathbf{v} = d\mathbf{x}/dt = v\cdot\mathbf{e}_\alpha = c\cdot\tanh\gamma_t\cdot\mathbf{e}_\alpha$ – coordinate velocity of the physical movement,

$\mathbf{v}^* = d\mathbf{x}/d\tau = v^*\cdot\mathbf{e}_\alpha = c\cdot\sinh\gamma_t\cdot\mathbf{e}_\alpha$ – proper velocity of the physical movement,

$\mathbf{s} = d\mathbf{x}/dt = s\cdot\mathbf{e}_\alpha = c\cdot\coth\gamma_t\cdot\mathbf{e}_\alpha$ – coordinate supervelocity, so, inside "black hole",

$\mathbf{s}^* = d\mathbf{x}/d\tau = s^*\cdot\mathbf{e}_\alpha = c\cdot\operatorname{csch}\gamma_t\cdot\mathbf{e}_\alpha$ – proper supervelocity, so, inside "black hole",

$\mathbf{P}_0 = P_0\cdot\mathbf{i}_\alpha = m_0\mathbf{c} = m_0c\cdot\mathbf{i}_\alpha$ – own 4×1-momentum of a particle M on a world line,

$P = mc = P_0\cdot\cosh\gamma_t$ – scalar cosine projection of $\mathbf{P}_0$ onto $\overrightarrow{ct}^{(1)}$ (total momentum),

$\mathbf{p} = m_0\mathbf{v}^* = P_0\cdot\sinh\gamma\cdot\mathbf{e}_\alpha$ – 3-vector sine projection of $\mathbf{P}_0$ into $\mathcal{E}^{3(1)}$ (real momentum),

$\mathbf{F} = F\cdot\mathbf{p}_\beta = m_0\mathbf{g}_\beta$ – 4×1 free inner force acting on a material point M in $\langle\mathcal{P}^{3+1}\rangle$ in $\tilde{E}_m$,

$\mathbf{g}_\beta = g_\beta\cdot\mathbf{p}_\beta$ – 4×1 free absolute inner acceleration of material point M in $\langle\mathcal{P}^{3+1}\rangle$,

$\mathbf{g}_\alpha = g_\alpha\cdot\mathbf{p}_\alpha$, $\mathbf{g}_\alpha^{(3)} = g_\alpha\cdot\cosh\gamma_t\cdot\mathbf{e}_\alpha$ – 4×1 tangential cosine acceleration with 3-projection,

$\mathbf{g}_\nu = g_\nu\cdot\mathbf{b}_\nu$ – 4×1 normal sine acceleration with zero time projection,

$\mathbf{j}_\kappa = j_\kappa\cdot\mathbf{p}_\kappa$ – 4×1 free absolute inner superacceleration of material point M in $\langle\mathcal{P}^{3+1}\rangle$,

$\mathbf{j}_\alpha = j_\alpha\cdot\mathbf{p}_\alpha$, $\mathbf{j}_\alpha^{(3)} = j_\alpha\cdot\sinh\gamma_t\cdot\mathbf{e}_\alpha$ – 4×1 tangential sine acceleration with 3-projection,

$\mathbf{j}_\mu = j_\mu\cdot\mathbf{b}_\mu$ – 4×1 normal cosine acceleration with zero time projection,

$\mathbf{k}_p = \overline{\overline{\mathbf{k}_p}} + \overset{\perp}{\mathbf{k}_p}$ – 4×1-vector of pseudocurvature K_β with quasiorthogonal decomposition,

$\mathbf{k}_q = \overline{\overline{\mathbf{k}_q}} + \overset{\perp}{\mathbf{k}_q}$ – 4×1-vector of quasicurvature Q_κ with pseudoorthogonal decomposition,

$\mathbf{i}_\alpha$, $\mathbf{i}_\kappa$ – unity 4×1-vectors of principal and free tangents with curvatures K_α, Q_κ,

$\mathbf{p}_\alpha$, $\mathbf{p}_\beta$ – unity 4×1-vectors of principal and free pseudonormal with curvatures K_α, K_β,

$\mathbf{b}_\nu$, $\mathbf{b}_\mu$ – unity 4×1-vectors of space-like sine and cosine binormal with curvatures K_ν, K_μ,

$\mathbf{i}_1$ or $\mathbf{t}_1$– 4×1 binormal of the cosine and sine orthoprocession along frame axis $\overrightarrow{ct}$ or $\overrightarrow{y}$,

$\mathbf{i}_\nu$ – tangent, perpendicular to principal one, for screwed curves,

$\mathbf{p}_\mu$ – pseudonormal, perpendicular to principal one, for screwed curves,

$\mathcal{Y}_{cos}, \mathcal{Y}_{sin}$ – cosine or sine *orthoprocession* at hyperbolic/spherical/orthospherical motions,

$\Pi(a) = \xi$ – countervariant spherical Lobachevsky parallel angle in the universal base $\tilde{E}_1$,

$P(a) = v$ – countervariant hyperbolic Special parallel angle correct in any admitted base $\tilde{E}_k$.

Chapter 1A

Space-times of Lagrange and space-time of Poincaré and of Minkowski as mathematical abstractions and physical reality

At first, consider the *conventionally trigonometric kinematic model* of a material point M physical movement in the 4-dimensional binary Lagrangian space-time $\langle\mathcal{L}^{3+1}\rangle$. Choose its simplest universal base $\tilde{E}_1 = I$ as an initial unity base with the origin O. In it all these four coordinate axes x_1, x_2, x_3, $\overrightarrow{t^{(1)}}$ are defined as if Euclidean orthonormal ones. The time arrow $\overrightarrow{t^{(1)}}$ at the origin O is the *time-like orthonormal axis*. The time-arrow $\overrightarrow{t^{(m)}}$ at the same origin O is the directed *time-like affine axis* under scalar slope $\tan\ \nu = v$ with the unity vector of the directional cosines $\mathbf{e}_\alpha$. It relates to the centered base $\tilde{E}_m$. But the three space axes x_1, x_2, x_3 form the Cartesian *space-like subbase* $\tilde{E}^{(3)}$ in $\tilde{E}_1$ and $\tilde{E}_m$. Its axes x_1, x_2, x_3 stay orthonormal under orthospherical rotations $rot\ \Theta$ in constant $\langle\mathcal{E}^3\rangle$, they form a right-handed triple in $\tilde{E}^{(3)}$. Hence, $3D$ Euclidean trigonometry with measureless orthospherical functions is applicable in $\langle\mathcal{E}^3\rangle$. Any universal base $\tilde{E}_{1u} = rot\ \Theta{\cdot}\tilde{E}_1$ corresponds to immovable Observer N_1. If the material point M moves with the vector velocity $\mathbf{v} = v\cdot\mathbf{e}_\alpha = \mathbf{const}$, then its proper centered base is $\tilde{E}_m = V\tilde{E}_1$, where its new time-arrow $\overrightarrow{t}^{(m)}$ have also the three particular slopes, with respect to the three space coordinates axes of $\tilde{E}_1^{(3)}$. The ratios of these space coordinates and the time arrow are characterized by the tangent vector $\mathbf{tan}\ \nu$ (as a world-line slope in $\langle\mathcal{L}^{3+1}\rangle$) identical to the vector velocity $\mathbf{v}$ of the material point M (if frame center O corresponds to zero $(\mathbf{x}_0 = \mathbf{0}, t_0 = 0)$ and then $\mathbf{x} = \Delta\mathbf{x},\ t = \Delta t > 0$):

$$\mathbf{tan}\ \nu = \tan\ \nu\cdot\mathbf{e}_\alpha = \mathbf{x}/t \equiv \mathbf{v} = v\cdot\mathbf{e}_\alpha, \quad \tan\nu_j = x_j/t \equiv v_j,\ j = 1,2,3. \qquad (1A)$$

Admissible transformations in linear $\langle\mathcal{L}^{3+1}\rangle$ form the group $\langle V_G\rangle$ of the homogeneous Galilean transformations. This is the mathematical foundation of the Galilean Principle of Relativity. The transformation V_G is continuous as $det\ V_G = +1$, and this condition guarantees preserving base orientation. In Cartesian-affine bases $\tilde{E}_k$, the space-time $\langle\mathcal{L}^{3+1}\rangle$ is represented as the direct sum of an Euclidean space and an affine time-arrow:

$$\langle\mathcal{L}^{3+1}\rangle \equiv \langle\mathcal{E}^3\rangle \oplus \overrightarrow{t}^{(k)} \equiv \langle\mathcal{E}^3\rangle \oplus \overrightarrow{t}^{(1)} \equiv \mathrm{CONST}, \quad (\Delta t > 0) \qquad (2A)$$

$$\langle\mathcal{E}^3\rangle \equiv \mathrm{CONST}\ '. \qquad (3A)$$

Seems, there is paradox: const' + variable=const, but it is not valid for a direct sum!

There holds analogy with binary spaces of Ch. 11 ($q = 1$), but (2A) is not an orthogonal sum! All time-arrows form the complete set of affine axes $\langle\overrightarrow{t}\rangle$ consisting of time-like lines with angular slopes to $\overrightarrow{t^{(1)}}$ ranging in $[0; \pm\pi/2]$. The invariant Euclidean space $\langle\mathcal{E}^3\rangle$ consists of space-like elements. All elements of the Lagrangian space-time are real numbers. The space-time properties are preserved under Galilean transformations, because ones in general $\langle\mathcal{L}^{3+1}\rangle$ are reduced to exactly three pure types:

1) automorphic orthospherical rotations $rot\ \Theta$ of the space $\langle\mathcal{E}^3\rangle$,
2) special *parallel* (or *middle*) rotations $f(\mathbf{tan}\ \nu)$ of $\overrightarrow{t}$, with respect to the space $\langle\mathcal{E}^3\rangle$,
3) linear space $\langle\mathcal{E}^3\rangle$ and $\overrightarrow{t}$ translations $\mathbf{p}$ due to this space-time homogeneity.

The general linear transformation V_G of a Cartesian-affine base $\tilde{E}_0$ is the following:

$$\overset{V_G}{\begin{bmatrix} R & \mathbf{a} \\ \mathbf{0}' & 1 \end{bmatrix}} \cdot \overset{\tilde{E}_0}{\begin{bmatrix} R_0 & \mathbf{a}_0 \\ \mathbf{0}' & 1 \end{bmatrix}} = \overset{\tilde{E}}{\begin{bmatrix} R\cdot R_0 & R\mathbf{a}_0 + \mathbf{a} \\ \mathbf{0}' & 1 \end{bmatrix}},\ R \in \langle rot\ \Theta_{3\times3}\rangle. \qquad (4A)$$

For the matrices of the bases, their first three columns determine the constant space $\langle\mathcal{E}^3\rangle$, the fourth column determines the variable time-arrow $\overrightarrow{t}$. If $\mathbf{a}_0 = \mathbf{0}$, then $\tilde{E}_0 = E_{1u}\rangle$ (the bases are universal), and in particular, if $R_0 = I$, then $\tilde{E}_0 = E_1$. In this case, the inverse matrix V_G^{-1} (of the same structure) maps a binary Cartesian-affine base $\tilde{E}$ into its simplest unity form, i. e., the original universal base $\tilde{E}_1$. The inverse matrix also realizes passive modal transformation of a linear element from $\tilde{E}_1$ into an admissible binary base $\tilde{E}$. A linear element of $\langle\mathcal{L}^{3+1}\rangle$ is represented in $\tilde{E}$ as the radius-vector:

$$\mathbf{r} = \mathbf{x} \oplus t = \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix}.$$

Thus *homogeneous affine-Euclidean Galilean transformations* in their trigonometric form are the non-commutative products of parallel and orthospherical rotations in the *polar forms*:

$$\begin{matrix} V_G = F(\Theta_{3\times3}, \mathbf{tan}\ \nu) & & f(\mathbf{tan}\ \nu) & & rot\ \Theta \end{matrix}$$

$$\begin{bmatrix} rot\ \Theta_{3\times3} & \mathbf{tan}\ \nu \\ \mathbf{0}' & 1 \end{bmatrix} = \begin{bmatrix} I_{3\times3} & \mathbf{tan}\ \nu \\ \mathbf{0}' & 1 \end{bmatrix} \cdot \begin{bmatrix} rot\ \Theta_{3\times3} & \mathbf{0} \\ \mathbf{0}' & 1 \end{bmatrix} = rot\ \Theta \cdot f[(\mathbf{tan}\ \nu)_\Theta], \quad (5A)$$

where $det\ V_G = +1$, and $f(\mathbf{tan}\ \nu)$ is the 4×4-matrix of principal parallel rotations,

$$f[(\mathbf{tan}\ \nu)_\Theta] = rot\ (-\Theta) \cdot f(\mathbf{tan}\ \nu) \cdot rot\ \Theta, \quad but(!)\ (\mathbf{tan}\ \nu)_\Theta = rot\ (-\Theta_{3\times3}) \cdot \mathbf{tan}\ \nu.$$

An inverse and passive homogeneous Galilean transformation is represented as

$$V_G^{-1} = \begin{bmatrix} rot\ (-\Theta_{3\times3}) & rot\ (-\Theta_{3\times3}) \cdot (-\mathbf{tan}\ \nu) \\ \mathbf{0}' & 1 \end{bmatrix} =$$

$$\begin{matrix} & rot\ (-\Theta) & & f[\mathbf{tan}\ (-\nu)] \end{matrix}$$

$$= \begin{bmatrix} rot\ (-\Theta_{3\times3}) & \mathbf{0} \\ \mathbf{0}' & 1 \end{bmatrix} \cdot \begin{bmatrix} I_{3\times3} & -\mathbf{tan}\ \nu \\ \mathbf{0}' & 1 \end{bmatrix} = f\{[\mathbf{tan}\ (-\nu)]_\Theta\} \cdot rot\ (-\Theta). \quad (6A)$$

Formula (5A) is the affine-Euclidean analog of polar representations (474) and (475) in sect. 11.3. On the other hand, transformation of the base E_1 is similar to (480), (481):

$$\tilde{E} = V_G \cdot \tilde{E}_1 = f(\mathbf{tan}\ \nu) \cdot rot\ \Theta \cdot \tilde{E}_1 = rot\ \Theta \cdot f[(\mathbf{tan}\ \nu)_\Theta] \cdot \tilde{E}_1. \quad (7A)$$

From the physical point of view, the subbase $\tilde{E}^{(3)}$ moves, with respect to the subbase $\tilde{E}_1^{(3)}$, at the velocity (1A).

Inverse matrix (6A) transforms passively the coordinates of a world point $\mathbf{r} \in \langle\mathcal{L}^{3+1}\rangle$ as follows:

$$\mathbf{r} = V_G^{-1} \cdot \mathbf{r}^{(1)} = F^{-1}(\Theta, \mathbf{tan}\ \nu) \cdot \mathbf{r}^{(1)} = \begin{bmatrix} rot\ (-\Theta_{3\times3}) \cdot (\mathbf{x}^{(1)} - \mathbf{tan}\ \nu \cdot t) \\ t \end{bmatrix}. \quad (8A)$$

If $\Theta = Z$ in (5A)–(8A), then we deal with pure parallel rotations in their conventional trigonometric and physical forms as the Galilean transformations of coordinates:

$$\left.\begin{aligned} \mathbf{x} &= \mathbf{x}^{(1)} - \mathbf{tan}\ \nu \cdot t = \mathbf{x}^{(1)} - \mathbf{v} \cdot t, \\ t &= t^{(1)}. \end{aligned}\right\} \quad (9A)$$

In $\langle\mathcal{L}^{3+1}\rangle$, the *scalar time* is invariant too and may be counted on the original axis $\overrightarrow{t^{(1)}}$ and $\overrightarrow{t^{(k)}}$ parallel to invariant $\langle\mathcal{E}^3\rangle$. Due to this fact, so called *parallel rotation* $f(\mathbf{tan}\ \nu)$ of the time-arrow $\overrightarrow{t}$ (as the ordinate) is geometrically intermediate between spherical and hyperbolic ones! Note, that $f(\mathbf{tan}\ \nu)$ is expressed above as a 4×4-matrix with the variable 3×1-vector element $\mathbf{tan}\ \nu$. The latter is the tangent of the angle ν. Multistep parallel rotations lead to the *classical law of tangents* $\mathbf{tan}\ \nu$ *or velocities* $\mathbf{v}$ *commutative geometric summation in the projective Euclidean vectorial space* $\{\langle\mathcal{E}^3\rangle\}$:

$$f(\mathbf{tan}\ \nu_{13}) = f(\mathbf{tan}\ \nu_{12}) f(\mathbf{tan}\ \nu_{23}) = f(\mathbf{tan}\ \nu_{23}) f(\mathbf{tan}\ \nu_{12}) = f(\mathbf{tan}\ \nu_{12} + \mathbf{tan}\ \nu_{23}) \rightarrow$$

$$\rightarrow f(\mathbf{tan}\ \nu) = f(\mathbf{tan}\ \acute{\nu}) = \prod f(\mathbf{tan}\ \nu_{kj}) = f(\sum \mathbf{tan}\ \nu_{kj}), \quad (\nu = \acute{\nu}). \qquad (10A)$$

The set $\langle\mathbf{tan}\ \nu\rangle$ is the commutative group in the projective vectorial space of velocities, i. e., "tangents". The set of parallel rotations $\langle f(\mathbf{tan}\ \nu)\rangle$ is the *kinematic commutative subgroup* of the homogeneous affine-Euclidean Galilean group $\langle V_G\rangle$. Its another subgroup is the *non-commutative group of orthospherical rotations*. Note, that $rot\ \Theta$ is expressed above as a 4×4-matrix with the variable 3×3-matrix element $rot\ \Theta_{3\times 3}$. The group $\langle V_G\rangle$ consisting of these two subgroups is the subgroup of the *general* affine-Euclidean Galilean group.

The Lagrange space-time is continuous, but not homogeneous and isotropic *entirely* (it is enough for this, that its space and time coordinates have different physical measures), however its space and time are homogeneous *separately* due to property of continuity and equivalence of all their point elements. In particular, any centralized 4×1 radius-vector element in $\tilde{E}_1$ can be chosen as the new origin of an admitted Cartesian-affine base, and the admissibility does not depend on this choice. Parallel translations in $\langle\mathcal{L}^{3+1}\rangle$ form the *commutative translating subgroup* of the general Galilean group. The relations (2A), (3A) give an affine nature of principal transformations and independence of space and time in it!

The Lagrange space-time has a lot of applications in non-relativistic physics. However, as long ago as to the end of XIX century, experimenters and theorists have encountered some facts that are inexplicable within its framework. Firstly, it is the "negative result" of the famous experiment of Michelson—Morley (1887), which contradicted the rule of velocities summing (in the near-light region). Secondly, the Maxwell electromagnetic wave equation were proved is no covariant in the Galilean inertial frames of reference, though the latter, due to Maxwell's theory, explains the nature and spreading of light. This non-covariance of the given equation to the Galilean transformations has mean the crisis of the fundamental physics to the end of XIX century. That is why, Lorentz suggested in 1892 the Special space and time transformations, initially for interpretation of the Michelson—Morley result [58]!

* * *

In 1904 Lorentz, taking into account *the Poincaré physical Postulate of Relativity* also from the same 1904, valid for all physical phenomena, showed that his space and time transformations follow from form-invariance of the Maxwell electromagnetic wave equation [59]!! And Henri Poincaré in his pioneer article from June 5 of 1905 established a group nature of new transformations, discovered before by Hendric Lorentz, and he named them as the *Lorentz transformations* [63], with introduction in the Physical Science of the new more perfect and united space-time of the Nature corresponding to them and having homogeneity and isotropy, similar to the Euclidean space!!! In addition to Galilei–Poincaré Postulate of Relativity (1636, 1904), *from the mathematical point of view* in June 1905, in fact, the following quite new physical-mathematical Postulates were introduced by Henri Poincaré.

Postulate 1: *By nature, the space-time with its various physical fields is homogeneous and isotropic entirely.* (These properties were valid due to speed scale factor "c", used by Henri Poincaré for the time-arrow as directed 4-th coordinate.)

Postulate 2: *This space-time is the binary complex-valued 4D quasi-Euclidean space with an index $q=1$, oriented by the time-arrow $\mathbf{i}\cdot\overrightarrow{ct}$, and with the main hyperbolic angle of motions.*

The new conception of space-time as STR with these two Postulates has no any more defects of the classical, non-relativistic one. And it realized the opportunity to transfer off the non-perfect and non-united Euclidean-affine space-time $\langle\mathcal{L}^{3+1}\rangle$ in the homogeneous and isotropic complex quasi- or real pseudo-Euclidean space-time with its quadratic metric! In the space-time, we use the opportunities of scalar, vector and tensor trigonometries! Thus, we may apply the principal hyperbolic angle of motion γ in the universal base $\tilde{E}_1$, with the use of specific tangent-tangent analogy (355), sect. 6.4, with velocity divided by constant c:

$$\left.\begin{array}{l}\mathbf{tan}\ \nu \to \mathbf{tan}\ \varphi_R = \mathbf{v}/c,\\ \mathbf{tan}\ \varphi_R \equiv \mathbf{tanh}\ \gamma = \mathbf{v}/c.\end{array}\right\}\ (t \to ct) \qquad (11A)$$

. However, through any initial *quasi-Cartesian and pseudo-Cartesian bases* $\tilde{E}$ with the common reflector tensor $I^{\pm}$ of their spaces $\langle\mathcal{Q}^{3+1}\rangle$ and $\langle\mathcal{P}^{3+1}\rangle$, we can introduce the principal hyperbolic angle with trigonometric functions immediately, with the use of the abstract analogy from the same sect. 6.2. Due to Postulate 1 and 2, with (322) and (323], there hold:

$$\left.\begin{array}{l}\mathbf{tan}\ (-\varphi_{\mathbf{R}}) = \mathbf{v}/ic \to \mathbf{tanh}\ (-i\varphi_R) = \mathbf{v}/c,\\ (1)\ \varphi_R \to i\gamma,\ \mathbf{tan}\ i\gamma = i\mathbf{v}/c;\ (2)\ -i\varphi_R \to \gamma,\ \mathbf{tanh}\ \gamma = \mathbf{v}/c.\end{array}\right\}\ (t \to ict) \qquad (12A)$$

Then, under logical development, the Euclidean vector subspace of tangents (velocities) are reduced into the hyperbolic tangent (or Kleinian) model inside the Cayley oval (sect.12.1).

Scalar trigonometric functions of $i\gamma$ in the *pseudospherical form* were first applied by Poincaré for presenting Lorentzian transformations in the 2-dimensional trigonometric form. Then for their realification, Minkowski used the real-valued scalar functions of γ in the 2-dimensional trigonometric form in $\langle\mathcal{P}^{1+1}\rangle$ [65]. Both used the plane trigonometry for presenting hyperbolic motions by 2×2 rotational matrices. Note (!), that the approach of Poincaré will give us clear opportunity for right operations with signs of quadric values from the internal and external cavities of isotropic cone and right chose of metric tensors $I^{\pm}$, $I^{\mp}$.

With Poincaré mathematical approach, STR was founded with his generalized Postulate of Relativity (1904) acting in the Galilean inertial frames of reference and introduction of his new complex-valued isotropic and homogeneous space-time. Logically the Galilean transformations were replaced and named by him as Lorentzian group of this space-time!

With Einstein physical approach, STR was appeared with the use of the similar Principle of Relativity (1905) and his Postulate of constancy of the light speed in the Galilean inertial frames of reference with the additional definition of events simultaneity.

The Principle of Relativity is traditionally applied only in its physical sense, although there exists its original mathematical prototype, see in sect. 12.3. Note, that physical space-time (here $\langle\mathcal{L}^{3+1}\rangle$ and $\langle\mathcal{P}^{3+1}\rangle$) is only a certain mathematical abstraction, and its admissible coordinates may be used for describing objective laws of matter movement. The *adequate interpretation of these laws in the coordinates maps the "reality" of the space-time.*

The new essential renovation of the real space-time conception is realizing in 1964 [82], by the eminent now Peter Higgs, within the framework of the Standard Model for the set of elementary particles, put forward a revolutionary, but up to 2012 still hypothetical theory, that during the formation of the Universe, according to the Big Bang Theory by the eminent George Gamow, at the stage when its full Mass appears, the latter creates in the Universe a certain new material field with its quantum particle "boson". It is this field creates the fundamental force of Nature under well-known name "inertia", which acts, due to Galileo, proportionally to the mass of any massive object (as its charge), but iff this object deviates from uniform and rectilinear motion. This theory was strictly confirmed with the discovery of the Higgs boson in 2012 at the Hadron Collider in Switzerland. What is very important, the Higgs material field on the whole is homogeneous and isotropic, with respect to acting Galilean inertia. The Poincaré – Minkowski space-time on the whole is also homogeneous and isotropic. Then, with the Newton's Principle of Equivalence of the inertial and gravitational masses, the Higgs theory proved very strictly a reality of this flat space-time of the Nature.

Thus, before the renovation, most difficult problem in relativistic theory of space-time was correct considerations of different world events taking into account gravitation. Historically first and up to now prevailing *geometric conception* was the Einsteinian GTR from 1916 [69] with curved by gravitation pseudo-Riemannian space-time. Alternative BMT conceptions (Bimetric Theories of Gravitation) are based on the nature of the gravitation as action of some tensor physical field in the Minkowsky space-time. Surprisingly, that historically the first version of BMT was proposed by theorist Nathan Rosen [78], an assistant to Einstein at Princeton University and later his close colleague! This shows how Albert Einstein was loyal to alternative points of view in science and even to his GTR. This is an example of the true and not just in words, attitude to the freedom of scientific thought. Beginning from the 1-st edition of our Tensor Trigonometry [15], we are following to similar bimetric point of view, i. e., all events are executed in the flat space-time by Poincaré – Minkowski, but any Observer see the same events through the lensed gravity field as distorted from curving by the pseudo-Riemannian metric tensor. Our approach is a good compromise that does not destroy the harmony of the Universe and excludes the positivism in real assessments of world events. Unfortunately, the aggressive behavior of specific apologists of a really curved space-time still resists such a peace-loving point of view and they continue to make from Albert Einstein the new Ptolemy as if from the middle Ages. (See the discussion in Ch. 9A).

Vector nature of space-time takes into account admissible directions to the *light* cone contains three isotropic geometric parts with respect to their pseudo-Euclidean metric. They are: (1) the external conic cavity consisting of the space-like elements with their Euclidean metric, (2) the internal conic cavity consisting of the time-like elements with an anti-Euclidean imaginary metric, and (3) the degenerated light conic dividing hypersurface with its zero metric: it separates these external and internal cavities. Therefore rotational and deformational linear transformations in the space-time may be represented as 4×4 tensor trigonometric functions of 4×4 tensor angles Γ and Θ (Chs. 6 and 10–12).

Generally, tensor trigonometric language (with hyperbolic and orthospherical functions) may be used for explaining all effects of STR connected with the time and the Euclidean subspace. *Tensor trigonometric functions of the angle* Γ, i. e., in their *hyperbolic form* in $\langle \mathcal{P}^{3+1} \rangle$ (they were described in Chs. 6, 11 and 12) give us the 4-dimensional tensor trigonometric forms for describing kinematics and dynamics of STR (see Chs. 5A, 7A, 10A).

The pseudo-Euclidean trigonometric rotations correspond to homogeneous continuous Lorentzian transformations. Hyperbolic rotations with the pseudo-Euclidean invariant $\sinh^2 \gamma - \cosh^2 \gamma = i^2$, $\cosh \gamma > 1$, interpret clarity the Einsteinian dilation of time. The tensor trigonometric hyperbolic deformations with the cross Euclidean quasi-invariant $\operatorname{sech}^2 \gamma + \tanh^2 \gamma = 1$, $\operatorname{sech} \gamma < 1$, interpret clarity the Lorentzian contraction of extent. If the two phenomena are considered in the pseudoplane corresponding to tensor angle Γ, then a pseudo-Euclidean right triangle for them is solved completely (see in sect. 6.4). Our special *mathematical* principle of relativity for admitted geometric transformations (sect. 12.3) is in one-to-one correspondence in $\langle \mathcal{P}^{3+1} \rangle$ with the Poincaré *physical* Postulate of relativity. The Poincaré–Einstein Law of mutual dependence of the space and the time and their relativity may be explained with the fact that the relativistic Euclidean space and the time-arrow are hyperbolically orthogonal direct complements of each to other, they change always together under hyperbolic rotations, and both do not change under orthospherical rotations:

$$\langle \mathcal{P}^{3+1} \rangle \equiv \langle \mathcal{E}^3 \rangle^{(k)} \boxtimes \overrightarrow{ct}^{(k)} \equiv \text{CONST}. \tag{13A}$$

This space-time is the united indivisible 4-dimensional continuum. As a whole set it is *an absolute*, but consisting of these two variable together relative summands of index k. The scaling coefficient "c", introduced by H. Poincaré for the time, is equal to the light speed in the cosmic vacuum. Note, this small, but great time modification led to identity of transformations in the homogeneous and isotropic space-time with the Lorentzian transformations adopted before for covariancy of the Maxwell electromagnetic wave equation [59].

Later Paul Dirac generalized the result in his relativistic covariant form of the Schrödinger quantum wave equation [61]. Moreover, the fundamental Law of Energy and Momentum Conservation, in accordance with the Noether Theorems [102], are inferred in STR strictly from homogeneity and isotropy of its basis Minkowski space-time in clear simplest tensor trigonometric form (see in Chs. 7A, 10A and in the Kunstkammer).

Also two Einsteinian postulates on maximality of moving matter velocity due to $v < c$ and on constancy of the light velocity c (only as scalar value) in all the Galilean inertial frames of reference follows directly from properties of the hyperbolic tangent modulus

$$||\mathbf{v}/c|| = ||\mathbf{tanh}\ \gamma|| < 1, \tag{14A}$$

and from these properties of the hyperbolic angle-argument of the physical velocity

$$\pm\infty \pm \gamma = \pm\gamma \pm \infty = \pm\infty, \tag{15A}$$

valid in any pseudo-Cartesian base $\tilde{E}_k$ of $\langle\mathcal{P}^{3+1}\rangle$ with relatively immobile Observer. Second rule (15A) implies also that the light velocity does not depend on its source movement. However, the instantaneous *proper velocity* v^* of a material object, from the point of view of Observer moving with it, changes due to relation $||\mathbf{v}^*|| < \infty$, as $\mathbf{v}^* = c \cdot \mathbf{sinh}\ \gamma$.

In Ch. 7A, we used our most general law of summing multistep motions in $\langle\mathcal{P}^{n+q}\rangle$ with polar decomposition proved by us before in Ch. 11 for inferring the relativistic non-commutative law of summing velocities in STR and segments in hyperbolic geometry in the general and complete forms. According to similar opportunities, we consider various relativistic motions with their kinematics and dynamics in Galilean and instantaneously Galilean accelerated frames of reference (see in detail in Chs. 5A, 6A, 7A and 10A).

* * *

Further, describe the trigonometric approach to representation of physical relativistic movements in its simplest form. Choose the right universal, i. e., *inertial* base $\tilde{E}_1 = \{I\}$ with immovable Observer N_1. Other right universal bases $\tilde{E}_{1u}$ are linked as follows:

$$\tilde{E}_{1u} = rot\ \Theta \cdot \tilde{E}_1 = \{rot\ \Theta\}, \tag{16A}$$

where $rot'\ \Theta \cdot I^{\pm} \cdot rot\ \Theta = I^{\pm} = rot\ \Theta \cdot I^{\pm} \cdot rot'\ \Theta$, according to (470).

The set of admissible pseudo-Euclidean bases are determined by the metric tensors of $\langle\mathcal{P}^{3+1}\rangle$ in two possible simplest forms – according to the Hermann Minkowski approach [65] (but with their right and clear chose on the base of the Poincaré initial approach with imaginary principal angle $i\gamma$ for conjugacy of Minkowski hyperboloids in Ch. 12 and further):

$$\{I^{\pm}\} = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \{I^{\mp}\} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} = -\{I^{\pm}\}. \tag{17A - I, II}$$

In the 1-st case, our natural Euclidean space is preserved, in the 2-nd case, it transforms into anti-Euclidean space, which is very strange for us – see in details in the last Chapter 10A.

In the base $\tilde{E}_1$ and all universal bases, coordinate axes are quasi-Euclidean and pseudo-Euclidean orthonormal, hence the specific spherical-hyperbolic analogy from sect. 6.2 may be used, and this is important from theoretical point of view. Till realification of space-time by Minkowski, it was as a complex quasi-Euclidean base of space-time by Poincaré (see before in detail in Ch. 10):

$$\tilde{E}' \cdot I^{\pm} \cdot \tilde{E} = I^{\pm} = (\sqrt{I^{\pm}} \cdot \tilde{E})' \cdot (\sqrt{I^{\pm}} \cdot \tilde{E}), \tag{18A}$$

where $\sqrt{I^{\pm}}$ is the arithmetic root of type (443). The latter gives an initial variant of Henri Poincaré [63] without $I^{\pm}$. A new base, according to our polar representations (480), (481), is the result of a unique combination of a hyperbolic rotation (in $\tilde{E}_1$) and orthospherical one (in $\tilde{E}_{1h}$), or in the reverse order, where the matrices are compatible with the reflector tensor $I^{\pm}(17A - I)$:

$$\tilde{E} = roth\ \Gamma \cdot rot\ \Theta \cdot \tilde{E}_1 = \{rot\ \Theta\}_{\tilde{E}_{1h}} \cdot \tilde{E}_{1h}. \tag{19A}$$

Suppose that a new pseudo-Cartesian base is the result of a pure hyperbolic rotation

$$\tilde{E}_{1h} = roth\ \Gamma \cdot \tilde{E}_1 = \{roth\ \Gamma\}. \tag{20A}$$

The new coordinate axes are, due to (363), completely spherically non-orthogonal as their scales in the Euclidean metric are dilated (this holds for at least two of the axes, one of them is time-arrow). These axes dilations in hyperbolic interpretation was introduced by Herman Minkowsky in [66]. Pure hyperbolic base rotation (20A) has the physical sense of uniform rectilinear movement of $\tilde{E}^{(3)}_{1h}$ with its N_{1h} relatively of $\tilde{E}^{(3)}_1$ with its N_1 at the velocity $v = c \cdot \tanh \gamma$. Hyperbolic rotation is elementary, it is performed in the rotation eigen pseudoplane $\langle \mathcal{P}^{1+1}\rangle \subset \langle \mathcal{P}^{3+1}\rangle$ determined here by the time-arrow $\overrightarrow{ct}^{(1)}$ and the vector $\mathbf{v} = c \cdot \mathbf{tanh}\ \gamma$ in $\langle \mathcal{E}^3\rangle^{(1)}$.

In the simplest case of 2×2-dimensional matrix (324), we have in the pseudoplane

$$\tilde{E}_{II} = \{roth\ \Gamma\}_{2\times 2} \cdot \tilde{E}_I = \begin{bmatrix} \cosh\gamma & \sinh\gamma\cdot\cos\alpha \\ \sinh\gamma\cdot\cos\alpha & \cosh\gamma \end{bmatrix}, \quad \cos\alpha = \pm 1. \tag{21A}$$

It is a hyperbolic rotation of the axes $x^{(1)}$ and $ct^{(1)}$ at the angle γ to the bisectrix of the 1-st quadrant if $\cos\alpha = +1$ and to the bisectrix of the 2-nd quadrant if $\cos\alpha = -1$).

Further, we begin to use the fundamental concept a "*world line*" as the curve-function $\mathbf{x}(\overrightarrow{ct})$, introduced in the Theory of Relativity by Hermann Minkowski in 1909 [65]. It is a geometric invariant – as two isotropic cones and both Minkowski hyperboloids (Chs. 11, 12). But all they can be expressed in relative admitted bases of the Minkowski space-time.

In first, consider the simplest relativistic physical uniform rectilinear movement of a material point M. At the moment $t = 0$ the point passes through the origin O of the frame of reference $\tilde{E}_1$, which here is the common origin for all centralized bases $\langle \tilde{E}_k\rangle$. Then this *world line* of M is a straight line inside the internal or "light" isotropic cone. The light cone is the locus of all central light rays proceeding from O. A certain pseudo-Cartesian base $\tilde{E}$, where M is immobile, has its own time-arrow $\overrightarrow{ct}$ coinciding with the straight world line of M mapped in the original base $\tilde{E}_1$. (In general, all the new coordinate axes are determined by columns of the matrix for a new base $\tilde{E}_k$.) This new time-arrow $\overrightarrow{ct}$ is completely determined in $\tilde{E}_1$ by the hyperbolic angle γ with $\overrightarrow{ct}^{(1)}$ and the fixed directional cosines of the vector $\mathbf{tanh}\ \gamma \in \langle \mathcal{E}^3\rangle^{(1)}$ or the point M velocity $\mathbf{v} = v \cdot \mathbf{e}_\alpha = c \cdot \mathbf{tanh}\ \gamma = \mathbf{const}$.

A world line may be, of course, arbitrary curvilinear one (as a geometric invariant), but its slope must be less than the slope of the light cone, i. e., of rays of light relatively to the time-arrow $\overrightarrow{ct}^{(1)}$. We represent world lines in the *universal base* $\tilde{E}_1 = \{I\}$ only for its geometric visuality and comparison with other world lines, as well as all the other pseudo-Cartesian bases $\tilde{E}$ are expressed also with respect to $\tilde{E}_1$! With these arguments, the base $\tilde{E}_1$ is defined initially as if Cartesian one too! Such approach was used before in Ch. 12 for representing the two Minkowskian Hyperboloids with the same purpose. (A universal base $\tilde{E}_1 = \{I\}$ is the relative notion defined by inertial immovable Observer N_1.)

In trigonometric kinematics of STR, the angles γ and Γ of motion tensor in (20A) for transformations of coordinates always have the sign $+$. The sign $-$ for the angles is possible only in *mental* motions to past with the use of *antipodal* hyperbolic geometry (sect. 12.1.) This is equivalent to the *Principle of determinism* for material phenomena. These facts distinguish to a some extent hyperbolic kinematics of STR and the laws of hyperbolic motions in the Lobachevsky–Bolyai geometry. The same time-arrow $\overrightarrow{ct}$ (and the same world straight lines) in the two cavities of the light cone are determined with the same matrices $roth\ \Gamma$ corresponding, from the physical point of view, to the same velocity vector and, from the geometrical point of view, to the same motion:

$$roth\ \Gamma = F(\gamma, \mathbf{e}_\alpha) \equiv F(-\gamma, -\mathbf{e}_\alpha). \tag{22A}$$

The last expression here is valid only in *antipodal* hyperbolic geometry. Another time-arrow that is symmetric to original one with respect to $\overrightarrow{ct}^{(1)}$ (and the parallel to it world straight line) is determined with the inverse matrix.

It has the physical sense of an additively opposite velocity vector and the corresponding to it geometric sense:

$$roth^{-1}\,\Gamma = F(\gamma, -\mathbf{e}_\alpha) = roth\,(-\Gamma) \equiv F(-\gamma, \mathbf{e}_\alpha). \qquad (23A)$$

In (22A), (23A), the principal angle γ is positive for directions of material points motions along the time arrow to the Future, it is formally negative for mental motions to the Past.

Formulae (20A), (21A) imply that due to hyperbolic rotations the *coordinate velocity* of physical movement v along $x^{(1)}$ is expressed trigonometrically from this relation:

$$\frac{v}{c} = \frac{\Delta x}{c\cdot\Delta t} = \frac{\sinh\gamma\cdot\cos\alpha}{\cosh\gamma} = \tanh\gamma\cdot\cos\alpha, \quad (\cos\alpha = \pm 1). \qquad (24A)$$

Generally, in $\langle\mathcal{P}^{3+1}\rangle$, the Euclidean *vector of coordinate velocity* $\mathbf{v}$ in $\langle\mathcal{E}^3\rangle^{(1)}$ is determined by its module $||\mathbf{v}||$ and the directional cosines $\cos\alpha_j$, $j = 1, 2, 3$; its three Euclidean projections onto the axes have also physical and trigonometric forms:

$$\frac{v_j}{c} = \frac{\Delta x_j}{c\cdot\Delta t} = \tanh\gamma\cdot\cos\alpha_j,\; j = 1,2,3, \quad (\mathbf{v} = \{v_j\} = v\cdot\mathbf{e}_\alpha = c\cdot\mathbf{tanh}\;\gamma), \qquad (25A)$$

where $v > 0$; $-1 \le \cos\alpha_j \le +1$ and $\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1$.

For mapping of the simplest physical uniform rectilinear movement of a point M at velocity v in the pseudoplane of motion $\langle\mathcal{P}^{1+1}\rangle$, its original base $\tilde{E}_1$, where M is immovable with coordinates $x^{(1)}$ and $\vec{ct}^{(1)}$, must be hyperbolically rotated at the angle $\gamma = \text{artanh}\; v/c$ with $-\mathbf{e}_\alpha$ into the base $\tilde{E}_2$, where M has new values of coordinates $x^{(2)}$ and $\vec{ct}^{(2)}$. Such description corresponds to the passive point of view onto modal transformation [21, p. 428].

The specific spherical–hyperbolic analogy between γ and φ in the universal base $\tilde{E}_1$ are usually either *sine-tangent* (331-I) or visual at graphic representations *tangent-tangent* (355):

$$dx^{(1)}/(dct^{(1)}) = v/c = \tanh\gamma \equiv \sin\varphi = \tan\varphi_R \;\; in\;\tilde{E}_1, \quad (\gamma > \varphi(\gamma) > \varphi_R(\gamma)).$$

There is no *infinitesimal distinctions* between the very small angles $\gamma, \varphi, \varphi_R$, when $\gamma \to 0$ ($v \ll c$). If we analyze in $\tilde{E}_1$ (with respect to immovable Observer) one-step absolute motion, then spherical and hyperbolic angles are equally applicable with right signs. But if we deal with combined or non-collinear principal motions, for example, some motion with respect to moving Observer, or with more complex multistep and integral motions, then only pseudo-Euclidean geometry with rotations and hyperbolic non-Euclidean geometry with motions should be applied with principal hyperbolic and secondary orthospherical angles γ and θ.

By this cause, in STR, with tensor trigonometric approach, the concepts of hyperbolic with spherical geometry may be useful. It concerns not only to motion and complementary angles, but and to the various types of parallelism angles, considered in sect. 6.4. Thus, the *spherical* parallel angle of Lobachevsky $\Pi(a)$ [40, 41] up to now is the initial fundament for construction of the real-valued *hyperbolic* non-Euclidean geometry. From the point of view of the enveloping space $\langle\mathcal{P}^{n+1}\rangle$ with interpretation on a hyperboloid II, the angular argument $\Pi(a)$ has a geometric sense on it and in STR only in universal bases $\tilde{E}_{1u}$ and only for one-step motions. However the analogous, but purely hyperbolic angle of parallelism υ in (364-IY), Ch. 6, is consistent in any pseudo-Cartesian bases with tensors $\{I^{\pm}\}$ or $\{I^{\mp}\}$:

$$\left.\begin{array}{l}\gamma,\ \varphi:\ \sinh\gamma \equiv \tan\varphi \;\Leftrightarrow\; \tanh\gamma \equiv \sin\varphi\;, (\varphi \ne \pm\pi/2) \;\Rightarrow\; \xi(\gamma) = \pi/2 - \varphi(\gamma),\\ \gamma,\ \xi:\ \sinh\gamma \equiv \cot\xi \;\Leftrightarrow\; \tanh\gamma \equiv \cos\xi\;,\ \text{sech}\;\gamma \equiv \sin\xi,\ (\xi \ne 0),\ d\xi = -d\varphi;\\ \Pi(a) \equiv \xi(\gamma) = \pi/2 - \varphi(\gamma) = \arccos(\tanh\gamma) = 2\arctan\,[\exp(-\gamma)],\\ P(a) = \upsilon(\gamma) = 2\text{artanh}\;[\exp(-\gamma) - \text{see both ngles from (360-II) in Ch. 6.}\end{array}\right\} \qquad (26A)$$

In relativistic factors: $v/c = \tanh\gamma \equiv \sin\varphi = \cos\;\xi$, $\sqrt{1-(v/c)^2} = \text{sech}\;\gamma \equiv \sin\xi = \cos\varphi$. Both relativistic factors, used up to now by physists in the Past, have not any geometric senses and are subjected to the operations of mathematical analysis with great difficulty.

We use all these motion and parallel angles for spherical and hyperbolic geometries and in STR as clear trigonometric arguments of our tensor trigonometry.
$\varphi = l/R$ – is the *covariant parallel angle* in spherical type of non-Euclid ean geometries,
$\gamma = \lambda/R$ – is the *covariant parallel angle* in hyperbolic type of non-Euclidean geometries.
They are correct either in universal bases $\tilde{E}_{1u}$ or in any $\tilde{E}_k$ in the same types of geometries.

In order to get absolute (i. e., not depending on the 5-th Euclid's Postulate) geometry, the spherical or hyperbolic nature of the parallel angle $\pm\alpha$ should not be fixed! Initially put $\alpha \neq 0$ is the angle between Euclidean and abstract parallels in the universal base $\tilde{E}_1$. (For example, in the hyperbolic geometry, the spherical type angle α is complementary to the Lobachevsky parallel angle $\Pi(a)$) till the right angle $\pi/2$.) And only after this formal first step, we become to the dilemma: what nature of the parallel angle α should be chosen us?

If $\alpha > 0$, it is chosen as a spherical angle, then non-Euclidean geometry of spherical type is gotten, and its parallels are intersected on the side of angle α due to G. Saccheri [35].

If $\alpha < 0$, it is chosen as a hyperbolic angle, then non-Euclidean geometry of hyperbolic type is gotten, and its parallels converge in ∞ on the side of angle α due to Lobachevsky [40].

If $\alpha < 0$, this corresponds to the Euclidean geometry, $\Pi(a) = \pi/2$.

Moreover, if in the universal base $\tilde{E}_1$ geodesic motions are realized from the center C on a hyperspheroid along a big circle or on a hyperboloid II along a hyperbola (see in Ch. 12, at Figure 4), then both principal angles change covariantly to the motion's direction as follows: $+\alpha(a) = \varphi \in [0 \cdots \pm\pi/2]$, $-\alpha(a) = \gamma \in [0 \cdots \pm\infty)$, realized on quasi- and pseudoplane.
In both plane variants, the single perpendicular to a given line at its given zero point determining the angle of parallelism $\Pi(a)$ at the point "a" off the perpendicular, and the single Euclidean parallel to a given line, passing perpendicularly through this point "a" of this perpendicular in order to determine in it the angle of parallelism α are found by application of Euclidean geometry, for example, using a compass and a ruler in the universal base $\tilde{E}_1$, with the universal relation $\Pi(a) - \alpha = \pi/2$!

* * *

Conclude this Chapter with the following very essential remark. The initial *mathematical approach* of Poincaré in 1905 [63] to constructing Theory of Relativity is logically quite perfect, contrary to the initial *physical approach* of Einstein in 1905 [67] based on his two Postulates acting in all Galileo's inertial frames of reference: (1) the extremum and equality of scalar speed of light "c" and (2) the Principle of Relativity, repeated the same Postulate of Poincaré from 1904 (without reference to it). However the Einsteinian Postulate (1) leads mathematically to constructing an infinite set of "trigonometries" and their quasiphysical isomorphisms with pseudo-Hölderian metrics of positive powers p (non-quadratic if $p \neq 2$):

$$|ds|^p = |dct|^p - \{|dx_1|^p + |dx_2|^p + |dx_3|^p\} \geq 0, \qquad 1 \leq p < \infty,$$

where $c = \max(||dx||_p/dt)$ and only for the speed of light $||dx||_p/dt = c, ds = 0$.

However, Einstein proposed the graceful physical manner for clear definition of events simultaneity with the use of two light rays, realizing the previous idea of Poincaré about using light rays for definition of events simultaneity from 1900 [62] (without reference to it), where also in first formula $m = E/c^2$ was inferred and published. Such *axiomatic definition of simultaneity* as if introduced implicitly the quadratic pseudo-Euclidean metric with $p = 2$ in the space-time of STR. But this Einsteinian definition is only a beautiful *theorem* (Ch. 4A) of the Minkowski pseudo-Euclidean geometry from 1909, when he renovated factually the original mathematical approach of Poincaré to new space-time. In Ch. 4A we chowed that the Einsteinian definition of events simultaneity leads strictly to discovery of the so-called *deformational transformations of coordinate* in the Poincaré-Minkowski space-time (i. e., non-Lorentzian ones). In this space-time, the concept of events simultaneity, with respect to the given frames of reference, is defined by tensor trigonometry highly simply and clarity. See about such deformational transformations in Ch. 4A, but initially in Chs. 5, 6, 12.

Chapter 2A

Tensor trigonometric model of Lorentzian homogeneous principal transformations

Let a particle M moves in the space-time $\langle\mathcal{P}^{3+1}\rangle$ uniformly and rectilinearly along its straight world line passing through the center O of $\tilde{E}_1$. Due to (21A), its 4 coordinates in the initial base $\tilde{E}_1$ and in $\tilde{E}$ tied with M are expressed in the *simplest trigonometric form* by the *passive* rotation at the hyperbolic angle Γ, identical to original Lorentzian transformation, found by him in 1895 [58] and named so by Poincaré in 1905 [63] as of new space-time group:

$$\begin{array}{ccc} roth\ (-\Gamma) & \mathbf{r}\{\tilde{E}_1\} & \mathbf{r}\{\tilde{E}\} \end{array}$$

$$\begin{bmatrix} \cosh\gamma & 0 & 0 & -\sinh\gamma\cdot\cos\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\gamma\cdot\cos\alpha & 0 & 0 & \cosh\gamma \end{bmatrix} \cdot \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ ct^{(1)} \end{bmatrix} = \begin{bmatrix} \cosh\gamma\cdot x_1^{(1)} - \sinh\gamma\cdot\cos\alpha\cdot ct^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ \cosh\gamma\cdot ct^{(1)} - \sinh\gamma\cdot\cos\alpha\cdot x_1^{(1)} \end{bmatrix}.$$

Represent the *hyperbolic* transformation in the 4-dimensional system $\{t=0, \mathbf{x}=\mathbf{0}\}$:

$$\left.\begin{array}{l} x_1 = \cosh\gamma\cdot x_1^{(1)} - \sinh\gamma\cdot\cos\alpha\cdot ct^{(1)} = \dfrac{x_1^{(1)} - \tanh\gamma\cdot\cos\alpha\cdot ct^{(1)}}{\operatorname{sech}\,\gamma}, \\ x_2 = x_2^{(1)}, \qquad x_3 = x_3^{(1)}, \\ ct = \cosh\gamma\cdot ct^{(1)} - \sinh\gamma\cdot\cos\alpha\cdot x_1^{(1)} = \dfrac{ct^{(1)} - \tanh\gamma\cdot\cos\alpha\cdot x_1^{(1)}}{\operatorname{sech}\,\gamma}. \end{array}\right\} \qquad (27A)$$

This is the initial Poincaré–Minkowski trigonometric form of the (in fact 2-dimensional) *Lorentz homogeneous transformations for space and time* in $\tilde{E}_1$ and $\tilde{E}$ [63, 65]. The multiplier $\cos\alpha = \pm 1$ determines two directions of the sine and tangent vectors. If (24A) are taken into account, they may be expressed in the *physical* form [58, 59, 63]:

$$x_1 = \frac{x_1^{(1)} - v\cdot t^{(1)}}{\sqrt{1-v^2/c^2}}, \quad x_2 = x_2^{(1)}, \quad x_3 = x_3^{(1)}, \quad ct = \frac{ct^{(1)} - (v/c)\cdot x_1^{(1)}}{\sqrt{1-v^2/c^2}}.$$

Take advantage of the hyperbolic rotational matrix with general canonical structure (363) in the base $\tilde{E}_1$, then we obtain the *general trigonometric linear transformations* (pure hyperbolic) of the four coordinates of M as the three scalar space-orthoprojections (at $i = 1, 2, 3$) and the time-orthoprojection

$$\left.\begin{array}{rl} x_i = & \cos\alpha_i\cdot[\cosh\gamma\cdot S - \sinh\gamma\cdot ct^{(1)}] + [x_i^{(1)} - \cos\alpha_i\cdot S], \\ ct = & \cosh\gamma\cdot ct^{(1)} - \sinh\gamma\cdot S, \\ (S = & \cos\alpha_1\cdot x_1^{(1)} + \cos\alpha_2\cdot x_2^{(1)} + \cos\alpha_3\cdot x_3^{(1)}), \end{array}\right\} \qquad (28A)$$

and their vectorial-scalar form with an arbitrary direction of sine and tangent vectors

$$\left.\begin{array}{rl} \mathbf{x} & = [\cosh\gamma\cdot \mathbf{e}_\alpha\mathbf{e}'_\alpha\cdot\mathbf{x}^{(1)} - \sinh\gamma\cdot\mathbf{e}_\alpha\cdot ct^{(1)}] + (I - \mathbf{e}_\alpha\mathbf{e}'_\alpha)\cdot\mathbf{x}^{(1)} = \\ & = [\cosh\gamma\cdot \overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}\cdot\mathbf{x}^{(1)} - \sinh\gamma\cdot\mathbf{e}_\alpha\cdot ct^{(1)}] + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}\cdot\mathbf{x}^{(1)}, \\ ct & = \cosh\gamma\cdot ct^{(1)} - \sinh\gamma\cdot\mathbf{e}'_\alpha\cdot\mathbf{x}^{(1)}. \end{array}\right\} \qquad (29A)$$

$$\overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha} = \mathbf{e}_\alpha\mathbf{e}'_\alpha = \overleftarrow{\mathbf{v}\mathbf{v}'} = \mathbf{v}\mathbf{v}'/|\mathbf{v}'\mathbf{v}| = \mathbf{v}\mathbf{v}'/||\mathbf{v}||^2, \quad I - \mathbf{e}_\alpha\mathbf{e}'_\alpha = \overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha} = \overrightarrow{\mathbf{v}\mathbf{v}'}$$

are the orthoprojectors in $\tilde{E}_1^{(3)}$ (see in sect. 2.5) into $\langle im\ \mathbf{v}\rangle$ and $\langle im\ \mathbf{v}\rangle^\perp$ in $\langle\mathcal{E}^3\rangle$.

In its general form, the vector of the directional cosines $\mathbf{e}_\alpha = \{\cos\alpha_i\}$ determines the direction of the sine and tangent vectors in $\tilde{E}_1^{(3)}$ of $\tilde{E}_1$ as well as of the velocity.

Transformations equivalent to (29A) were derived by G. Herglotz [84; 76, p. 27] as

$$\mathbf{x} = \mathbf{x_v} + (\mathbf{x}^{(1)} - \mathbf{x_v}^{(1)}) = \frac{\overleftarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha{}'} \cdot \mathbf{x}^{(1)} - \mathbf{v} \cdot t^{(1)}}{\sqrt{1 - ||\mathbf{v}||^2/c^2}} + \overrightarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha{}'} \cdot \mathbf{x}^{(1)}, \quad ct = \frac{ct^{(1)} - (\mathbf{v}/c)' \cdot \mathbf{x}^{(1)}}{\sqrt{1 - ||\mathbf{v}||^2/c^2}}.$$

He decomposed $\mathbf{x}^{(1)}$ in $\langle \mathcal{E}^3 \rangle$ as the relativistic and non-relativistic projections onto $\mathbf{v}$ (*the Principle of Herglotz*). They are turned into the form (29A) with $\mathbf{v}/c = \mathbf{tanh}\gamma$.

The clear interpretation of these general trigonometric and physical transformations are seen from their comparison with (27A). When the base $\tilde{E}_1$ is hyperbolically rotated in the pseudoplane $\langle \mathbf{v}, ct^{(1)} \rangle$, then only the time projection and the relativistic space projection $\overleftarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha{}'}\mathbf{x}^{(1)}$ are subjected to the modal transformation. The non-relativistic space-projection $\overrightarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha{}'}\mathbf{x}^{(1)}$ orthogonal to $\mathbf{v}$ is invariant under Lorentzian and Galilean transformations.

In the projective non-Euclidean vectorial tangent subspace of radius $R = 1$ there hold: $||\mathbf{tanh}\ \gamma|| = \tanh\gamma = ||\mathbf{v}||/c = \sqrt{\tanh^2\gamma_1 + \tanh^2\gamma_2 + \tanh^2\gamma_3}$ $\quad (\gamma \geq 0)$, and

$$\mathbf{tanh}\ \gamma = \tanh\gamma \cdot \mathbf{e}_\alpha = \mathbf{v}/c \to \tanh\gamma_i = \cos\alpha_i \cdot \tanh\gamma = v_i/c, \quad (i = 1, 2, 3), \qquad (30A)$$

where γ_i are the partial hyperbolic angles with their values in the Euclidean orthoprojections $\tanh\gamma_i = \cos\alpha_i \cdot \tanh\gamma$ of the vector $\mathbf{tanh}\ \gamma$ in the subbase $\tilde{E}_1^{(3)}$.

The same we get for sine is $\sinh\gamma_i = \cos\alpha_i \cdot \sinh\gamma = \cosh\gamma \cdot \tanh\gamma_i$. But the projective vectorial sine space is Euclidean one, because for it $R \to \infty$. In both these especial vectorial spaces (of tangents and sines), the Pythagorean Theorem for moduli of the projections is inferred. (By multiplier c, they are transformed into the velocities spaces – see in Ch. 3A).

In the transformations of coordinates of a particle M moving along its world line, as a rule, two kinds of bases are used: $E_{1u} = rot\ \Theta \cdot \tilde{E}_1 = \{rot\ \Theta\}$ and $\tilde{E} = roth\,\Gamma \cdot \tilde{E}_1 = \{roth\,\Gamma\}$. The first base is one of the universal ones (16A). In STR, the initial universal base $\tilde{E}_1 = \{I\}$ is a relative notion too. However it is tied to the given immovable in it inertial Observer, say N_1 as if in the Cartesian subbase $\tilde{E}_1^{(3)}$. Canonical trigonometric matrix forms are expressed initially usually in terms of the base $\tilde{E}_1$! The base determines a relation between Observer N_1 and other pseudo-Cartesian base $\tilde{E}_k = T_{1k} \cdot \tilde{E}_1$ with Observer N_k.

The following two pure variants are possible.

(1) $T'_{1k} \cdot T_{1k} = I$. Then $\tilde{E}_{1k} \in \langle rot\ \Theta \rangle$, it is another universal base, but for N_{1k}.

(2) $T_{2k} = T'_{2k}$. Then $\tilde{E}_{2k} \in \langle roth\ \Gamma \rangle$, this base is another one for inertially moving N_{2k}.

In variant (1), the subbase $\tilde{E}_k^{(3)}$ is immovable with respect to N_1, it is the result of orthospherical rotating $\tilde{E}_1^{(3)}$ at the angle Θ_{1k}. In variant (2), the subbase $\tilde{E}_k^{(3)}$ is moving at the velocity $\mathbf{v} = c \cdot \mathbf{tanh}\ \gamma$ with respect to N_1. Any general homogeneous Lorentzian transformation of bases in $\langle P^{3+1} \rangle$ may by represented as the product of the two pure types transformation (1) and (2) due to the polar decomposition (19A).

Lorentzian transformations in matrix form T can be applied actively to pseudo-Cartesian bases for expression in them of all space-time coordinate for the given particle M or other some objects, as *passive point of view*. Then these coordinates are calculated with the inverse matrix T^{-1}. Lorentzian transformations can be applied with the same matrix T to the space-time coordinate of the given moving particle M or other some objects in the fixed base, as *active point of view*.

The Special physical-mathematical principle of relativity (sect. 12.3) takes place for them. It consists, for example, in form-invariance of Lorentzian transformations of pseudo-Cartesian bases for a moving uniformly and rectilinearly material point on a straight world line. Of course, it is the simplest case for relativistic–geometric transformations in $\langle P^{3+1} \rangle$.

Due to homogeneity and isotropy of the Minkowskian space-time, all Lorentzian transformations may be expressed in the clear *trigonometric forms*. However, if we deal with a moving *non-point* geometric object, then, in addition, the quite another trigonometric type of relativistic transformations may be used. It determines relativistic contraction of the object with geometric parameters in the direction of its physical movement. Generally, in scalar and tensor variants of a trigonometry, projective characteristics of two kinds, either sine–cosine or tangent–secant, are evaluated. Their kind depends on a problem being solved. So, in tensor trigonometry of the space-time, the rotational as deformational elementary trigonometric matrix-functions are used. Their canonical forms in the base $\tilde{E}_1$ were given by formulae (362), (363) and (364), (365), for example, in $\langle \mathcal{P}^{n+1} \rangle$ generally in these *fourth-block forms*:

$$roth\,\Gamma = F_h(\gamma, \mathbf{e}_\alpha), \quad (F = F'), \qquad def h\,\Gamma = D_h(\gamma, \mathbf{e}_\alpha), \quad (D \neq D')$$

$$\left\| \frac{\cosh\gamma \cdot \overleftarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha{}'} + \overrightarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha{}'}}{\sinh\gamma \cdot \mathbf{e}'_\alpha} \middle| \frac{\sinh\gamma \cdot \mathbf{e}_\alpha}{\cosh\gamma} \right\| \cdots \left\| \frac{\mathrm{sech}\,\gamma \cdot \overleftarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha{}'} + \overrightarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha{}'}}{+\tanh\gamma \cdot \mathbf{e}'_\alpha} \middle| \frac{-\tanh\gamma \cdot \mathbf{e}_\alpha}{\mathrm{sech}\,\gamma} \right\|. \quad (31A - I, II)$$

(31A-I) represents Lorentzian transformation as a pure hyperbolic and hyperbolically orthogonal bivalent tensor in more general $4D$ trigonometric form, and (31A-II) represents the bivalent tensor of trigonometric deformation for expression of the Lorentzian contraction. See more in detail in Chs. 3A, 4A, 5A, 7A. As the next developing of this topic, we'll represent in Ch.'7A the homogeneous Lorentzian transformation in its general pseudo-Euclidean form.

These matrices express corresponding symmetric and anti-symmetric tensors of specific transformations in $\langle \mathcal{P}^{n+1} \rangle$: $roth\,\Gamma$ realizes as well as principal hyperbolic rotations at the angle γ as orthosperical rotations of the unity vector $\mathbf{e}_\alpha$ of the directional cosines of the tensor angle Γ with corresponding rotations in the current Euclidean subspace of $\langle \mathcal{P}^{n+1} \rangle$; but $def h\,\Gamma$ realizes trigonometric deformation at the angle γ in the direction $\mathbf{e}_\alpha$ (see Ch. 4A).

Rotational hyperbolic matrix (31A - I) and orthospherical matrix (497) from the sect. 12.2 in these elementary forms are the two pure types of the homogeneous Lorentz transformations in both their canonical forms with respect to the universal base $\tilde{E}_1$. And all their compositions in pseudo-Cartesian bases admissible with reflector tensor (17A - I) form the group of continuous homogeneous Lorentz transformations. Such transformations may be reduced to their polar forms as products of these two matrices of pure types. All orthospherical rotations form their proper subgroup of the Lorentz group. (In STR and in non-Euclidean hyperbolic geometry, these two pure types of rotations are geometric motions and used only in elementary forms with $q = 1$, and, more clearly, as (362), (363), (497).

The term "Lorentz transformations" was introduced by Henri Poincaré in his pioneer paper on the new relativity theory [63] in June of 1905. These rotational homogeneous transformations play the essential role in his previously suggested in 1904 Physical Principle of Relativity as development of the classical Galilean Principle of Relativity from 1636.

In two next Chs. 3A and 4A, we give trigonometric interpretations (sine–cosine and tangent–secant) of the space-time relativistic effects of STR. They take place in the internal and external cavities of the light cone, the latter only with respect to the original base $\tilde{E}_1$.

In Ch. 7A, (153A), we give the most general and developed $(n{+}1){\times}(n{+}1)$ matrix (mainly, in particular, at $n = 3$) canonical and polar forms of arbitrary Lorentzian pseudo-Euclidean homogeneous transformation also in the original base $\tilde{E}_1$ in $\langle \mathcal{P}^{n+1} \rangle$. And in Ch. 8A, (202A), we addded it by corresponding forms for Special quasi-Euclidean transformations in $\langle \mathcal{Q}^{n+1} \rangle$ (in particular, at $n = 2$).

Chapter 3A

Minkowskian real kinematic dilation of time as a consequence of the time-arrow hyperbolic rotation

A world line in $\langle \mathcal{P}^{3+1} \rangle$ is connected at each point M with the instantaneous light cone with its center – a world point M, where two internal cavities of the cone diverge as these *cone of past* and *cone of future*. Any relativistic motion is directed along own proper time-arrow from past to future. Hence, it is performed inside the light *cone of future*, where a slope of a world line at any point satisfies inequalities $\mathbf{0} \leq |\mathbf{tanh}\,\gamma| \leq \mathbf{1}$ – Figure 1A(1). In $\langle \mathcal{P}^{3+1} \rangle$, all physical movements are represented by world lines in homogeneous coordinates [63, 65], and more clear by Minkowski diagrams. The straight lines represent uniform rectilinear movement, because the relativistic effects of STR mentioned in Chs. 1A and 2A need in differentiation of 1-st order with 1-st differentials of increments of space-time coordinates!

In the beginning, let us outline briefly the historical aspects in discovery of this discussed relativistic effect. In 1887 Michelson–Morley ultra-precise physical experiment in the USA did not reveal absolute motion of light relatively to the Earth. A crisis arose in the classical kinematics. In 1895 Lorentz modified Maxwell's equations by introducing a contraction in size of the electron along its moving. In addition, he introduced the so-called "local time" for the same coordinate system associated with the moving electron [58]. Factually 1985, Lorentz during creation of the correct theory of moving electron, hypothesized the local mutual contractions time and space intervals in the direction of moving, led further to his relativistic well-known transformations. In 1900, in his article "La Théorie de Lorentz et le Principe de réaction" [62], Henri Poincaré, with formula $m = E/c^2$, gives interpretation to the "temps local de Lorentz" as: "C'est le temps d'observateurs mobiles qui réglent leurs horloges par des signaux optiques en ignorant le mouvement de traduction dont ils sont animés." (It is the time of mobile observers who regulate their clocks by optical signals, ignoring the translational movement by which they are animated.) This idea became basis for Einstein in paper "Zur Elektrodynamik beweter Köper." of June 30, 1905 [67], without references to previous Poincaré and Lorentz well-known works. Young Albert Einstein accepted reality of time slowdown in moving systems, however after the pioneer article by Poincaré "Sur la dynamique de l' électron." of June 5, 1905 [63], where fundamental relativistic notions – complex pseudo-Euclidean space-time with its Lorentz group were introduced.

The material point representing a real lengthy object is the object inertia center (the barycenter), i. e., as a particle. A material point M (see Figure 1A) in $\langle \mathcal{P}^{3+1} \rangle$ is physically immovable with respect to a certain frame of reference $\tilde{E}_2$ and is physically moving with respect to $\tilde{E}_1$. The straight world line of the particle M in $\langle \mathcal{P}^{3+1} \rangle$ with respect to $\tilde{E}_1$ is its time-arrow parallel to $\overrightarrow{ct}^{(2)}$ (the light cone inclination does not depend on the base chosen, as it is invariant). For the movement, the bases $\tilde{E}_1$ and $\tilde{E}_2$ are connected by the hyperbolic rotation $\tilde{E}_2 = roth\,\Gamma_{12} \cdot \tilde{E}_1$. From the point of view of Observer N_1, the particle M is moving in $\langle \mathcal{E}^3 \rangle^{(1)}$ at velocity $v_{12} = c \cdot \mathbf{tanh}\,\gamma_{12}$. In a neighborhood of M, a certain process may take place. By the clock of Observer N_2, the process takes time interval $\Delta t^{(2)}$ determined by segment $M'M''$ of the world line parallel to $\overrightarrow{ct}^{(2)}$ with taking into account the scale in the time-arrow. It is, according to STR, the *proper time* $\Delta\tau = \Delta t^{(2)}$ of the process, as it is counted by a relatively immovable clock. Proper time in any moving object is its absolute characteristic, or a pseudo-Euclidean metric invariant inside the cone of future. With respect to its rest base $\tilde{E}_2$, it is identical to coordinate time $\Delta t^{(2)}$. With respect to $\tilde{E}_1$, coordinate time of the process counted by Observer N_1 is determined by projection of the segment $M'M''$ onto $ct^{(1)}$ with taking into account the scale, it is equal to $\Delta t^{(1)}$ [76, p. 109]. *Coordinate time* $\Delta t^{(1)}$ of the process in moving object is its *relative characteristic* [67].

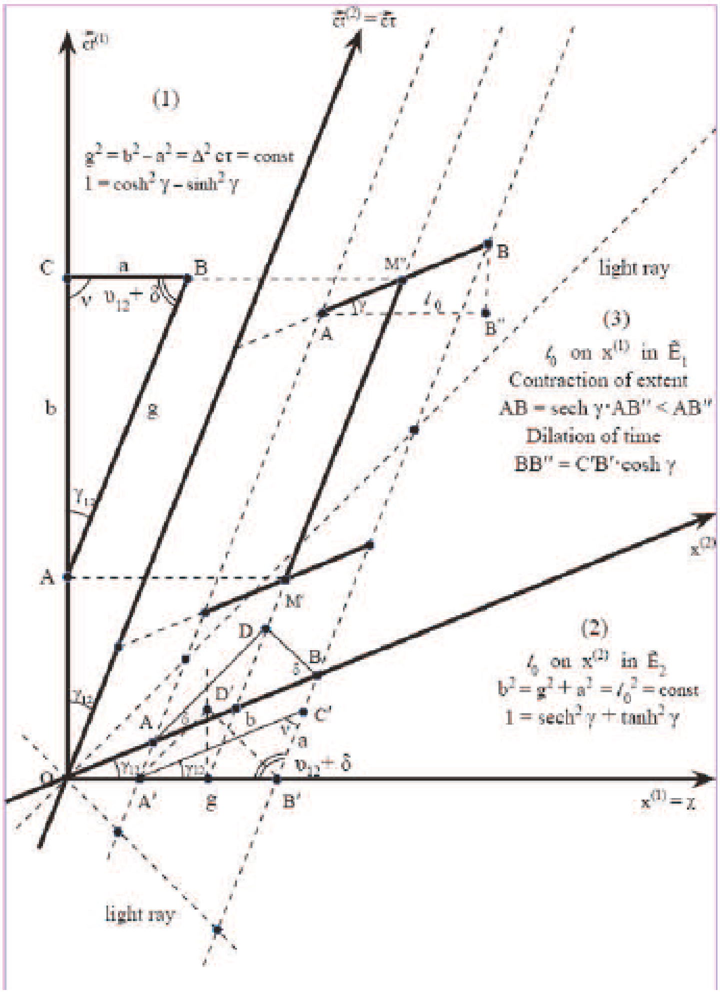

Figure 1A. Trigonometric interpretations of the STR relativistic effects inside and outside the light cone in coordinates $\{\mathbf{x}, ct\}$ with angles γ, υ, δ and ν (Ch. 6) in the Minkowski space-time $\langle\mathcal{P}^{3+1}\rangle$, according to Poincaré's and Einstein's different interpretations.

(1). Relativistic dilation of time of a moving object with its Poincaré interpretation on the pseudo-Euclidean plane (in interior right triangle ABC); coordinate and proper velocities:

$$g^2 = b^2 - a^2 = \Delta^2 c\tau = \text{const} \ \sim 1 = \cosh^2\gamma - \sinh^2\gamma,$$
$$b = \Delta ct^{(1)} = \cosh\gamma \cdot g > g = ct^{(2)} \ \rightarrow \ \Delta ct^{(2)} = \Delta c\tau = \Delta ct^{(1)}/\cosh\gamma < \Delta ct^{(1)},$$
$$a = \sinh\gamma \cdot g = \tanh\gamma \cdot b = \Delta x^{(1)} = \Delta\chi,$$
$$v = \Delta x^{(1)}/\Delta t^{(1)} = \Delta\chi/\Delta t^{(1)} = c\cdot\tanh\gamma, \ v^* = \Delta\chi/\Delta\tau = c\cdot\sinh\gamma \ \rightarrow v^* > v.$$
$$v < c \ \text{ and } \ v < v^* < \infty.$$

(2). Lorentzian contraction of a moving rod extent with its interpretation in the pseudo-Euclidean exterior right triangle $A'B'C'$; supervelocity of two moving rods contacts:

$$b^2 = g^2 + a^2 = l_0^2 = \text{const} \ \sim 1 = \text{sech}^2\gamma + \tanh^2\gamma \equiv \cos^2\varphi(\gamma) + \sin^2\varphi(\gamma),$$
$$g = l = \text{sech}\ \gamma\cdot b < b = l_0 \ \rightarrow \ l = \text{sech}\ \gamma\cdot l_0 \equiv \cos\varphi(\gamma)\cdot l_0 < l_0,$$
$$a = \tanh\gamma\cdot b = \tanh\gamma\cdot l_0 = \Delta ct^{(2)} \neq 0, \ w = l_0/\Delta t^{(2)} = c\cdot\coth\gamma = c^2/v > c.$$

(3). The Einsteinian approach to STR on the basis of his definition of simultaneity, but with the use of trigonometrically bonded cross bases (Ch. 4A).

Additionally, the accelerational and gravitational dilations of proper time by Einstein will be considered us in Chs. 5A ang 9A with both their equivalent cosines.

For example, with respect to $\tilde{E}_1$, this time is evaluated with the use of passive rotational transformation as well as one in the hyperbolic angle Γ_{12} of $\tilde{E}_1$ into $\tilde{E}_2$:

$$\Delta\mathbf{r}^{(1)} = roth\ \Gamma_{12} \cdot \Delta\mathbf{r}^{(2)} =$$

$$= roth\ \Gamma_{12}) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta c\tau \end{bmatrix} = \begin{bmatrix} \sinh\gamma_{12}\cdot\cos\alpha_1\cdot\Delta c\tau \\ \sinh\gamma_{12}\cdot\cos\alpha_2\cdot\Delta c\tau \\ \sinh\gamma_{12}\cdot\cos\alpha_3\cdot\Delta c\tau \\ \cosh\gamma_{12}\cdot\Delta c\tau \end{bmatrix} = \begin{bmatrix} \Delta x_1^{(1)} \\ \Delta x_2^{(1)} \\ \Delta x_3^{(1)} \\ \Delta ct^{(1)} \end{bmatrix}, \qquad (32A)$$

where $\Delta c\tau = \Delta ct^{(2)}$, and from the matrices fourth rows we obtain:

$$\Delta ct^{(1)} = \cosh\gamma_{12}\cdot\Delta c\tau \rightarrow \Delta c\tau = \Delta ct^{(1)}/\cosh\gamma_{12} < \Delta ct^{(1)}. \qquad (33A)$$

In STR relativistic effect (33A) is called *Einsteinian dilation of time* [67; 76, p. 30, 48]. *The notions "proper time" and "time dilation" (see the term interpretation in sect. 12.3) were introduced by H. Minkowski in his fundamental article* [66]. The segment $\Delta c\tau$ of the straight world line, i. e., of the process time in M, is expressed in the coordinates of its base $\tilde{E}_2 = \{x^{(2)}, \overrightarrow{ct}^{(2)}\}$. Geometrically this segment of the world line is a linear tensor element as the time-like oriented vector in $\langle\mathcal{P}^{3+1}\rangle$. Its quadratic pseudo-Euclidean imaginary invariant in the four-dimensional form of coordinates with respect to any pseudo-Cartesian base $\tilde{E}$ is

$$-(\Delta c\tau)^2 = -(\Delta ct)^2 + (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 = \text{const}, \qquad (34A)$$

where $\Delta t > 0,\ \Delta\tau > 0$. Since $\Delta c\tau = const$, invariant (34A) may be reduced in the base $\tilde{E}_1$, to its sine-cosine form-invariant trigonometric expression, which may be interpreted locally by the tangent of the unity hyperboloid I, identical to the pseudonormal of the conjugated unity hyperboloid II from Ch. 12 (see about this correspondence in Chs. 7A and 10A):

$$(i)^2 = -1 = -\cosh^2\gamma + (\sinh^2\gamma_1' + \sinh^2\gamma_2' + \sinh^2\gamma_3') = -\cosh^2\gamma + \sinh^2\gamma. \qquad (35A)$$

Here γ_j' (at $j = 1,2,3$) are the particular hyperbolic angles with their values in the Euclidean orthoprojections $\sinh\gamma_j' = \cos\alpha_j\cdot\sinh\gamma$ of the space-like sine vector $\mathbf{sinh}\ \gamma$ in the base $\tilde{E}_1$. Formula (35A) gives trigonometric quadratic invariant -1 under Lorentzian transformations of an unit time-like linear element $\Delta\mathbf{i}$.

Invariant scalar proper time is expressed in any pseudo-Cartesian base $\tilde{E}$ as

$$\Delta\tau = \Delta t/\cosh\gamma = \min\langle\Delta t^{(k)}\rangle. \qquad (36A)$$

When one deals with a *curvilinear world line*, the similar rotational transformation is *instantaneous*, and (32A) is applied to its arc differential as a linear element:

$$d\mathbf{r}^{(1)} = \{roth\ \Gamma\}^{(m)} d\mathbf{r}^{(m)} = \{roth\ \Gamma\}^{(m)} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ dc\tau^{(m)} \end{bmatrix} = \begin{bmatrix} dx_1^{(1)} \\ dx_2^{(1)} \\ dx_3^{(1)} \\ dct^{(1)} \end{bmatrix}. \qquad (37A)$$

Here the linear element $d\mathbf{r}^{(m)}$ is expressed also in coordinates of the *instantaneous base* $\tilde{E}_m = \{x^{(m)}, \overrightarrow{c\tau}^{(m)}\}$. In STR the instantaneous bases, *on the differential level, are always inertial*, but only from the point of view of inertial Observer, say N_1 in $\tilde{E}_1$. This has place, because the axes $\overrightarrow{c\tau}^{(m)}$ and $x^{(m)}$ are *instantaneous tangent and pseudo-normal* to a world line at a point M. Hence, the differential form similar to (36A) is

$$d\tau^{(m)} = dt^{(1)}/\cosh\gamma = d\lambda^{(m)}/(ic) = \min\langle dt^{(k)}\rangle. \qquad (38A)$$

Integrating (38A), one obtains $\Delta\tau = \Delta\lambda/(ic)$, where $\Delta\lambda$ is the pseudo-Euclidean length of a world line segment [76, p. 110]. Formulae (36A), (38A) express in the clear trigonometric form the relativistic effect of the Minkowski dilation of time at moving object *with respect to immovable Observer*, namely of some time process in the object [66]. *The effect may be easily interpreted as a consequence of the hyperbolic rotation of* $\overrightarrow{ct}^{(m)}$! But simultaneously the same dilation of space coordinate $x^{(m)}$ acts in the pseudoplane of this rotation for conservation of the Minkowski space-time structure!. The effect of time dilation was first established by V. Voight in 1887 [80] in his light elasticity theory and correctly by H. Lorentz in 1895 [58].

The segment of a world line $\Delta\mathbf{r}^{(2)}$ in $\tilde{E}_1$, due to (32A), has else the space-like projection $a = \Delta\chi$ into $\langle\mathcal{E}^3\rangle^{(1)}$ – Figure 1A. It is the space trajectory of the object M. It is expressed *in terms of coordinate time as well as proper time* with the two definitions of velocity:

$$\Delta\,\chi = \sqrt{\Delta^2 x_1^{(1)} + \Delta^2 x_2^{(1)} + \Delta^2 x_3^{(1)}} = \tanh\gamma\;\Delta ct^{(1)} = v\cdot\Delta t^{(1)} = \sinh\gamma\;\Delta c\tau = v^*\cdot\Delta\tau.$$

The *proper velocity* v^* is defined in addition to a *coordinate velocity* v as a concomitant relativistic effect. It is measured in proper distance $d\chi = dx^{(1)}$ by proper time $d\tau$:

$$\left.\begin{aligned} v^* &= c\cdot\sinh\gamma = d\chi/d\tau = v\cdot\cosh\gamma = d\chi/d\tau > v = c\cdot\tanh\gamma;\\ v_j^* &= c\cdot\sinh\gamma_j' = c\cdot\cos\alpha_j\cdot\sinh\gamma > v_j = c\cdot\tanh\gamma_j\;(j=1,2,3). \end{aligned}\right\} \qquad (39A)$$

The four vectors $\mathbf{v}, \mathbf{v}^*, \mathbf{tanh}\gamma, \mathbf{sinh}\,\gamma$ are collinear. The hyperbolic angles γ_j and γ_j' in (30A) – Ch. 2A and (35A) are related as follows:

$$(v_j = c\cdot\tanh\gamma_j = v\cdot\cos\alpha_j,\;\; v_j^* = c\cdot\sinh\gamma_j' = v^*\cdot\cos\alpha_j)\;\rightarrow\;\sinh\gamma_j' = \cosh\;\gamma\cdot\tanh\gamma_j.$$

In the pseudoplane of hyperbolic rotation, the given problem is reduced to solving an "interior hyperbolically right pseudo-Euclidean triangle" (see in sect. 6.4), where $\Delta c\tau$ is similar to the hypotenuse g, and $\Delta\chi$, $\Delta ct^{(1)}$ are similar to the legs a, b.

In products (32A), (37A), the hyperbolic rotational matrix is formally truncated, only its last row is used, because the original linear element $\Delta\mathbf{r}^{(2)}$ is parallel to its time-arrow $\overrightarrow{ct}^{(2)}$, and all its points in $\tilde{E}_2$ have *zero abscissa*. The whole matrix is used if the original element is on another time-arrow $\overrightarrow{ct}^{(3)}$ under an additional angle γ_{23} from the time-arrow $\overrightarrow{ct}^{(2)}$. It is valid for two- and multistep motions (see in Ch. 7A).

The following important theorem of STR and Minkowski Geometry is enough obvious.

Let M' and M'' be two causally-connected world points in $\langle P^{3+1}\rangle$. Then the straight-line segment $M'M''$ inside the light cone of future has the maximal pseudo-Euclidean length (proper time) among all continuous world lines (directed in future) connecting M' and M'':

$$ct_2 - ct_1 = ct|_{t_1}^{t_2} = \Delta ct\; > \Delta ct' = \int_{t_1}^{t_2} dct/\cosh\gamma(t) = \int_{t_1'}^{t_2'} dct' \geq 0,$$

where t is the time of immovable Observer N_1 in $\tilde{E}_1$, t' is the time of moving Observer N_m in $\tilde{E}_m$ – see at Figure 1A. But a continuous world line $M'M''$ can have the *minimal pseudo-Euclidean length* $\lambda = 0$ if the points M' and M'' are connected by the light segments, and only two of them are enough. The inequality above is also the clear trigonometric illustration to the well-known relativistic "twins paradox" [85], when in its left part Δt is interpreted as the *Earth time* and in its right part $\Delta t'$ is counted by astronauts. In the end of Ch. 5A, we comment it on the example of the imaginary cosmic travel to the nearest star system. Since world lines in $\langle P^{3+1}\rangle$ are invariants of the Lorentzian homogeneous transformations, then the pseudo-Euclidean length of segment $M'M''$ in $\tilde{E}_1$ and as a world line $M'M''$ in the pseudo-Cartesian $\tilde{E}_m$ is invariant too. This kinematic twin paradox has a place only by the cause, that we compare two different world ways between M' and M'' with their smaller and bigger slopes in $\langle P^{3+1}\rangle$, with respect to the time-arrow, for example, $\overrightarrow{ct}^{(1)}$.

Chapter 4A

Lorentzian seeming contraction of moving object extent as a consequence of the moving Euclidean subspace hyperbolic deformation

The Lorentzian seeming contraction of moving object's extent with the coefficient sech $\gamma(v)$ (see Figure 1A (2), (3)) is interpreted correctly on the basis of Einsteinian physical definition of simultaneity. The latter is caused by geometric theorem in $\langle\mathcal{P}^{3+1}\rangle$ only due to its pseudo-Euclidean metric! So, in the external cavity of the light cone in $\langle\mathcal{P}^{3+1}\rangle$ – see at Figure 1A(2), one usually considers some set of world points belonging on the whole to a certain Euclidean space $\langle\mathcal{E}^3\rangle^{(j)}$. In the simplest practical variant, the set consists of two world points as two events with a space-like interval between them. In general variant, important for subject of this Chapter, the set consists of points of a concrete geometric object immovable in a certain Euclidean space $\langle\mathcal{E}^3\rangle^{(j)}$ and moving with its projective map in another certain base $\tilde{E}_i$ from the point of view of Observer N_i. Of course, in the base $\tilde{E}_j$ all the geometric object's points are simultaneous, as they have always the same time coordinate on its own time-arrow $\overrightarrow{ct}^{(j)}$.

From the other hand, all world points of a given geometric object belong to their world lines in $\langle\mathcal{P}^{3+1}\rangle$. If the object is immovable with respect to the base $\langle\mathcal{E}^3\rangle^{(j)}$ and it is in uniform rectilinear movement with respect to the base $\tilde{E}_i$, then the world lines of all its points are parallel to the time-arrow $\overrightarrow{ct}^{(j)}$. Observer N_i fixes the moving object points in his own $\langle\mathcal{E}^3\rangle^{(i)}$ at a certain value of time on his own time-arrow $\overrightarrow{ct}^{(i)}$ although simultaneously, but with the object's sizes distortion along the moving direction. This space-like phenomenon is defined as *an improper world fixation* of the world points or of the moving object (as a set of its world points fixed in $\langle\mathcal{E}^3\rangle^{(i)}$).

$\langle\mathcal{E}^3\rangle^{(i)}$ and $\overrightarrow{ct}^{(j)}$ are hyperbolically orthogonal in $\langle\mathcal{P}^{3+1}\rangle$ iff the object is physically immovable just in $\langle\mathcal{E}^3\rangle^{(i)}$ also. Then $i = j$ and *the world fixation of the object is proper*. It corresponds to true sizes of the object as immovable one. And this graphical way for constructing fixations defines simultaneity of the world points in a certain base.

The Einstein's definition of simultaneity [67] is caused by a graceful geometric theorem in $\langle\mathcal{P}^{3+1}\rangle$ adopted by him implicitly. In 2-, 3, 4-dimensional cases, it is expressed as follows.

Theorem 1. *If a triangle ABC* (see Figure 1A) *is formed by a space-like segment AB and two light segments AC and BC (i. e., isotropic zero legs) coming from the opposite directions, then its median and height passing through the point C are identical.*
Corollary. If ABC is such a light triangle in a certain pseudoplane, then its median (a height) and its base (a hypotenuse) are the time-arrow $\overrightarrow{ct}^{(k)}$ and the space axis $x^{(k)}$.

Theorem 2. *In the cone obtained with any elliptic cut of a light cone in $\langle\mathcal{P}^{3+1}\rangle$, the median passing through its apex C and 2- or 3-dimensional base are hyperbolically orthogonal to each other, hence its height and median passing through the point C are identical.*

The theorems with the Einsteinian as if only physical definition of simultaneity, motivate the pseudo-Euclidean *quadratic* metric in his version of STR! Of course, simultaneity of events as world points fixation is a relative notion. It is defined with respect to a certain Euclidean space $\langle\mathcal{E}^3\rangle^{(i)}$ and a certain time-arrow $\overrightarrow{ct}^{(j)}$ in $\langle\mathcal{P}^{3+1}\rangle$. This is illustrated clearly at Figure 1A(2). Here a rod as a geometric object is immovable on the axis x_2 (i. e., $j = 2$), and it is moving physically along the axis x_1 (i. e., $i = 1$) at velocity $\pm v$ ($\tanh\ \gamma = ||v||/c$). The world lines of this rod's points are parallel to the time-arrow $\overrightarrow{ct}^{(j)}$. That is why, Observer N_i fixes the rod's points on its axis x_1 as their oblique projections parallel to time-arrow $\overrightarrow{ct}^{(j)}$. From the mathematical point of view, this improper fixation is *a cross projection* onto x_1 parallel to $\overrightarrow{ct}^{(j)}$ – see first definition of cross projections in sect. 5.10. Here we have the hyperbolic type deformation. Due to this, the moving rod contraction seems to Observer N_i.

In general, an improper world fixation, with respect to a certain pseudo-Cartesian base $\tilde{E}_\imath$, is defined as a graphically *simultaneous* cut of a geometric object world trajectory parallel to $\langle\mathcal{E}^3\rangle^{(\imath)}$ at a certain moment of time $t^{(\imath)}$. If the object is physically immovable in $\langle\mathcal{E}^3\rangle^{(\jmath)}$, then its world trajectory in $\langle\mathcal{P}^{3+1}\rangle$ is parallel to time-arrow $\overrightarrow{ct}^{(\jmath)}$. Hence definition of an object's world fixation in $\tilde{E}_\imath$ is reduced to its projecting into $\langle\mathcal{E}^3\rangle^{(\imath)}$ parallel to $\overrightarrow{ct}^{(\jmath)}$, i. e., to a space-like projection in the cross base $\tilde{E}_{\imath,\jmath} \equiv \{x_k^{(\imath)}, \overrightarrow{ct}^{(\jmath)}\}$ (sect. 5.10). Single cross projecting is expressed trigonometrically as the hyperbolic deformation in the pseudoplane of rotation. The pseudoplane at cross projecting has some properties of a quasi-Euclidean plane, but only in the universal base, usually in initial $\tilde{E}_\imath$, as then the *cross quasi-Euclidean invariant* under trigonometric deformations is valid in this pseudoplane – sect. 5.10 and 12.2. For a geometric object, the volume of its fixation is maximal iff the fixation is proper:

$$V = v^{(\imath,\jmath)}/\text{sech}\ \gamma = \max\langle v^{(\imath,\jmath)}\rangle = const. \tag{40A}$$

If a k-dimensional ($k \le n$) geometric object is moving rectilinearly and uniformly, then exactly four variants of its world trajectory are possible:
1) a **line** if $k = 0$, the object is a particle as a world point;
2) a **band** if $k = 1$, the object is a rod as a directed segment (a vector);
3) a **3-dimensional band** if $k = 2$, the object is a triangle or a parallelogram;
4) a **4-dimensional band** if $k = 3$, the object is a tetrahedron or a parallelepiped.
We consider only simplest objects, they are represented by $4 \times k$-*lineors*, see sect. 5.1.

The set of all world fixations for a given object is, from geometrical point of view, equivalent to the set of all space-like cuts of its *world trajectory*. So, relatively immovable Observer N_1 fixes a rod simultaneously as its projection into $\langle\mathcal{E}^3\rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(2)}$ (see Figure 1A). A world fixation, as well as a world trajectory, is a tensor notion, their valency is 1. World fixations of objects pointed out above are expressed as either 4×1-vectors, or 4×2-lineors, or 4×3-lineors. If an object is immovable in $\langle\mathcal{E}^3\rangle^{(\jmath)}$, then its proper world fixation is defined with respect to $\tilde{E}_\jmath$.

In the base $\tilde{E}_\jmath$, these one-, two-, and three-dimensional immovable geometric objects reduced to a current world point (the barycenter of a material body) are expressed initially as the following space-like $4 \times k$-*lineors* in the Minkowskian linear space-time:

$$\mathbf{a}^{(\jmath)} = \begin{bmatrix} \Delta x_1^{(\jmath)} \\ \Delta x_2^{(\jmath)} \\ \Delta x_3^{(\jmath)} \\ 0 \end{bmatrix}; \quad A_{4\times 2}^{(\jmath)} = \begin{bmatrix} \Delta x_{11}^{(\jmath)} & \Delta x_{12}^{(\jmath)} \\ \Delta x_{21}^{(\jmath)} & \Delta x_{22}^{(\jmath)} \\ \Delta x_{31}^{(\jmath)} & \Delta x_{32}^{(\jmath)} \\ 0 & 0 \end{bmatrix}; \quad A_{4\times 3}^{(\jmath)} = \begin{bmatrix} \Delta x_{11}^{(\jmath)} & \Delta x_{12}^{(\jmath)} & \Delta x_{13}^{(\jmath)} \\ \Delta x_{21}^{(\jmath)} & \Delta x_{22}^{(\jmath)} & \Delta x_{23}^{(\jmath)} \\ \Delta x_{31}^{(\jmath)} & \Delta x_{32}^{(\jmath)} & \Delta x_{33}^{(\jmath)} \\ 0 & 0 & 0 \end{bmatrix}. \tag{41A}$$

With respect to the cross base $\tilde{E}_{\jmath,\imath}$, we take out only Euclidean images in $\langle\mathcal{E}^3\rangle^{(\jmath)}$ of the lineors as their proper fixations, because they are immovable with respect to $\tilde{E}_\jmath$:

$$\mathbf{a}^{(\jmath,\imath)} = \mathbf{a}^{(\jmath)}; \quad A_{4\times 2}^{(\jmath,\imath)} = A_{4\times 2}^{(\jmath)}; \quad A_{4\times 3}^{(\jmath,\imath)} = A_{4\times 3}^{(\jmath)}. \tag{42A}$$

If the coordinates of these tensors are subjected to deformational transformation $defh\,\Gamma_{\imath\jmath}$ from $\tilde{E}_{\jmath,\imath}$ into another cross base $\tilde{E}_{\imath,\jmath}$ (see below), then *hyperbolic (tangent-secant) one-time pseudo-Euclidean quasi-invariant* from sect. 12.2 holds (for the one-time transformation). This quasi-invariant is expressed as follows:

$$[\mathbf{a}^{(\jmath)}]' \cdot \mathbf{a}^{(\jmath)} = [\mathbf{a}^{(\imath,\jmath)}]' \cdot \mathbf{a}^{(\imath,\jmath)} = ||\mathbf{a}||_E^2 = l_0^2 = \text{const} > 0, \tag{43A}$$

$$[A^{(\jmath)}]' \cdot A^{(\jmath)} = [A^{(\imath,\jmath)}]' \cdot A^{(\imath,\jmath)} = |A|^2 = \text{Const}, \tag{44A}$$

where $|A|$ is the $k \times k$-matrix Euclidean module of the $4 \times k$-lineor A (sect. 9.4).

This one-step quasi-invariant is similar to Euclidean invariant due to *spherical-hyperbolic analogy* (341) with respect to the base $\tilde{E}_i$ for Observer N_i fixed the Lorentzian contraction:

$$\boxed{\tilde{E}_j = roth\ \Gamma_{ij}\tilde{E}_i \rightarrow \tilde{E}_{i,j} = defh\ \Gamma_{ij}\cdot\tilde{E}_{j,i},\ defh\ \Gamma_{ij} \equiv rot\ \Phi(\Gamma_{ij}) \equiv defh^{-1}\Gamma_{ji}.} \quad (45A)$$

Express with the passive modal transformation the new coordinates of lineors (41A) with initial equalities (42A) in terms of both the modal matrices:

$$\mathbf{a}^{(i,j)} = defh\ \Gamma_{ij}\cdot\mathbf{a}^{(j)} = rot\ \Phi_{ij}\cdot\mathbf{a}^{(j)} = \begin{bmatrix} \Delta x_1^{(i,j)} \\ \Delta x_2^{(i,j)} \\ \Delta x_3^{(i,j)} \\ \Delta ct^{(j,i)} \end{bmatrix}, \quad (46A)$$

$$A_{4\times 2}^{(i,j)} = defh\ \Gamma_{ij}\cdot A_{4\times 2}^{(j)} = rot\ \Phi_{ij}\cdot A_{4\times 2}^{(j)} = \begin{bmatrix} \Delta x_{11}^{(i,j)} & \Delta x_{12}^{(i,j)} \\ \Delta x_{21}^{(i,j)} & \Delta x_{22}^{(i,j)} \\ \Delta x_{31}^{(i,j)} & \Delta x_{32}^{(i,j)} \\ \Delta ct_1^{(j,i)} & \Delta ct_2^{(j,i)} \end{bmatrix}, \quad (47A)$$

$$A_{4\times 3}^{(i,j)} = defh\ \Gamma_{ij}\cdot A_{4\times 3}^{(j)} = rot\ \Phi_{ij}\cdot A_{4\times 3}^{(j)} = \begin{bmatrix} \Delta x_{11}^{(i,j)} & \Delta x_{12}^{(i,j)} & \Delta x_{13}^{(i,j)} \\ \Delta x_{21}^{(i,j)} & \Delta x_{22}^{(i,j)} & \Delta x_{23}^{(i,j)} \\ \Delta x_{31}^{(i,j)} & \Delta x_{32}^{(i,j)} & \Delta x_{33}^{(i,j)} \\ \Delta ct_1^{(j,i)} & \Delta ct_1 2^{(j,i)} & \Delta ct_3^{(j,i)} \end{bmatrix}. \quad (48A)$$

Thus we have two equivalent trigonometric definitions of a general world fixation with one-time cross projecting, and respectively two kinds of the modal matrices in relation (45A): hyperbolic deformational one and spherical rotational one. In the spherical rotational variant, the angle $\mathbf{\Gamma}$ should be transformed into the angle-analog $\mathbf{\Phi}(\mathbf{\Gamma})$ by this analogy. The second variant is used for visual graphical interpretation of the Lorentz contraction. We choose mainly the first variant with angle $\mathbf{\Gamma}_{ij}$ connected simply with velocity $\mathbf{v}$. For example, express by passive modal transformation (46A) the new coordinates of the rod in terms of original ones from (41A), (42A) with the use of canonical structure (364) for the hyperbolic deformational modal matrix:

$$\mathbf{a}^{(i,j)} = \begin{bmatrix} \Delta x_1^{(i)} \\ \Delta x_2^{(i)} \\ \Delta x_3^{(i)} \\ \Delta ct^{(j)} \end{bmatrix} = \begin{bmatrix} \Delta x_1^{(j)} - \cos\alpha_1\cdot\cos\varepsilon\cdot l_0\cdot(1-\operatorname{sech}\,\gamma) \\ \Delta x_2^{(j)} - \cos\alpha_2\cdot\cos\varepsilon\cdot l_0\cdot(1-\operatorname{sech}\,\gamma) \\ \Delta x_3^{(j)} - \cos\alpha_3\cdot\cos\varepsilon\cdot l_0\cdot(1-\operatorname{sech}\,\gamma) \\ \cos\varepsilon\cdot l_0\cdot\tanh\gamma \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{e}_\alpha\cdot[1-\cos\varepsilon\cdot(1-\operatorname{sech}\,\gamma)]\cdot l_0 \\ \cos\varepsilon\cdot\tanh\gamma\cdot l_0 \end{bmatrix}, \quad (49A)$$

where in the rod fixation, the first three rows determine its new Cartesian coordinates in the base $\tilde{E}_i$, the fourth row determines its non-zero time-like projection onto $\overrightarrow{ct}^{(j)}$ as the additional *time-like effect* (explanation in details will be lower);
$l_0 = ||\mathbf{a}^{(j)}||$ is the Euclidean length of the rod in its rest state in the subbase $\tilde{E}_j^{(3)}$,
ε is the angle in $\tilde{E}_j^{(3)}$ between the rod and the **antivelocity vector** $\mathbf{v}_{ji} = (-\mathbf{e}_\alpha\cdot v_{ij})^{(j)}$ with the same unity vector of the directional cosines (formally these cosines are equal to ones of $\mathbf{v}_{ij}$, but expressed in the base $\tilde{E}_i^{(3)}$). And there holds

$$\cos\alpha_1\cdot\Delta x_1^{(j)} + \cos\alpha_2\cdot\Delta x_2^{(j)} + \cos\alpha_3\cdot\Delta x_3^{(j)} = \mathbf{e}'_\alpha\cdot\mathbf{a}^{(j)} = \cos\varepsilon\cdot l_0 = ||\overleftarrow{\mathbf{v}\mathbf{v}'}\cdot\mathbf{a}^{(j)}||. \quad (50A)$$

Note one more relativistic effect: *the hyperbolic angle between the velocity and antivelocity is non-zero and equal to* γ_{ij}. If the velocity and the axis x_1 are parallel, then $\cos\alpha_1 = 1 = \cos\varepsilon$, $\cos\alpha_2 = \cos\alpha_3 = 0$, and the new rod coordinates are

$$\mathbf{a}^{(i,j)} = \begin{bmatrix} \Delta x_1^{(i)} \\ \Delta x_2^{(i)} \\ \Delta x_3^{(i)} \\ \Delta ct^{(j)} \end{bmatrix} = \begin{bmatrix} 0 + \text{sech}\,\gamma \cdot \Delta x_1^{(j)} \\ \Delta x_2^{(j)} \qquad + 0 \\ \Delta x_3^{(j)} \qquad + 0 \\ 0 + \tanh\gamma \cdot \Delta x_1^{(j)} \end{bmatrix}, \quad (\Delta x_1^{(j)} = \cos\varepsilon \cdot l_0 = l_0). \tag{51A}$$

Here the non-relativistic and relativistic parts are pointed out as the summands from the left and from the right respectively. More generally, if in (49A) also the rod and the velocity are formally coaxial ($\cos\varepsilon = 1$) in $\tilde{E}_j^{(3)}$, then there holds

$$\mathbf{a}^{(i,j)} = \begin{bmatrix} \mathbf{e}_\alpha \cdot \ \text{sech}\ \gamma \cdot l_0 \\ \tanh\gamma \cdot l_0 \end{bmatrix}. \tag{52A}$$

The Cartesian coordinates in (51A, 52A) express the relativistic effect of so-called *Lorentzian contraction of extent* [58, 59; 76, p. 109], which realizes coaxially to velocity:

$$l^{(i,j)} = l^{(i)} = \text{sech}\ \gamma_{ij} \cdot l_0 = \sqrt{1 - (v/c)^2} \cdot l_0 < l_0. \tag{53A}$$

Other coordinates are normal to the velocity, they do not change. The original and new *four* coordinates of the rod in (49A) and special cases (51A) satisfy (43A), i. e., they form quasi-Euclidean invariant, this follows from (45A). The sum of all *three space* coordinates squares is the squared Euclidean length module of the rod contracted. In this most general case, for the Lorentzian contracted oriented rod, there holds:

$$l^{(i,j)} = l^{(i)} = ||\Delta\mathbf{x}^{(i)}|| = l_0\sqrt{\cos^2\varepsilon \cdot \text{sech}^2\gamma_{ij} + \sin^2\varepsilon} =$$

$$= l_0\sqrt{1 - \cos^2\varepsilon \cdot \tanh^2\gamma_{ij}} = l_0\sqrt{1 - \cos^2\varepsilon \cdot (v/c)^2} < l_0. \tag{54A}$$

Apply the Herglotz Principle and evaluate its relativistic and non-relativistic summands. The non-relativistic part (that is normal to the velocity vector) is the Euclidean invariant:

$$\mathbf{a}_{inv}^{(i,j)} = \mathbf{a}^{(j)} - \cos\varepsilon \cdot l_0 \cdot \begin{bmatrix} \mathbf{e}_\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \Delta\mathbf{x}^{(j)} - \mathbf{e}_\alpha \cos\varepsilon \cdot l_0 \\ 0 \end{bmatrix}. \tag{55A}$$

Subtracting (49A) and (55A) gives the relativistic part:

$$\mathbf{a}_{rel}^{(i,j)} = \begin{bmatrix} \mathbf{e}_\alpha \cdot \cos\varepsilon \cdot \ \text{sech}\ \gamma \cdot l_0 \\ \cos\varepsilon \cdot \tanh\gamma \cdot l_0 \end{bmatrix} = \begin{bmatrix} \Delta\mathbf{x}_{rel}^{(i)} \\ \Delta ct^{(j)} \end{bmatrix}. \tag{56A}$$

Apply the Pythagorean Theorem to its Cartesian part and obtain the relativistic part $|\cos\varepsilon \cdot \text{sech}\ \gamma \cdot l_0|$ for the Euclidean length of a moving rod. From (55A) and (50A) the non-relativistic part $|\sin\varepsilon \cdot l_0|$ is evaluated too. This is the algebraic way for explaining structure of (54A), another way is graphical. The Euclidean length of a moving rod is, due to (54A), the orthogonal sum in $\langle\mathcal{E}^3\rangle^{(i)}$ of non-relativistic projection $\sin\varepsilon \cdot l_0$ and relativistic projection $\cos\varepsilon \cdot \text{sech}\ \gamma \cdot l_0$. *The first summand* is normal projection of the rod relatively to the antivelocity $\mathbf{v}_{ji}$. It is invariant under hyperbolic deformation. That is why, this part of the rod fixation is spherically orthogonal to both vectors $\mathbf{v}_{ij}$ in $\langle\mathcal{E}^3\rangle^{(i)}$ and $\mathbf{v}_{ji}$ in $\langle\mathcal{E}^3\rangle^{(j)}$. The second *relativistic summand* is obtained from parallel projection of the rod with its cross projecting into $\langle\mathcal{E}^3\rangle^{(i)}$ parallel to $\overrightarrow{ct}^{(j)}$ under condition in (52A) onto velocity $\mathbf{v}_{ij}$.

Squared Euclidean lengths of relativistic fixations (52A) and (56A) for the rod, due to (43A) and (45A), are hyperbolic quasi invariants under one-step hyperbolic deformation. They are space-like hyperbolic quadratic *one-step cross invariants as if Euclidean ones*:

$$[l^{(\jmath)}]^2 = ||\Delta \mathbf{x}^{(\imath,\jmath)}||^2 + \Delta^2 ct^{(\jmath,\imath)} = [l^{(\imath,\jmath)}]^2 + \Delta^2 ct^{(\jmath,\imath)} = l_0^2 = \text{const}, \qquad (57A)$$

$$[l^{(\jmath)}]^2_{rel} = ||\Delta \mathbf{x}^{(\imath,\jmath)}||^2_{rel} + \Delta^2 ct^{(\jmath,\imath)} = [l^{(\imath,\jmath)}]^2_{rel} + \Delta^2 ct^{(\jmath,\imath)} = l_0^2 \cos^2 \varepsilon = \text{const}. \qquad (58A)$$

The trigonometric secant-tangent form of invariant (58A) is

$$(\text{sech}^2\gamma_1'' + \text{sech}^2\gamma_2'' + \text{sech}^2\gamma_3'') + \tanh^2\gamma = ||\mathbf{sech}^2\gamma|| + \tanh^2\gamma = 1, \qquad (59A)$$

where γ_k'' is the hyperbolic angle between vector $-\mathbf{v}_{\jmath\imath}$ in the subbase $\tilde{E}_\jmath^{(3)}$ and the axis x_k in the subbase $\tilde{E}_\imath^{(3)}$ and sech $\gamma_k'' = \cos\alpha_k \cdot \text{sech } \gamma$. This is an invariant for a unit space-like linear element. The proper length of a rod (in the rest state) is a quasi-Euclidean metric invariant in all other cross bases $\tilde{E}_{k\jmath}$, in particular, in $\tilde{E}_{\imath\jmath}$:

$$l_0 = \frac{l^{(\imath,\jmath)}}{\sqrt{1 - \cos^2\varepsilon \cdot \tanh^2\gamma_{\imath\jmath}}} = \max\langle l^{(\imath,\jmath)}\rangle. \qquad (60A)$$

This follows from (54A). The Lorentzian seeming contraction as the relativistic effect has coordinate nature, i. e., it does not lead to any mechanical stretch. Formally, contraction of moving objects of type (53A) was first established by G. FitzGerald in 1889 [89] in frame of interpretation of the Michelson-Morley experiment – see above, and later by H. Lorentz in 1895 [58] in frame of interpretation of the Maxwell electromagnetic wave equation.

The set of all world fixations of a moving rod is semiopen, as it does not contain extremal cuts of its world trajectory by the hypersurface of the light cone, see Figure 1A. These extremal cuts for a rod have zero Euclidean length of the relativistic space cross projection, ones for objects of rank greater than 1 have zero Euclidean norms of order 1 and 2 for their relativistic space cross projection and order 3 for their space volume fixation. These cuts correspond to objects as if moving at the velocity c.

Furthermore, this rod, in addition, has the time-like projection in the same cross base $\tilde{E}_{\imath\jmath}$, this follows from (56A). Projecting is performed into the time-arrow $\overrightarrow{ct}^{(\jmath)}$, thus it is expressed in the base $\tilde{E}_\jmath$. This effect has the following relativistic explanation. Observer $N_\jmath$ can see the analogous rod as immovable on the axis $x_1^{(\imath)}$ and moving at the same velocity $\mathbf{v}_{\jmath\imath}$ in $\tilde{E}_\jmath$, with seeming Euclidean length (54A). In the general case, when the two identical rods meet, their two left ends and two right ends considered separately meet, according to (56A), with the following time lag:

$$\Delta ct^{(\imath,\jmath)} = \Delta ct^{(\jmath)} = \cos\ \varepsilon \cdot l_0 \cdot \tanh\gamma_{\imath\jmath} \neq 0. \qquad (61A)$$

It is the *relativistic effect of non-synchronous meeting of two identical immovable and moving coaxial rods paired points.* Contact of the points pairs of meeting rods (if $\varepsilon = 0$) is spreading at the left to the right along the axis $x_1^{(\jmath)}$ at *supervelocity* w greater than c:

$$s = l_0/\Delta t^{(\jmath)} = c/\tanh\gamma_{\imath\jmath} = c\cdot\coth\gamma_{\imath\jmath} = c\cdot\cosh\upsilon_{\imath\jmath} = c^2/v > c. \qquad (62A)$$

(See connections of these complementary hyperbolic angles γ and υ in (360), sect. 6.4.) During this accelerated movement the *coordinate supervelocity* decreases from ∞ to c (for the angle γ) and increases from c to ∞ (for the complementary angle υ). However, in the classic mechanics, the pairs of points meet simultaneously.

Note, that the full set $\langle w \cdot \mathbf{e}_\alpha \rangle$ forms the hyperbolic cotangent vector space that is the cotangent models outside the trigonometric circle or ball of radius $\mathbf{1}$ (the unity Cayley's oval) or $\boldsymbol{c}$ for supervelocity, where the motion angles γ is on a hyperboloid I (see Ch. 12, 6A, 7A).

In products (46A)–(48A) the hyperbolic deformational matrix is formally truncated, only three first rows are used (compare with rotational matrices in products (32A), (37A) in Ch. 3A, because the original objects (lineors) in forms (41A) are parallel to their proper Euclidean space $\langle \mathcal{E}^3 \rangle^{(\jmath)}$.

In the common pseudoplane of the hyperbolic rotation $\boldsymbol{roth}\,\Gamma_{\imath\jmath}$ in the base $\tilde{E}_\imath$ and the hyperbolic deformation $\boldsymbol{defh}\,\Gamma_{\imath\jmath}$ in the cross base $\tilde{E}_{\imath\jmath}$ at Figure 1A, the problem is reduced to solving the *exterior right triangles*: either *pseudo-Euclidean* one ABC (sect. 6.4), where $l^{(\imath)}$ is similar to hypotenuse $AB = g$ and l_0, $\Delta ct^{(\jmath)}$ are similar to legs a, b; or *quasi-Euclidean* one $A'B'D'$ (Figure 1A(2)), where $a = A'D'$ is similar to hypotenuse as l_0, $g = A'B'$ is similar to leg $l^{(\imath)}$ as contracted rod length, $b = B'D' = \Delta ct^{(\jmath)}$; $i = 1$, $j = 2$). Then Lorentzian contraction is expressed formally in the *quasiplane* by the spherical rotation $\boldsymbol{rot}\,\Phi(\Gamma_{\imath\jmath})$ in (45A) in the universal base $\tilde{E}_\imath$, and hyperbolic cross projections are determined due to the Pythagorean theorem.

In a cross base $\tilde{E}_{\imath\jmath}$, for two vectors (rods) applied in one world point M, there holds

$$\cos\beta_{12}^{(\imath,\jmath)} = [\mathbf{a}_1^{(\imath,\jmath)}]' \cdot \mathbf{a}_2^{(\imath,\jmath)} / ||\mathbf{a}_1^{(\imath,\jmath)}|| \cdot ||\mathbf{a}_2^{(\imath,\jmath)}|| = [\mathbf{e}_1^{(\imath,\jmath)}]' \cdot \mathbf{e}_2^{(\imath,\jmath)}, \quad (\beta_{12} \in [0;\pi]).$$

Here the algebraic formula for the cosine of the angle between two vectorial fixations in $\langle \mathcal{E}^3 \rangle^{(\imath)}$ is given. Apply (54A) to this expression. The result is the trigonometric formula for the cosine of the angle between two moving vectors (rods) applied in one point M:

$$-1 \le \cos\beta_{12}^{(\imath)} = \frac{\cos\beta_{12}^{(\jmath)} - \cos\varepsilon_1 \cdot \cos\varepsilon_2 \cdot \tanh^2\gamma}{\sqrt{1-\cos^2\varepsilon_1 \cdot \tanh^2\gamma} \cdot \sqrt{1-\cos^2\varepsilon_2 \cdot \tanh^2\gamma}} \le +1, \qquad (63A)$$

where $\beta_{12}^{(\jmath)}$ and $\beta_{12}^{(\imath)}$ are the scalar angle between the vectors measured by Observers $N_\jmath$ and $N_\imath$. Two the initial vectors with the antivelocity vector form a triple in $\langle \mathcal{E}^3 \rangle^{(\jmath)}$.

According to the Hadamard Inequality (see in Ch. 3), for their unity vectors Gram determinant, there holds

$$\boxed{0 \le \det\{[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]' \cdot [\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]\} = s_{123}^2 \le 1.}$$

And from here the triple trigonometric inequality follows:

$$2\cos\alpha_{12}\cdot\cos\alpha_{13}\cdot\cos\alpha_{23} \le \cos^2\alpha_{12}+\cos^2\alpha_{13}+\cos^2\alpha_{23} \le 1+2\cos\alpha_{12}\cdot\cos\alpha_{13}\cdot\cos\alpha_{23}.$$

In our case, we have $\alpha_{13} = \varepsilon_1$, $\alpha_{23} = \varepsilon_2$, $\alpha_{12} = \beta_{12}$. These inequalities and condition $\tanh^2\gamma < 1$ infer (63A) *as inequality* too.

If the initial angle between the vectors is $\beta_{12}^{(\jmath)} = \pi/2 \to \cos\beta_{12}^{(\jmath)} = 0$, then the new angle $\beta_{12}^{(\imath,\jmath)}$ is either acute ($\cos\varepsilon_1 \cdot \cos\varepsilon_2 < 0$), or obtuse ($\cos\varepsilon_1 \cdot \cos\varepsilon_2 > 0$), or zero ($\cos\varepsilon_1 \cdot \cos\varepsilon_2 = 0$).

If $\beta_{12}^{(\jmath)} = 0$, then $\varepsilon_1 = \varepsilon_2$ and $\beta_{12}^{(\imath,\jmath)} = 0$.

If both the vectors (and the angle between them) are orthogonal to the antivelocity vector, then the relativistic effect of the angle changing does not take place; namely we have: $\cos\varepsilon_1 = \cos\varepsilon_2 = 0 \;\to\; \beta_{12}^{(\imath,\jmath)} = \beta_{12}^{(\jmath)}$.

If one of these two vectors is collinear to the antivelocity vector, then $|\cos\beta_{12}|$ decreases, and the acute angle increases, the obtuse angle decreases ($\varepsilon_1 = 0 \to \beta_{12}^{(\jmath)} = \varepsilon_2$):

$$0 < \cos\beta_{12}^{(\imath)} = \cos\beta_{12}^{(\jmath)} \cdot \sqrt{\frac{1-\tan^2\gamma}{1-\cos^2\varepsilon_2\cdot\tanh^2\gamma}} < \cos\beta_{12}^{(\jmath)}. \qquad (64A)$$

Relativistic area of the parallelogram on two the vectors is

$$S_{12}^{(i,j)} = l_1^{(i,j)} \cdot l_2^{(i,j)} \cdot \sin\beta_{12}^{(i,j)} =$$

$$= \frac{S_{12}^{(j)}}{\sin\beta_{12}^{(j)}} \cdot \sqrt{\sin^2\beta_{12}^{(j)} - (\cos^2\varepsilon_1 + \cos^2\varepsilon_2 - 2\cos\beta_{12}^{(j)} \cdot \cos\varepsilon_1 \cdot \cos\varepsilon_2) \cdot \tanh^2\gamma}. \qquad (65A)$$

The diagonals of the moving parallelogram are subjected to Lorentzian contraction unless they are orthogonal to the velocity. In general, for the length of the diagonals, there holds:

$$[L^{(i,j)}]^2_{1,2} = [L^{(j)}]^2_{1,2} - [l_1^{(j)} \cdot \cos\varepsilon_1 \pm l_2^{(j)} \cdot \cos\varepsilon_2]^2 \cdot \tanh^2\gamma. \qquad (66A)$$

The volume of a parallelepiped (as well as of other body) decreases proportionally to the secant of the hyperbolic angle γ of motion - see in (40A). With the use of (54A), (40A) and the Hadamard Inequality the sine norm of a moving 3-dimensional lineor angle is evaluated:

$$s_{123}^{(i,j)} = \frac{s_{123}^{(j)} \cdot \operatorname{sech}\gamma}{\sqrt{1-\cos^2\varepsilon_1 \cdot \tanh^2\gamma} \cdot \sqrt{1-\cos^2\varepsilon_2 \cdot \tanh^2\gamma} \cdot \sqrt{1-\cos^2\varepsilon_3 \cdot \tanh^2\gamma}},$$

$$s_{123}^{(i,j)} \in (0;1). \qquad (67A)$$

Inequalities $0 < s_{123}^{(i,j)} < 1$ may be inferred by another way, with the use of formulae (63A) and the Hadamard Inequality, because we have:

$$[s_{123}^{(i,j)}]^2 = 1 + 2\cdot\cos\beta_{12}^{(i,j)} \cdot \cos\beta_{13}^{(i,j)} \cdot \cos\beta_{23}^{(i,j)} - \cos^2\beta_{12}^{(i,j)} - \cos^2\beta_{13}^{(i,j)} - \cos^2\beta_{23}^{(i,j)}.$$

The essential distinction in STR between the Lorentzian contraction of extent and the Minkowskian dilation of time consists in the following. For polysteps motions, the latter may be always expressed through multiplication of rotational matrices of all particular motions with evaluating its summarized motive tensor angle after polar decomposition (see in Chs. 5A and 7A). However, Lorentzian contraction, for polysteps motions, is not expressed similarly through multiplication of all particular deformational matrices, because their hyperbolic tensor angles are not summable. But it may be expressed through deformational matrix-function of the final motive tensor angle in the rotational matrix-function obtained after multiplication of particular rotational matrices and following polar decomposition of a result.

Moreover, due to (45A), the geometric result of one-step Lorentzian contraction is visually similar to massive object's spherical rotation at the angle $\Phi(\Gamma)$ with the following spherical cosine projecting. Also, from the point of view of our tensor trigonometry, the equivalent spherical matrices $defh\ \Gamma_{ij} \equiv rot\ \Phi(\Gamma_{ij})$ mathematically clear interpret relativistic effect as the "Terrell-Penrose visual rotation of moving objects" under Lorentzian contraction (in the base $\tilde{E}_1$ of an immovable Observer). For general 4D analogy, we obtain:

$$\{defh\ (\pm\Gamma)\}_{(n+1)\times(n+1)} \qquad\qquad \{rot(\pm\Phi)\}_{(n+1)\times(n+1)}. \qquad (68A)$$

$$\begin{array}{|c|c|}\hline I_{n\times n} + (\operatorname{sech}\gamma - 1)\cdot \mathbf{e}_\alpha \mathbf{e}'_\alpha & \mp\tanh\gamma\cdot\mathbf{e}_\alpha \\ \hline \pm\tanh\gamma\cdot\mathbf{e}'_\alpha & \operatorname{sech}\gamma \\ \hline\end{array} \equiv \begin{array}{|c|c|}\hline I_{n\times n} + (\cos\varphi - 1)\cdot \mathbf{e}_\alpha \mathbf{e}'_\alpha & \mp\sin\varphi\cdot\mathbf{e}_\alpha \\ \hline \pm\sin\varphi\cdot\mathbf{e}'_\alpha & \cos\varphi \\ \hline\end{array}.$$

$$(\mathbf{e}_\alpha \mathbf{e}'_\alpha = \overleftarrow{\mathbf{e}_\alpha \mathbf{e}'_\alpha})$$

We have an important peculiarity: the Lorentzian seeming contraction is a typical *artefact*, i. e., it is a really observational but seeming to N_1 space-like phenomenon evaluated in a certain universal base $\tilde{E}_1$ (in contrast to the mutual dilation of the space coordinate together with the time coordinate as a result of the Lorentz transformations – see in sect. 12.3). When the object returns to the rest state, its geometric sizes and angles are preserved. Any internal mechanical stretches in an object, according only to inertial movements, are impossible!

Chapter 5A

Trigonometric models of two-steps, polysteps, and integral collinear motions in STR and two hyperbolic geometries

Consider in details trigonometric modeling of the various rectilinear relativistic physical movements. They are described mathematically by hyperbolic rotational matrix functions of tensor angles in their elementary form (Ch. 2A). In process of the rectilinear movement its changing tensor angle must preserve trigonometric compatibility. Due to **Rule 2** (sect. 5.7), *compatible rotational matrices commute, in their multiplications the tensor argument angles of motive type form an algebraic sum.* Hence, in this Chapter, we use *mainly* the scalar form for these motion angles and connected with them trigonometric functions and velocities. The latters may be subjected also to operations of integration (into some distances) and differentiation (into some accelerations), and what's more, these operations are realized inside of a certain pseudoplane of these compatible hyperbolic type motions! Some examples of similar relativistic physical movements for the following analysis are exposed at Figure 2A.

By this reason, the relativistic Poincaré–Einstein Law of two velocities summation [63], [67] as well as hyperbolic tangents summation for *collinear summands* has the following trigonometric interpretation as *compatible rotations in the hyperbolic angles* $\mathbf{\Gamma}_{jk}$:

$$\left.\begin{aligned} &roth\ \Gamma_{13} = roth\ \Gamma_{12}\cdot roth\ \Gamma_{23} = roth\ (\Gamma_{12}+\Gamma_{23}) \Rightarrow \\ &\Rightarrow \cos\alpha_{(13)}\cdot\gamma_{13} = \cos\alpha_{(12)}\cdot\gamma_{12} + \cos\alpha_{(23)}\cdot\gamma_{23}, \end{aligned}\right\}(\cos\alpha = \pm 1, \gamma > 0) \qquad (69A)$$

$$\left.\begin{aligned} &\cos\alpha_{(13)}\cdot\tanh\ \gamma_{13} = \tanh\left[\cos\alpha_{(12)}\cdot\gamma_{12} + \cos\alpha_{(23)}\cdot\gamma_{23}\right] = \\ &= \frac{\cos\alpha_{(12)}\cdot\tanh\gamma_{12} + \cos\alpha_{(23)}\cdot\tanh\gamma_{23}}{1+\cos\varepsilon\cdot\tanh\gamma_{12}\cdot\tanh\gamma_{23}} \Rightarrow \\ &\Rightarrow\ v_{13} = \frac{\cos\alpha_{(12)}\cdot v_{12} + \cos\alpha_{(23)}\cdot v_{23}}{1+\cos\varepsilon\cdot v_{12}v_{23}/c^2}, \end{aligned}\right\} \qquad (70A)$$

where $\cos\varepsilon = \cos\alpha_{(12)}\cdot\cos\alpha_{(23)}$.

Hyperbolic Sommerfeld's form of this Law was first derived by eminent physicist and mathematician Arnold Sommerfeld with geometric inferring as if on a *sphere of imaginary radius* ic [86, 76, p. 111], i. e., in fact on the Minkowskian hyperboloid II (see in sect. 12.1). This is based on the rule for summation through the tangents-functions of trigonometrically compatible hyperbolic angles. The relativistic law of summing *several collinear velocities* is expressed also in the simplest hyperbolic form:

$$\cos\alpha\cdot\gamma = \sum_{t=1}^{m}\cos\alpha_{(t)}\cdot\gamma_{(t)}, \quad (\cos\alpha = \pm 1, \gamma > 0) \qquad (71A)$$

$$v = c\cdot\cos\alpha\cdot\tanh\ \gamma = c\cdot\tanh\sum_{t=1}^{m}\cos\alpha_{(t)}\cdot\text{artanh}\ v_t/c. \qquad (72A)$$

The term "collinear" has here and further rather conventional character, it means merely that all these summarized particular velocities $\mathbf{v_t}$ are directed in their common 3-dimensional Euclidean vectorial space coaxially with the non-directed vector $\mathbf{e}_\alpha = \langle\cos\alpha_i\rangle = \mathbf{const}$, $(i = 1, 2, 3)$. Hence, the particular velocity $\mathbf{v_t}$ can have only one of two values of directed vector of directional cosines $\pm\mathbf{e}_\alpha$, i. e., in contrary directions. In (69A)–(72A), this condition corresponds to values $\cos\alpha = \pm 1$.

An integral collinear motion as a curve world line in $\langle\mathcal{P}^{3+1}\rangle$ is projected hyperbolically into some Euclidean sub-space $\langle\mathcal{E}^3\rangle^{(m)}$ as a rectilinear physical movement. More in details, such motion is realized in some only one pseudoplane $\langle\mathcal{P}^{1+1}\rangle$ with its specific directional vector $\mathbf{e}_\alpha$, but physically the motion is projected hyperbolically as a straight line into any its space axis, for example, $x^{(1)} = \chi$ in parallel to $\overrightarrow{ct}^{(1)}$. Hence, speaking strictly, "rectilinear movement" is a physical term, which has rather conventional character too in $\langle\mathcal{P}^{3+1}\rangle$. (In the Lagrangian space-time, a collinear motion is projected always into its Euclidean subspace as single one for all the bases in parallel to any $\overrightarrow{ct}$.)

Continuous summation of collinear motion angle differentials $d\gamma = d\gamma^{(m)}$ is accomplished with integrating either along instantaneous axis $x^{(m)}$ as differentials $d\gamma = dv^{(m)}/c$ of its inclination to the Euclidean sub-space $\langle\mathcal{E}^3\rangle^{(1)}$ or along instantaneous tangent to a world line as differentials $d\gamma$ of its inclination to the time-arrow $\overrightarrow{ct}^{(1)}$. Note, that these 1-st differentials $d\gamma$ and $dv^{(m)}$, as always, only are linear parts of curve increments $\Delta\gamma$ and $\Delta v^{(m)}$ (and in the current point M there holds: $v^{(m)} = 0$).

The space axis $x^{(1)}$ collinear with $\pm\mathbf{e}_\alpha$ and the time arrow $\overrightarrow{ct}^{(1)}$ determine the constant pseudoplane $\langle\mathcal{P}^{1+1}\rangle$ with this two-dimensional universal base. Such base $\tilde{E}_1$ corresponds to the rest state of inertial Observer N_1 of STR. In the base $\tilde{E}_1$, we have the specific spherical-hyperbolic analogy (26A) between hyperbolic and spherical motion angles for very important applications. Further, we shall describe two-steps, polystep and integral collinear motions mainly in the universal base $\tilde{E}_1$ – see at Figure 2A.

In the tensor trigonometric version of STR, the *principal hyperbolic angle of motion* γ has also relative nature as well as the time-arrow and the space. Here this angle is counted in the base $\tilde{E}_1$ off $\overrightarrow{ct}^{(1)}$ unless another condition is accepted. So, for a straight world line, the relative velocity between Observers N_1 and N_2 determines the hyperbolic tangent of the angle of motion γ_{ij} from two opposite points of view – Figure 2A(1):

$$\tanh\gamma_{12} = \frac{v_{12}}{c} = \frac{\Delta x^{(1)}}{\Delta(ct^{(1)})} = \frac{\Delta x^{(1)}/\cosh\gamma_{12}}{\Delta(ct^{(1)})/\cosh\gamma_{12}} = \frac{-\Delta x^{(2)}}{\Delta(ct^{(2)})} = -\tanh\gamma_{21}. \qquad (73A)$$

The same takes place on the level of differentials if a material object M is moving physically rectilinearly with acceleration or deceleration along Euclidean directions $\pm\mathbf{e}_\alpha$ with its instantaneous pseudo-Cartesian base $\tilde{E}_m$ (from the point of view of Observer N_1 in the initial universal base $\tilde{E}_1$). For each point M of its world line, the origin of $\tilde{E}_m$ is associated with the barycenter of the moving object M. We have

$$\tilde{E}_m = roth\,\Gamma\cdot\tilde{E}_1 = F_1(\gamma,\mathbf{e}_\alpha)\cdot\tilde{E}_1. \qquad (74A)$$

The slope of a world line $\tanh\gamma$ is determined by the *coordinate velocity* of movement, and this velocity may be expressed by two ways: from points of view of Observers N_1 and N_m:

$$\tanh\gamma = \frac{v}{c} = \frac{d\chi}{d(ct^{(1)})} = \frac{dx^{(1)}/\cosh\gamma}{d(ct^{(1)})/\cosh\gamma} = \frac{-dx^{(m)}}{d(c\tau)} = -\tanh\,(-\gamma). \qquad (75A)$$

This formula corresponds to the *Minkowski dilations of the space and time intervals in the moving system of reference* $\tilde{E}_m$ (Ch. 3A) with the equal "relativistic factor" as $\cosh^{-1}\gamma$. (This can be explained by the fact that both these dilations are caused by the hyperbolic projection from the coordinate system at rest $\tilde{E}_1$ into the moving coordinate system $\tilde{E}_m$ – see in sect. 12.3.) Indeed, Observer N_1 is in the relative $\langle\mathcal{E}^3\rangle^{(1}$, Observer N_m is in relative $\langle\mathcal{E}^3\rangle^{(m}$, and both have own coordinate parameters of the space and the time for velocity (with its equal module). (Further similar Greek notations $\chi = x^{(1)}$, $c\tau = ct^{(2)}$ stand for *proper coordinates*.) The proper time differential $dc\tau$ is also the differential of pseudo-Euclidean length of a world line arc, i. e., along any continuous world line (see in details in Ch. 3A).

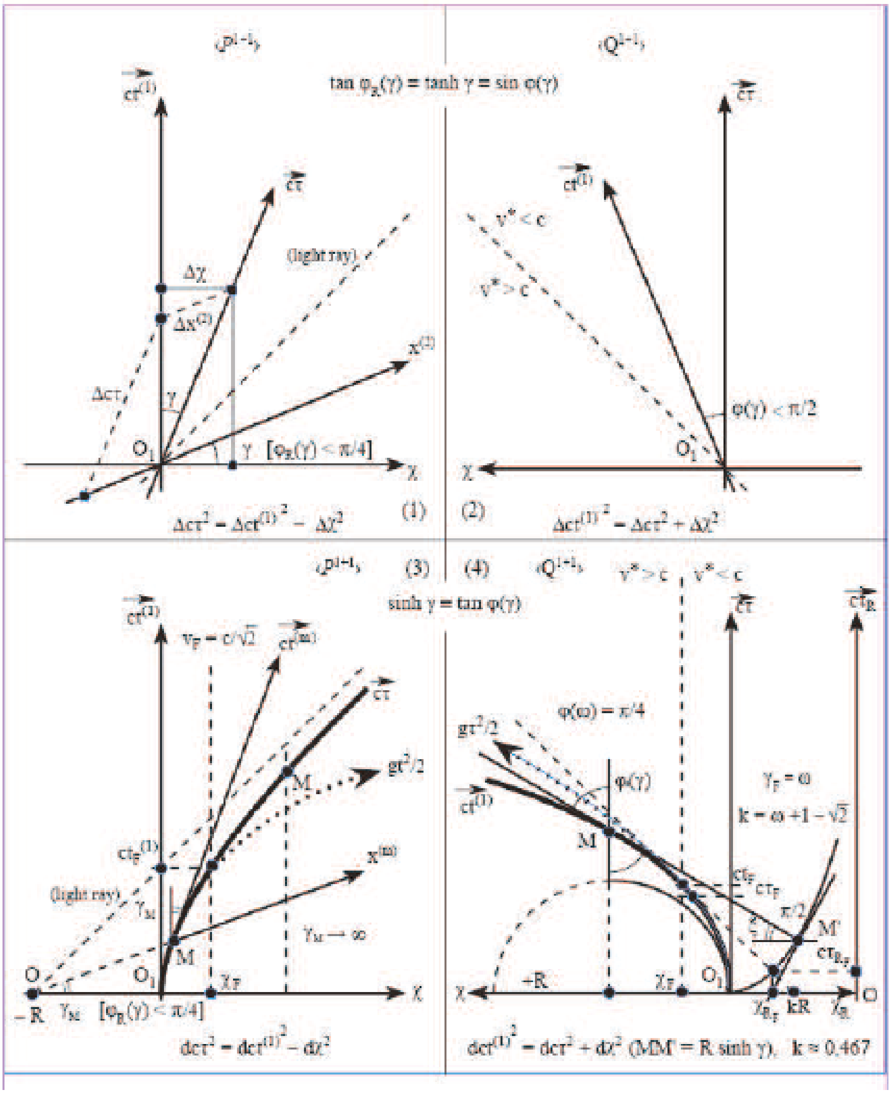

Figure 2A. The world lines of a material point M for simplest kinds of rectilinear relativistic physical movements, represented in universal proper and compressed bases:

(1), (2) – uniform rectilinear relativistic movement,

(3), (4) – uniformly accelerated rectilinear relativistic movement (hyperbolic motion).

Note (!), that at our Picture 2A(4), we combined and displayed [15, p. 223] the *Triade I* from three bonded geometric objects – the catenoid I with its generating time-like catenary, the tractricoid I (as Beltrami pseudosphere) with its generating Minding tractrix with the same radius-parameter R and revolving axis $\overrightarrow{c\tau_R}$ (or $\overrightarrow{y}$), and the adjacent torus with its generating circle also of radius R. This Triade was produced by us from the Minkowski pro-hyperboloid I with generating time-like hyperbola at Picture 2A(3). Its three objects are bonded by the same hyperbolic and spherical angles in result of using evolute-involute metric's transfer (Ch. 6A). The tractricoid I is one-step isometric to the hyperboloid I in the universal base $\tilde{E}_1$ of their enveloping binary spaces. With our tensor trigonometric approach, one may also produce and display the Triade II from the Minkowski pro-hyperboloid II.

For the moving object, its curvilinear world line is identical to its proper-time-arrow $\overrightarrow{\int_0^t d(ct^{(m)})} \equiv \overrightarrow{c\tau}(\gamma)$, see Figure 2A(3). A pseudo-normal and a tangent to a curvilinear world line at point M form instantaneous directed axes $x^{(m)}$ and $\overrightarrow{ct}^{(m)}$ of the base $\tilde{E}_m$.

In (73A), (75A), the relative velocity v_{12} in $\tilde{E}_1$ of Observer N_2 with respect to N_1 is evaluated with the use of its coordinate time $t^{(1)}$ and its proper distance $x^{(1)} = \chi$. Similarly, the relative velocity v_{21} in $\tilde{E}_2$ of Observers N_1 with respect to N_2 is evaluated with the use of its decreased proper time $t^{(2)}$ ($dt^{(2)} = \text{sech}\ \gamma_{21}\ dt^{(1)}$) and its moving coordinate distance $x^{(1)}$ ($dx^{(1)} = \text{sech}\ \gamma_{21}\ dx^{(2)}$) – the latter is formally analogous to Einstein's dilation of time. Hence, the notion v is, in fact, the *coordinate* velocity.

The *proper velocity* of physical movement (39A) is defined with the use of also *proper coordinates*, i. e., proper time $d(c\tau)$ in a moving Euclidean subspace in $\tilde{E}_m$ and immovable proper distance $d\chi = dx^{(1)}$ in $\tilde{E}_1$. It is expressed by the hyperbolic sine:

$$v^* = \frac{dx^{(1)}}{d(t^{(m)})} = \frac{d\chi}{d\tau} = c \cdot \cosh\gamma \cdot \tanh\gamma = c \cdot \sinh\gamma > v. \tag{76A}$$

In the following, we use an asterisk in notation of proper characteristics! The proper velocity of a light ray is infinite, because $d(c\tau) = 0$. Hence, the relativistic law of proper velocities summation for collinear summands has the following hyperbolic sine interpretation, though hyperbolic angles are summed as before, see in (70A):

$$\left.\begin{array}{l} v^*_{13} = c \cdot \sinh[\cos\alpha_{(13)} \cdot \gamma_{13}] = c \cdot \sinh[\cos\alpha_{(12)} \cdot \gamma_{12} + \cos\alpha_{(23)} \cdot \gamma_{23}] = \\ = c \cdot [\cos\alpha_{(12)} \cdot \sinh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\alpha_{(23)} \cdot \sinh\gamma_{23} \cdot \cosh\gamma_{12}], \Rightarrow \\ \Rightarrow v^*_{13} = v^*_{12} \cdot \sqrt{1 + (v^*_{23}/c)^2} + v^*_{23} \cdot \sqrt{1 + (v^*_{12}/c)^2}, \\ v_{12}v_{23} > 0 \ \leftrightarrow \ |v^*_{13}| > |v^*_{12} + v^*_{23}|. \end{array}\right\} \tag{77A}$$

Thus, there holds: $v^* = v/\sqrt{1-(v/c)^2} \ \rightarrow \ 1/c^2 = 1/v^2 - 1/(v^*)^2$. The latter is equivalent to the trigonometric identity: $1 = \coth^2\gamma - \text{csch}^2\gamma$. It is an invariant of cotangent-cosecant rotational matrix, for example, of (361) from Γ, in particular, in the the right triangle of supervelocities (so, see in Chs. 6, 6A)! The directed cosines of vectors $\mathbf{v}^*$ and $\mathbf{sin}\ \gamma$ are equal to those of $\mathbf{v}$ and $\mathbf{tan}\ \gamma$, as they are obtained from the same differential $d\mathbf{x}$ in the numerator of their derivatives.

Let the frame of reference with Observer N_m moves also rectilinearly, but non-uniformly. Then N_m has the instantaneous coordinate velocity with respect to N_1 as

$$v^{(m)}_{21} = \frac{dx^{(m)}}{d\tau} = \frac{-dx^{(1)}/\cosh\gamma}{dt/\cosh\gamma} = \frac{-dx^{(1)}}{dt} = -v^{(1)}_{12}.$$

However, the instantaneous coordinate velocity of N_m in $\tilde{E}_m$, as its increment from zero value in a certain previous current origin M of a world line, is expressed as $\dfrac{d^2x^{(m)}}{d\tau} = dv^{(m)}$ and exactly in M it is zero: $v^{(m)}_M = 0$. The *inner velocity* $v^{(m)} \rightarrow 0$ has another sense in contrast to above one. For the world trajectory passing through the point M, consider a neighborhood of M and introduce in it two hyperbolic angles: $\gamma^{(1)} = \gamma$ is a *general motion angle in* $\tilde{E}_1$, and $\gamma^{(m)}$ is a *additional infinitesimal motion angle in the base* $\tilde{E}_m$ determined by the *inner acceleration or deceleration of movement* in the neighborhood of M. For differentials of the two coordinate velocities with respect to $\tilde{E}_1$ and $\tilde{E}_m$ in the neighborhood of M, their trigonometric forms are expressed as:

$$\left.\begin{array}{l} d\left(\dfrac{dx^{(1)}}{d(ct^{(1)})}\right) = d\left(\dfrac{d\chi}{d(ct^{(1)})}\right) = d\tanh\gamma = \text{sech}^2\gamma\ d\gamma = d\gamma/\cosh^2\gamma, \\ d\left(\dfrac{dx^{(m)}}{d(ct^{(m)})}\right) = d\left(\dfrac{dx^{(m)}}{d(c\tau)}\right) = d\tanh\gamma^{(m)} = d\gamma^{(m)} = d\gamma, \end{array}\right\} \tag{78A}$$

where $\gamma^{(m)} \rightarrow 0$ is counted in the base $\tilde{E}_m$ from the current point M, but the angle γ is counted in the base $\tilde{E}_1$ from the origin O along the same world line. The angle γ is counted also from the axes $x^{(1)}$ and $\overrightarrow{ct}^{(1)}$ of $\tilde{E}_1$ up to $x^{(m)}$ and $\overrightarrow{c\tau}$ of $\tilde{E}_m$ applied to the point M.

For a curve world line segment, the infinitesimal angle $\gamma^{(m)}$ is counted in the current point M from the time-arrow $\overrightarrow{c\tau}$ (as a tangent) or from the axis $x^{(m)}$ (as a pseudo-normal) in these two opposite directions to the light cone between them. In a neighborhood of the point M, there holds $\gamma^{(m)} \to 0$ as $v_M^{(m)} = 0$. (For a straight world line segment, angles $d\gamma$ and $\gamma^{(m)}$ are zero.) For a collinear motion in its pseudoplane, $d\gamma$ is expressed in the same instantaneous base $\tilde{E}_m$ as $d\gamma^{(m)} = d\gamma$. At M the *inner 3-acceleration in* $\tilde{E}_m$ is

$$\frac{d^2x^{(m)}}{d\tau^2} = \frac{dv^{(m)}}{d\tau} = c \cdot \frac{d(\tanh \gamma^{(m)})}{d\tau} = c \cdot \frac{d(\tanh\ d\gamma)}{d\tau} = c \cdot \frac{d\gamma}{d\tau} = c \cdot \eta^*_\gamma = g^{(m)}(\tau). \quad (79A)$$

From here, for collinear motions, we obtain *the fundamental trigonometric formulae:*

$$\boxed{d^2\mathbf{x}^{(m)} = d\gamma \cdot d(c\tau) = dv^{(m)} \cdot d\tau = g^{(m)} d\tau^2, \quad dv^{(m)} = c\, d\gamma = g^{(m)} d\tau, \quad dx^{(m)} = 0.} \quad (80A)$$

in $\langle \mathcal{P}^{3+1} \rangle$: $d^2\mathbf{x}^{(m)} = d^2x^{(m)} \cdot \mathbf{e}_\alpha = d\gamma \cdot d(c\tau) \cdot \mathbf{e}_\alpha,\ d\mathbf{x}^{(m)} = \mathbf{0},\quad \mathbf{e}_\alpha = \pm\, \mathbf{const},\quad d(c\tau) \neq 0.$

$d(c\tau) = d\lambda$ is 1-st differential of the pseudo-Euclidean length of a world line segment; $d\gamma$ is space-like or time like. It is counted from M along the current $x^{(m)}$ or tangent. Formulae (80A) connect three differential parameters of curvilinear collinear motion. So, we obtain the *inner velocity and acceleration* as $v^{(m)} = c \cdot \gamma^{(m)} \to 0$ and $g^{(m)} = dv^{(m)}/d\tau = c\, d\gamma/d\tau$.

We use the trigonometric opportunities in the Minkowskian space-time $\langle \mathcal{P}^{3+1} \rangle$ for clear descriptions of relativistic motions, in particular here collinear ones, with their kinematics and dynamics in inertial and uninertial (accelerated or decelerated) frames of reference, but from the point of view of inertial (Galilean) universal frame of reference $\tilde{E}_1$. Thus in STR the base $\tilde{E}_m$ may be considered in $\tilde{E}_1$ as instantaneously inertial [76]. At a moment of the time τ, an *inner 3-force* F acted on $\mathbf{M}$, with caused by it the inner 3-acceleration $g^{(m)}$ and the inner 3-velocity $dv^{(m)}$ (i. e., as collinear 3-vectors in $\langle \mathcal{E}^3 \rangle^{(m)}$), are directed in $\tilde{E}_m$ along the $x^{(m)}$-axis. Hence in $\tilde{E}_m$ *these three instantaneous characteristics are always collinear with common directive vector* $\mathbf{e}$. According to the 2-nd Newton's Law of mechanics and relation (79A) with the relativistic dilated time τ and, in addition, with the instantaneous radius of the pseudo-curvature $\overline{\overline{R}} = 1/\overline{\overline{K}}$ (pure hyperbolic here) along a world line, we get:

$$\overline{\overline{\mathbf{g}}}(\tau) = \frac{F(\tau) \cdot \mathbf{e}_\alpha}{m_0} = \frac{d^2x^{(m)} \cdot \mathbf{e}_\alpha}{d\tau^2} = c^2 \frac{d\gamma}{d(c\tau)} \cdot \mathbf{e}_\alpha = c \cdot \eta^*_\gamma,\ \ d\gamma = \overline{\overline{K}}\, d(c\tau)\ \Rightarrow\ \overline{\overline{g}} = c^2/\overline{\overline{R}}. \quad (81A)$$

$|\mathbf{F}| = m_0 \cdot |\mathbf{g}|$ is the same for Observers in all inertial bases. (If $\mathbf{F}$ is an active force, then $|\mathbf{F}|$ is the number showed at the scale of a dynamometer in $\tilde{E}_m^{(3)}$.) The rest *own mass* $m_0 \neq 0$ of a material point (object) M does not depend on the given frame of reference. The *absolute value of inner acceleration* determined by (79A) and (81A) is *an invariant* (strongly at constant temperature $m_0 = \text{const}$). In $\tilde{E}_m$, it does not depend on γ (or velocity of movement) contrary to corresponding *relative characteristics*. Due to this, exactly $\mathbf{g}(\tau)$ is considered in STR as the inner 3-acceleration in own Cartesian sub-base $\tilde{E}_m^{(3)}$ (here as collinear one, but generally as non-collinear – see in (145A), Ch. 7A). If $\mathbf{g}$ is collinear to velocity $\mathbf{v}$, then the world line stays in the same own pseudoplane (the motion is *coplanar*). The constant collinear to $\mathbf{v}$ inner acceleration $\mathbf{g}$ determines rectilinear uniformly accelerated or decelerated physical movement. Such absolute motion is described in a certain pseudoplane along a time-like hyperbola with the constant radius of pseudo-curvature $\overline{\overline{R}} = d\lambda/d\gamma = 1/\overline{\overline{K}}$, where $d\gamma \neq 0$, $d(c\tau) = d\lambda = \overline{\overline{R}}\, d\gamma$ is the hyperbolic arc with its radius-vector of pseudo-normal radiated out of the hyperbola center O along vector $\mathbf{g}$. For *colllinear motion* with instantaneous parameters, including *hyperbolic velocity* η, (79A) gives the inner acceleration as follows

$$\overline{\overline{g}} = \frac{d^2x^{(m)}}{d\tau^2} = \frac{dv^{(m)}}{d\tau} = c \cdot \frac{d\gamma}{d\tau} = c \cdot \eta^*_\gamma = c^2 \frac{d\gamma}{d\lambda} = c^2 \overline{\overline{K}} = c^2/\overline{\overline{R}} = \text{const} \ \Rightarrow\ d^2\mathbf{x}^{(m)} = \overline{\overline{R}}\, (d\gamma)^2.$$

In general, there are else two types of parallel inner accelerations for collinear motions. The *proper 3-acceleration* in $\tilde{E} = (\chi, \vec{c\tau})$, with taking into account (76A) and (80A), is

$$\overline{\overline{g^*}}(\tau) = \frac{d^2\chi}{d\tau^2} = \frac{dv^*}{d\tau} = c \cdot \frac{d\sinh\gamma}{d\tau} = c \cdot \cosh\gamma \cdot \frac{d\gamma}{d\tau} = \cosh\gamma \cdot \overline{\overline{g}}(\tau) > \overline{\overline{g}}(\tau). \qquad (82A)$$

It is greater than inner 3-acceleration in (79A), as the differentials $d^2x^{(m)}$ is decreased proper differential $d^2\chi$ due to relativistic dilation as result of rotation of the axis $x^{(m)}$. Contrary, the *coordinate acceleration* in $\tilde{E}_1$ due to (78A) is very less than inner one:

$$\overline{\overline{g^{(1)}}}(t^{(1)}) = \frac{dv}{dt^{(1)}} = \frac{d^2\chi}{(dt^{(1)})^2} = c\cdot\frac{d\tanh\gamma}{dt^{(1)}} = c\cdot\operatorname{sech}^2\gamma\cdot\frac{d\gamma}{dt^{(1)}} = c\cdot\operatorname{sech}^3\gamma\cdot\frac{d\gamma}{d\tau} = c\cdot\frac{d\gamma}{d\tau}/\cosh^3\gamma \Rightarrow$$

$$\Rightarrow \overline{\overline{g^{(1)}}}(t^{(1)}) = \overline{\overline{g}}[\tau(t)^{(1)}]/\cosh^3\gamma \;\Rightarrow\; \{\overline{\overline{g}}[\tau(t)^{(1)}] \ll \overline{\overline{g}}[\tau(t)^{(1)}] < \overline{\overline{g^*}}[\tau(t)^{(1)}]\}. \qquad (83A)$$

The formula for tangential 3-acceleration $\overline{\overline{g}}^{(1)}(t)$ is known in STR *in physical form, but not in this simplest clear trigonometric form (!)*. The parameters $ct^{(1)}$ and $c\tau$ are used as arguments of various functions. Both are synchronous in the universal base $\tilde{E}_1$ if they are fixed with clocks of N_1 and N_m simultaneously. Simultaneity is defined in differential and integral forms derived from projecting in parallel to proper time into $\langle\mathcal{E}^3\rangle^{(1)}$ (see in Ch. 4A):

$$d(c\tau) = d(ct^{(1)})/\cosh\gamma < d(ct^{(1)}), \qquad c\tau = \int_0^{ct^{(1)}} d(ct^{(1)})/\cosh\gamma < ct^{(1)}; \qquad (84A)$$

$$d(ct^{(1)}) = \cosh\gamma\; d(c\tau) > d(c\tau), \qquad ct^{(1)} = \int_0^{c\tau} \cosh\gamma\; d(c\tau) > c\tau. \qquad (85A)$$

They are obtained with cut parallel to the axis $x^{(1)} = \chi$. Here $c\tau$ is, according to (84A), the pseudo-Euclidean length of a world line counted from the base $\tilde{E}_1$ origin.

If motion is *integral* and, as before, $\mathbf{v}$ and $\mathbf{g}$ in the Euclidean $3D$-subspace are collinear, then the angle γ, v and v^* vary continuously wth $\mathbf{e}_\alpha = \langle\cos\alpha_t\rangle = \mathbf{const}$. In particular, for *hyperbolic motion*, uniformly accelerated or decelerated (as the 1-st type of such motion) there holds $g = \text{const}$. This first simplest kind of relativistic accelerated movement was first analyzed by H. Minkowski [76, p. 111], M. Born [83] and A. Sommerfeld [86]. Second kind see in Ch. 10A. We give here and in Ch. 10A our author's variants with the tensor trigonometric approach. Thus, according to (79A) and (85A), we have for it a lot of important relations.

$$\left.\begin{array}{l} d\gamma = \overline{\overline{g}}\; d\tau/c \;\Rightarrow\; d(c\tau) = \overline{\overline{R}}\; d\gamma,\; \tau_0 = 0, \\ \gamma = \overline{\overline{g}}\tau/c \;\Rightarrow\; c\tau = \overline{\overline{R}}\cdot\gamma \;\; (\overline{\overline{g}}\tau = c\cdot\gamma), \end{array}\right\} \;(\overline{\overline{g}} = c^2/\overline{\overline{R}} = \text{const}). \qquad (86A)$$

$$\left.\begin{array}{l} d\sinh\gamma = \overline{\overline{g}}\; dt/c \;\Rightarrow\; \overline{\overline{R}}\; d\sinh\gamma = d(ct),\; t_0 = 0, \\ \sinh\gamma = \overline{\overline{g}}t/c \;\Rightarrow\; ct = \overline{\overline{R}}\cdot\sinh\gamma \;(\overline{\overline{g}}t = c\cdot\sinh\gamma), \end{array}\right\} \;(\overline{\overline{g}} = c^2/\overline{\overline{R}} = \text{const}). \qquad (87A)$$

From (86A) and (87A), we get the analogous relations with synchronized time parameters.

$$\left.\begin{array}{l} d(c\tau) = \overline{\overline{R}}\; d\gamma = \dfrac{d(ct^{(1)})}{\cosh\gamma} = \dfrac{d(ct^{(1)})}{\sqrt{1+[\overline{\overline{g}}t^{(1)}/c]^2}} = \dfrac{d(ct^{(1)})}{\sqrt{1+\left[ct^{(1)}/\overline{\overline{R}}\right]^2}}, \\ c\tau = \overline{\overline{R}}\cdot\gamma = (c^2/\overline{\overline{g}})\cdot\gamma = (c^2/\overline{\overline{g}})\cdot \operatorname{arsinh}[\overline{\overline{g}}\cdot t^{(1)}/c] = \overline{\overline{R}}\cdot\operatorname{arsinh}\left[ct^{(1)}/\overline{\overline{R}}\right]. \end{array}\right\} \qquad (88A)$$

$$\left.\begin{array}{l} d(ct^{(1)} = \overline{\overline{R}}\cdot\cosh\gamma\; d\gamma = \cosh\gamma\; d(c\tau) = \cosh(\overline{\overline{g}}\cdot\tau/c)\; d(c\tau), \\ ct^{(1)} = \overline{\overline{R}}\cdot\sinh\gamma = (c^2/\overline{\overline{g}})\cdot\sinh\gamma = (c^2/\overline{\overline{g}})\cdot\sinh(g\cdot\tau/c). \end{array}\right\} \qquad (89A)$$

With these relations for the hyperbolic motion and for the equivalent physical movement, the coordinate and proper velocities are functions in coordinate and proper time expressed also synchronically:

$$\left.\begin{aligned} v &= v(t^{(1)}) = c\cdot\tanh\gamma = \frac{\overline{\overline{g}}\cdot t^{(1)}}{\sqrt{1+\left[\overline{\overline{g}}\cdot t^{(1)}/c\right]^2}} < \overline{\overline{g}}\cdot\tau < \overline{\overline{g}}\cdot t^{(1)}, \\ v^* &= v^*(\tau) = c\cdot\sinh\gamma = c\cdot\sinh(\overline{\overline{g}}\cdot\tau/c) \equiv v_t^*(t^{(1)}) = \overline{\overline{g}}\cdot t^{(1)} > \overline{\overline{g}}\cdot\tau. \end{aligned}\right\} \qquad (90A)$$

These inequalities may be interpreted trigonometrically as: $\tanh\gamma < \gamma < \sinh\gamma < \cosh\gamma$.

Let's find out how also two types of distances x^* in $\tilde{E}_m$ and χ in $\tilde{E}_1$ are integrated up in hyperbolic motion? According to (75A), coordinate velocities in them are equal $v_{12} = v_{21}$, but, with respect to the inertial base $\tilde{E}_1$, their times ct in (86A) and $c\tau$ in (87A) are different. The 2-nd distance as a function in time τ is counted with the clock of N_m as follows:

$$x^* = \int_0^\tau v(c\tau)\ d\tau = \overline{\overline{R}}\int_0^\gamma \tanh\gamma(\tau)\ d\gamma(\tau) = \mathcal{L}_R(\gamma) = \mathcal{L}_R(\tau) = \overline{\overline{R}}\cdot\ln\cosh\gamma(\tau). \qquad (91A)$$

We established, that such a way is expressed by the *Huygens tractrix*. It is generating curve for construction of the tractricoid II by its revolving around time axis in the uninertial Special quasi-Euclidean space – see in Ch. 6A with the *Minding tractrix* – generating line for the tractricoid I. Both tractrices have equal length from angular argument γ or $\varphi(\gamma)$, but their uninertial Special quasi-Euclidean spaces, Euclidean sub-spaces and slopes are different! They have not invariants of motion in their spaces, but only one-step quasi-invariants.

The proper distance as functions in time $t^{(1)} = t$ or τ by the clocks of N_1 or N_m are:

$$\chi = \int_0^t v(t)\ dt = \overline{\overline{R}}\cdot\left[\sqrt{1+\left(ct/\overline{\overline{R}}\right)^2} - 1\right] \equiv \int_0^\tau v^*(\tau)\ d\tau = \overline{\overline{R}}\cdot[\cosh(c\tau/\overline{\overline{R}} - 1)]. \qquad (92A)$$

with (86A) and (87A). We established, that, in the 1-st case, it is direct equation of the kinematic time-like hyperbola. In the 2-nd case, it is direct equation of the catenary, i. e., such a way is expressed by the time-like catenary. It is generating curve for construction of the catenoid I by its revolving around time axis in the uninertial Special quasi-Euclidean space – see below. Both catenaries also have not invariants, but only one-step quasi-invariants. Note in (91A) and (92A) the common approximation at the beginning of these ways if $\gamma \to 0$: $\{x^*(\tau)\ \&\ \chi(\tau)\} \to R\gamma^2/2 = g\tau^2/2$, where $g = F/m_0 = c^2/R$ is inner acceleration (81A). This time-like hyperbola has the cosine-sine poly-steps invariant in the constant inertial pseudoplane $\langle\mathcal{P}^{1+1}\rangle$, it follows clarity from its parametric equations in γ as below:

$$\left.\begin{aligned} \chi + \overline{\overline{R}} &= \overline{\overline{R}}\cdot\cosh\gamma, \\ ct^{(1)} &= \overline{\overline{R}}\cdot\sinh\gamma, \end{aligned}\right\} \Rightarrow (\chi + \overline{\overline{R}})^2 - (ct^{(1)})^2 = \overline{\overline{R}}^2\cdot(\cosh^2\gamma - \sinh^2\gamma) = \overline{\overline{R}}^2. \qquad (93A)$$

It relates also to the *hyperbolic motion* as the uniform relativistic motion on a pseudoplane with its *sine-cosine time-like invariant* $\sinh^2\gamma - |\mathbf{cosh}\gamma|^2 = i^2 = -1$ in any base $\tilde{E} = \{x, \overrightarrow{ct}\}$. It has constant *pseudo-curvature* $\overline{\overline{K}}_R = 1/\overline{\overline{R}}$ and *hyperbolic angular proper velocity* as:

$$\eta_\gamma^* = d\gamma/d\tau = c/\overline{\overline{R}} = c\overline{\overline{K}}_R = \overline{\overline{g}}/c\ (rad/sec).$$

It expresses the velocity of hyperbolic rotation of tangent $\mathbf{i}$ with pseudonormal $\mathbf{p}$ radiated from the center O (Figure 2A(3)) for hyperbolic type of collinear motions. The second kind of the simplest uniformly accelerated relativistic motion as the *pseudoscrewed motion* will be considered in last Ch. 10A, because it is executed with rotated principal angle of motion.

In relation (86A), we have the parametric hyperbola with the angle-argument γ as a parameter of geometric and relativistic motions. So, it is the angular argument in tensor trigonometric representations of the two hyperbolic geometries (Ch. 12) and STR (Ch. 1A). Such time-like ans space-like hyperbolae are generatrices of the hyperboloids I and II of Minkowxki. With (86A), (87A), we obtain various forms for the coefficient of similarity $\overline{\overline{R}}$:

$$\overline{\overline{R}} = \frac{\chi}{\cosh\gamma - 1} = \frac{\chi + \overline{\overline{R}}}{\cosh\gamma} = \frac{ct^{(1)}}{\sinh\gamma} = \frac{c\tau}{\gamma} = \frac{c^2}{\overline{\overline{g}}} = \text{const.} \qquad (94A)$$

The kinematic hyperbola is intermediate between the Newtonian kinematic parabola in $t^{(1)}$ and an isotropic straight line of the light ray going out of the point O, see Figure 2A(3):

$$ct^{(1)} - \overline{\overline{R}} \ < \ \chi = \chi_t(t^{(1)}) \ < \ \overline{\overline{g}} \cdot (t^{(1)})^2/2 \quad (\sinh\gamma - 1 < \cosh\gamma - 1 < (\sinh^2\gamma)/2).$$

* * *

Contrary to pseudo-Euclidean approach, function $\chi(c\tau)$ in (93A), (94A), measured by a clock of N_m, produces Euclideanly the time-like *catenary* with the same radius-parameter R

$$\chi(\tau) = \int_0^\tau v^*(\tau)d\tau = c\int_0^\tau \sinh\gamma(\tau)d\tau = R\int_0^\gamma \sinh\gamma\, d\gamma =$$

$$= R\cdot[\cosh(c\tau/R) - 1] = R\cdot(\cosh\gamma - 1) \equiv R\cdot[\sec\varphi(\gamma) - 1] \Rightarrow \qquad (95A - I)$$

with very important *Consequence from hyperbolic motion* $\boxed{\cosh\gamma = 1 + \chi/R = 1 + g\chi/c^2}$! For instance, if in it we exchange inertial acceleration g_a into gravitational intensity g_f of astronomical mass M, then this produces equivalent *gravitational cosine* $\cosh\gamma_{(f)} \equiv \cosh\gamma_{(a)}$ with identical influences on time from inertia and gravitation – see more in Ch. 9A, where only with these cosines we'll explain *the Mercury perihelion relativistic shift* in frame of STR.

In Ch. 6 we established, that in the quasi-Cartesian and pseudo-Cartesian so-called *universal bases* of their binary spaces, between rotations and deformations there are angular and metric connections with general tensor correspondences (334), (335) in the quart circle (341) due to the covariant and countervariant specific spherical–hyperbolic analogies (331).

We extend this concept onto one-step isomorphic transformation $\langle\mathcal{P}^{1+1}\rangle \Rightarrow \langle\mathcal{Q}^{1+1}\rangle$ in relation (95A-I) – see visually on Figure 2A(3→4) by rectification with orthogonalization of initial regular curves, for example, line $\overrightarrow{c\tau}$ into straight axis $\overrightarrow{c\tau}$ (or $\overrightarrow{y_2}$) instead of previous $\overrightarrow{ct}$ (or $\overrightarrow{y_1}$). From the physical point of view, in $\langle\mathcal{P}^{1+1}\rangle$ we have velocity $v = d\chi/dt = \tanh\gamma\cdot c$ and, with respect to proper time, it is $v^* = d\chi/d\tau = \sinh\gamma\cdot c \equiv \tan\varphi(\gamma)\cdot c$ with tangent slope in the Euclidean quasiplane $\langle\mathcal{Q}^{1+1}\rangle \subset \langle\mathcal{Q}^{3+1}\rangle$. In $\langle\mathcal{P}^{1+1}\rangle$ and $\langle\mathcal{Q}^{1+1}\rangle$, these universal original and new bases are: $E_1 = \{\chi, ct\}$ and $E_C = \{\chi, c\tau\} = E_{(1,2)}$. From the mathematical point of view, we did transformation of initial polystep invariant of motion in $\langle\mathcal{P}^{3+1}\rangle$ in the one-step quasi-invariant in $\langle\mathcal{Q}^{3+1}\rangle$‡ (see earlier the same in Chs. 5, 6 and 4A):

$$[d(c\tau)]^2 = [d(ct(\gamma))]^2 - [d\chi(\gamma)]^2 \to [d(ct)]^2 = \{d(c\tau[\varphi(\gamma)])\}^2 + \{d\chi[\varphi(\gamma)]\}^2.$$

In particular, this gives one-to-one correspondence between tangent-secant hyperbolic differentials of the time-like pro-hyperbola and sine-cosine spherical differentials of the time-like catenary, confirmed again relations (87A), now on a tensor level and with angles γ and $\varphi(\gamma)$:

$$\left.\begin{array}{l} d\chi = R\sinh\gamma\, d\gamma, \\ dc\tau = Rd\gamma, \\ \chi = R(\cosh\gamma - 1), \\ c\tau = R\gamma. \end{array}\right\} \Rightarrow \left\{\begin{array}{l} (d\chi)^2 + (dc\tau)^2 = (dct)^2 = R^2\cosh^2\gamma\, d\gamma^2 = \\ = d\mathcal{C}_R^2(\gamma) \equiv d\mathcal{C}_R^2[\varphi(\gamma)] = R^2\sec^4\varphi(\gamma)\, d[\varphi(\gamma)]^2 \to \\ \to \mathcal{C}_R(\gamma) = ct = R\sinh\gamma \equiv \mathcal{C}_R[\varphi(\gamma)] = R\tan[\varphi(\gamma)]. \\ \text{Under } \gamma_0 = 0,\ \varphi_0 = 0 - \text{from } O_I \text{ at Figure 2A(4).} \end{array}\right. \qquad (95A - II)$$

All this is interpreted as passage into *Special quasi-Euclidean binary space or uninertial space-time* with time-like catenaries and catenoid I. Such binary space is direct spherically orthogonal sum of the Euclidean subspace $\langle\mathcal{E}^3\rangle^{(1)}$ and the new *rectified time-arrow* $\overrightarrow{c\tau}$:

$$\langle\mathcal{Q}_C^{3+1}\rangle^{\ddagger} \equiv \langle\mathcal{E}^3\rangle^{(1)} \boxplus \overrightarrow{c\tau}, \quad \text{where } \langle\mathcal{E}^3\rangle^{(1)} \equiv \text{CONST}, \ \overrightarrow{c\tau} \equiv \text{Const}. \qquad (95A-III)$$

Space-times $\langle\mathcal{Q}_C^{3+1}\rangle^{\ddagger}$ and $\langle\mathcal{P}^{3+1}\rangle$ have the same *reflector tensor* $\{I^{\pm}\}$ and orthospherical rotations $rot\ \Theta$. As if Euclidean length of world line $\overrightarrow{c\tau}$, as the new time axis $\overrightarrow{c\tau}$ in $\tilde{E}_C$, corresponds to proper time; Euclidean length of world line $\overrightarrow{ct}$ corresponds to coordinate time. With analogous to (95A-II) procedure, from these space-like pro-hyperbola and Minkowski pro-hyperboloid II, we obtain these space-like catenaries and Catenoid II in space $\langle\mathcal{Q}_C^{2+1}\rangle^{\leftrightarrow}$.

With (95A-II), by analogy (331-I), in addition to the hyperbolic invariant of kinematic pro-hyperbola (93A), we obtain in $\langle\mathcal{Q}_C^{2+1}\rangle^{\ddagger}$ the one-step hyperbolic secant-tangent quasi-invariant of the kinematic catenary for its points off initial O_1 with the same parameter R (due to its true hyperbolic nature on the pro-hyperbola), and its one-step cosine-sine quasi-invariant as the spherical analogue, transferred to a circle tangent to it at O_1 (Figure 2A(4)):

$$\left.\begin{array}{l}\left[\dfrac{R^2}{\chi+R}\right]^2+\left[\dfrac{\tanh\gamma}{\gamma}\cdot c\tau\right]^2 = R^2 = R^2\cdot(\text{sech}^2\gamma+\tanh^2\gamma) = \\ = R^2\cdot(\tanh^2 v+\text{sech}^2 v) \equiv R^2\cdot[\cos^2\varphi(\gamma)+\sin^2\varphi(\gamma)].\end{array}\right\}(c\tau = R\gamma, \varphi\neq\pm\pi/2) \quad (96A)$$

The equation is an invariant to orthospherical rotations in $\langle\mathcal{E}^2\rangle \subset \langle\mathcal{Q}_C^{2+1}\rangle^{\ddagger}$ with the same reflector tensor. Along the time-like catenary, it is *one-step tangent-secant quasi-invariant of the time-like pro-hyperbola* in $\langle\mathcal{P}^{1+1}\rangle$. In $\langle\mathcal{Q}_C^{1+1}\rangle^{\ddagger} \subset \langle\mathcal{Q}_C^{2+1}\rangle^{\ddagger}$ it is one-step sine-cosine quasi-invariant with $\varphi(\gamma)$, expressed by equation of the circle tangent to the catenary in point O_I, as situated on a torus around and tangent to the catenoid I – Figure 2A(4). [At it angle γ is expressed by φ with (360-II).] Along the circle spherical angle $\varphi(\gamma)$ is summarized! Analogy (331) breaks at $\varphi = \pm\pi/2$ in C_{II}. Acute angles γ and v are bonded by (360-IY).

By rotation of time-like catenary around $y_1 = \overrightarrow{c\tau} = R\gamma$ we get one sheet "horn shaped" catenoid I (of Euler); and by rotation of the space-like catenary around $y_2 = R\cosh\gamma$ we get two sheets "symmetric cups shaped" catenoid II as the *minimal suprfaces* formed also by the *line of sag*. Below we give hyperbolic and spherical equations of *spatial* time-like and space-like catenaries *by seemning rotation at the right angle* $\Pi/2$ *with exchange of their space and time coordinates* as $\tilde{E}_{(C1)} = \{\chi, c\tau\} = \{x_1, y_1\} \leftrightarrow \tilde{E}_{(C2)} = \{R\gamma, R\cosh\gamma\} = \{x_2, y_2\}$ with construction of catenoids I and II. Hence, the values of both catenaries radius and length are the same for both catenoids. After curve's rotation at $\Pi/2$, the Meusnier angle changes in complementary, but in both quasi-Euclidean enveloping spaces $\langle\mathcal{Q}_C^{2+1}\rangle^{\ddagger}$ and $\langle\mathcal{Q}_C^{2+1}\rangle^{\leftrightarrow}$, Euclidean metric and orthogonal differentiation are preserved with φ and ξ. Now we can obtain for catenoids I and II their equations even in $(n+1)$-dimensional quasi-Cartesian bases $\tilde{E}_C$ and calculate 1-st metric forms with two variant of parameterization in γ and φ.

For the catenoid I in $\langle\mathcal{Q}_C^{2+1}\rangle^{\ddagger}$ (at Meusnier angle φ between normal to catenary and $\mathbf{r}_1$), we get subsequently its metric form in its *vector-scalar (vs) form* from zero on the Equator:

$$\left.\begin{array}{l}\mathbf{x}_{(I)} = \chi\cdot\mathbf{e}_\alpha = R\cdot\cosh\gamma\cdot\mathbf{e}_\alpha \equiv R\cdot\sec\varphi\cdot\mathbf{e}_\alpha, \\ y_{(I)} = c\tau = \pm R\cdot\gamma \equiv \pm R\cdot\gamma(\varphi).\end{array}\right\}\Rightarrow \qquad (97A-I)$$

$$\left.\begin{array}{l}d\mathbf{x}_{(I)} = d(\chi\cdot\mathbf{e}_\alpha) = Rd(\cosh\gamma\cdot\mathbf{e}_\alpha) = R(\sinh\gamma\ d\gamma\cdot\mathbf{e}_\alpha+\cosh\gamma\ d\alpha\cdot\mathbf{e}_\mu), \\ dy_{(I)} = dc\tau = R\ d\gamma.\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{l}d\mathbf{x}_{(I)} = d(\chi\cdot\mathbf{e}_\alpha) = Rd(\sec\varphi\cdot\mathbf{e}_\alpha) = R(\sec\varphi\cdot\tan\varphi\ d\varphi\cdot\mathbf{e}_\alpha+\sec\varphi\ d\alpha\cdot\mathbf{e}_\mu), \\ dy_{(I)} = dc\tau = R\ d\gamma(\varphi) = R\sec\varphi\ d\varphi.\end{array}\right\}\Rightarrow$$

$$dl_1^2(\gamma) = [-R_C(d\gamma)]^2 d\gamma^2+[Rn(\gamma)]^2 d\alpha^2 = R^2(\cosh^2\gamma\ d\gamma^2+\cosh^2\gamma\ d\alpha^2) = R^2\{[d\mathcal{C}_{(I)}(\gamma)]^2+\cosh^2\gamma\ d\alpha^2\} \equiv$$

$$\equiv dl_1^2(\varphi) = [-R_C(d\varphi)]^2 d\varphi^2+[Rn(\varphi)]^2 d\alpha^2 = R^2(\sec^4\varphi\ d\varphi^2+\sec^2\varphi\ d\alpha^2) = R^2\{[d\mathcal{C}_{(I)}(\varphi)]^2+\sec^2\varphi\ d\alpha^2\}.$$

Then $R_1 = -R_C(d\varphi) = -R\sec^2\varphi,\ R_2 = Rn(\varphi)/\cos\varphi = +R\sec^2\varphi,\ 1/K_G = R_1R_2 = -R^2\sec^4\varphi.$

For the catenoid II in $\langle \mathcal{Q}_C^{2+1}\rangle^{\leftrightarrow}$ (at Meusnier angle ξ between normal to catenary and $\mathbf{r}_2$), we get subsequently its metric form in its *vector-scalar (vs) form* from zero on the Pole:

$$\left.\begin{array}{l}\mathbf{x}_{(II)} = R\cdot\gamma\cdot\mathbf{e}_\alpha \equiv R\cdot\gamma(\varphi)\cdot\mathbf{e}_\alpha = R\cdot\ln\cot[(\pi/2-\varphi)/2]\cdot\mathbf{e}_\alpha,\\ y_{(II)} = R\cdot\cosh\gamma \equiv R\cdot\sec\varphi.\end{array}\right\}\Rightarrow \qquad (97A-II)$$

$$\left.\begin{array}{l}d\mathbf{x}_{(II)} = R\,d(\gamma\cdot\mathbf{e}_\alpha) = R(d\gamma\cdot\mathbf{e}_\alpha + \gamma\,d\alpha\cdot\mathbf{e}_\nu),\\ dy_{(II)} = R\cdot\sinh\gamma\,d\gamma.\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{l}d\mathbf{x}_{(II)} = R\,d[\gamma(\varphi)\cdot\mathbf{e}_\alpha] = R\,d\{\ln\cot[(\pi/2-\varphi)/2]\cdot\mathbf{e}_\alpha\} = R[\sec\varphi\,d\varphi\cdot\mathbf{e}_\alpha + \ln\cot[(\pi/2-\varphi)/2]\,d\alpha\cdot\mathbf{e}_\nu],\\ dy_{(II)} = R\,d\sec\varphi = R\cdot\sec\varphi\cdot\tan\varphi\,d\varphi.\end{array}\right\}\Rightarrow$$

$$dl_2^2(\gamma) = R^2(\cosh^2\gamma\,d\gamma^2 + \gamma^2\,d\alpha^2) = R^2\{[d\mathcal{C}_{(II)}(\gamma)]^2 + \gamma^2\,d\alpha^2\} \equiv$$
$$\equiv dl_2^2(\varphi) = R^2\{\sec^4\varphi\,d\varphi^2 + \ln^2\cot[(\pi/2-\varphi)/2]\,d\alpha^2\} = R^2\{[d\mathcal{C}_{(II)}(\varphi)]^2 + \ln^2\cot[(\pi/2-\varphi)/2]\,d\alpha^2\}.$$

Then $R_1 = R_C(d\varphi) = R\sec^2\varphi,\ R_2 = Rn(\varphi)/\cos\xi = R\cdot\gamma(\varphi)/\sin\varphi,\ 1/K_G = R_1R_2 = R^2\sec\varphi\tan\varphi\cdot\gamma(\varphi).$

In next Ch. 6A we'll obtain naturally, with our tensor trigonometric approach and the same angles-analogues γ and φ (with countervariant specific spherical-hyperbolic analogy), two kinds of tractrices with tractricoids I and II as the following derivative objects from both two hyperbolae and hyperboloids I and II with the common coefficient of similarity R.

At the focal point χ_F of the time-like catenary, the *focal hyperbolic angle* of inclination for these catenary and hyperbola (see at Figure 2A (3) and (4)) are $\gamma_F = \omega = \text{arsinh}\ 1 \approx 0.881$ and $\varphi_F(\gamma_F) = \pi/4$. They are defined by the same covariant sine-tangent analogy, where $\omega = \text{arsinh}\ 1$ is the Especial hyperbolic angle introduced in sect. 6.4, as the hyperbolic analog of the Especial spherical number $\pi/4$. The proper distance for the catenary $\chi = R\cdot(\cosh\gamma - 1)$ tends to parabola $f(c\tau) = g\tau^2/2 = R\gamma^2/2$ (at $\tau\to\infty$) due to $(\cosh\gamma - 1)\approx\gamma^2/2$. The time-like catenary lies under the kinematic parabola in τ and the focal tangent to catenary (with inclination $\pi/4$), but it lies above the tangent circle (up to $\chi = R$) in $\langle\mathcal{Q}_C^{1+1}\rangle^{\updownarrow}$:

$$g\tau^2/2 < \chi,\ \ c\tau - kR\ \le\chi,\ \ \chi = \chi_\tau(\tau) < R - \sqrt{(R)^2 - (c\tau)^2}\ \text{ if }\ c\tau\le|R|.$$

These inequalities are interpreted as follows: $\gamma^2/2 < \cosh\gamma - 1 < 1 - \sqrt{1-\gamma^2}\ (\gamma\le 1)$.

In pseudo- and quasi-Cartesian bases $\tilde{E}_1$, both world lines of hyperbolic motion lie at different sides of two kinematic parabolae, see at Figure 2A(3), (4). If the angle of motion γ is equal to $\gamma_F = \omega$ (and $\varphi(\gamma_F) = \pi/4$), then the coordinate velocity v achieves value $v_F = c\cdot\tanh\ \omega = c/\sqrt{2}$ (for the hyperbola), and the proper velocity v^* achieves value $v_F^* = c\cdot\sinh\ \omega = c$ (for the catenary). Furthermore, $v^* > c$ if $\gamma > \omega$ and $\varphi(\gamma) > \pi/4$. But proper velocity of light is infinite. The maximum proper velocity of matter is $v^*\to\infty$! (It is a velocity of astronauts by their clocks – see to the end of this Chapter.) The coordinates of time-like catenary point at its focus χ_F in these bases are expressed in terms of the hyperbolic characteristic radius $R = c^2/g$:

$$\chi_F = (\sqrt{2}-1)R\approx 0.41R;\ \ ct_F^{(1)} = R,\ \ c\tau_F = \omega R\approx 0.881R,\ \text{(but } c\tau = R \text{ at } \gamma = 1);$$

$$kR = c\tau_F - \chi_F\ \Rightarrow\ k = \omega + 1 - \sqrt{2}\approx 0.467,\ \text{as } \gamma = \omega \text{ and } \varphi(\omega) = \pi/4 \text{ at } F.$$

Let us pay attention to the fact that in our Triad I all the objects discussed above have common spherical and hyperbolic angles. To visually display them, we use an adjacent torus, on whose generating small circles these angles are displayed with specific analogy (331) and quasi-invariants (96A), (105A) in $\tilde{E}_1$ off the radius-perpendicular directed to the tangent (and normal) MM' at Picture 2A(4). The *Principle of Correspondence* by Niels Bohr in our STR-tensor trigonometric interpretation (and further GTR!) means that the kinematic hyperbola, catenary, and two classic parabolae (of t and of τ) have the same tangent circle of radius R at their zero point O_1, see at Figure 2A(3), (4). This is equivalent to the fact that these curves have at point O_1 the same derivatives of the 1-st (zero) and 2-nd orders, these time-like hyperbola and catenary with two approximating them parabolae have the common radius of curvature R in the own coordinates of $\tilde{E}_1$ with the approximating relations:

$$|\,(g\cdot\tau^2)/2 < \chi = (c^2/g)\cdot\{\cosh\,[g\cdot\tau(t)/c] - 1\} < (g\cdot t^2)/2\,|\Rightarrow (g\cdot t^2)/2,\ \ \text{if } v/c\to 0.$$

* * *

Trigonometric approach can be used for clear and simplest introducing of main dynamical relativistic characteristics too. So, for rectilinear progressive physical movement of mass M, we define scalar, vector and tensor trigonometric expressions of these characteristics as Newtonian ones both in the original base $\tilde{E}_1$ and in the current base $\tilde{E}_m$ with the proper time (36A). The moving material body is reduced to its barycenter as a material point M. Then, with the use of the 2-nd Newtonian Law, we obtain in the relativistic space-time:

$$F = F_\tau(\tau) = m_0 g(\tau) = m_0 \cdot \frac{dv^{(m)}}{d\tau} = \frac{d[m_0 v^{(m)}]}{d\tau} = \frac{dp^{(m)}}{d\tau} = m_0 c \cdot \frac{g(\tau)}{c} = m_0 c \cdot \frac{d\gamma}{d\tau} \equiv$$
$$\equiv m_0 c \cdot \frac{\cosh\gamma\; d\gamma}{dt^{(1)}} = \frac{d(m_0 c \cdot \sinh\gamma)}{dt^{(1)}} = \frac{d[(\cosh\gamma \cdot m_0)\cdot(\tanh\gamma \cdot c)]}{dt^{(1)}} =$$
$$= \frac{d(m_0 v^*)}{dt^{(1)}} = \frac{d(mv)}{dt^{(1)}} = \frac{dp^{(m)}}{d\tau} = \frac{dp^{(1)}}{dt^{(1)}} = F_t(t^{(1)}). \qquad (A)$$

Formulae of the first row hold only in an instantaneous pseudo-Cartesian base where $m_0 = \text{const}$ is the *own mass*. Hence, this form of the 2-nd Newtonian Law is covariant! (An active inner force $|\mathbf{F}|$ is the number showed as if at the scale of a dynamometer in $\tilde{E}_m^{(3)}$.) Capacity of this inner force, due to the Newtonian mechanics, is presented in the base $\tilde{E}_1$ as

$$N = F \cdot v = m_0 c^2 \cdot \frac{\cosh\gamma\; d\gamma}{dt^{(1)}} \cdot \tanh\gamma = \frac{d(\cosh\gamma \cdot m_0 c^2)}{dt^{(1)}} = \frac{d(mc^2)}{dt^{(1)}} = \frac{dE}{dt^{(1)}}. \qquad (B)$$

First both these STR equations were obtained in *physical forms* by Henri Poincaré [63, 64]. These expressions allow to introduce instantaneous dynamical characteristics in $\tilde{E}_m$ and $\tilde{E}_1$: the *own 4-momentum* $\mathbf{P}_0 = P_0\mathbf{i} = m_0\mathbf{c}$, the *total scalar momentum* $P = mc = \cosh\gamma \cdot P_0$ and the *real 3-momentum* $\mathbf{p} = m\mathbf{v} = m_0\mathbf{v}^* = \sinh\gamma \cdot m_0 c\mathbf{p} = \sinh\gamma \cdot P_0\mathbf{p}$. Here $\mathbf{c}$ is 4-velocity by Poincaré, $\mathbf{i}$ and $\mathbf{p}$ are the principal tangent and pseudonormal to a world line. The time-like proportional total parameters ($P = mc, m, E = mc^2$) are *cosine* orthoprojections onto the time-arrow $\overrightarrow{ct^{(1)}}$, the space-like real momentum is a *sine* orthoprojection into the Euclidean subspace $\langle\mathcal{E}^3\rangle^{(1)}$ from a world line. Contrary to non-invariant total momentum, $\mathbf{P}_0 = m_0\mathbf{c}$ is invariant given by the *4D pseudo-Euclidean Pythagorean Theorem of three momenta* in the internal right triangle for the dynamics of M with hypotenuse $\mathbf{P}_0$, legs P and $\mathbf{p}$ in $\langle\mathcal{P}^{3+1}\rangle$:

$$\mathbf{P}_0 = P_0 \cdot \mathbf{i} = P \cdot \mathbf{i}_1 + p \cdot \mathbf{j} \;\Rightarrow\; (iP_0)^2 = (iP)^2 + p^2 \;\Rightarrow\; \{P_0^2 = P^2 - p^2\} \qquad (98A-I)$$

– here under metric tensor $\{I^{\mp}\}$ with invariant $1 = \cosh^2\gamma - \sinh^2\gamma$ (see in Chs.7A, 10A).

The own energy on a world line is $E_0 = P_0 c = m_0 c^2$. An increment of the non-invariant total energy $E = Pc = mc^2$ is expressed by exact trigonometric formula with approximation:

$$\boxed{k_E = (\cosh\gamma - 1) = \Delta P/P_0 = \Delta E/E_0 = (E - E_0)/E_0 = A/E_0} \approx \tanh^2\gamma/2. \qquad (99A)$$

These cosine formulae are very important for energetics interpretations in Chs. 7A, 9A, 10A. From here the Poincaré–Einstein formula for non-invariant mass-energy follows [62], [68]:

$$E = Pc = mc^2 = \sqrt{E_0^2 + (pc)^2} \approx E_0 + m_0(v^*)^2/2 \approx E_0 + m_0 v^2/2. \qquad (C)$$

With such mechanical way, it was inferred by G. Lewis in 1908 [88]. The former approximate values in this formula for m, P and E are upper bounds for the characteristics, second ones are lower bounds. This follows from inequalities: $1 + \sinh^2(\gamma/2) > \cosh\gamma > 1 + \tanh^2(\gamma/2)$.

Note essentially, that the use of homogeneous dynamic characteristics given above ($\mathbf{P}_0, P, \mathbf{p}$) *in the Theory of Relativity instead of heterogeneous ones with* (m_0, E_0, m, E) *has the obvious advantage that they are all reduced to a common physical dimension as the original invariant characteristic with its cosine and sine projections here along a world line.*

Last expression is the *cosine* energetic Hamilton function of γ as $E = \sqrt{E_0^2 + (pc)^2} = \sqrt{E_0^2 + (E_0 \cdot ||\mathbf{sinh}\ \gamma||)^2} = \cosh\gamma \cdot E_0 = E_0 + A$. But both *pseudo-Euclidean proportional invariants* in $\langle\mathcal{P}^{3+1}\rangle$ are $P_0 = m_0 c = +\sqrt{P^2 - (p)^2} \sim E_0 = m_0 c^2 = +\sqrt{E^2 - (pc)^2}$.

Besides, we express trigonometrically the phase velocity of the de Broglie wave as the supervelocity $s = E/p = \coth\gamma \cdot c = c^2/v$ and its real velocity as $v = dE/dp = \tanh\gamma \cdot c$.

And total momentum (98A-I) as the principal dynamical characteristic in STR can be represented on an invariant world line in the space-time $\langle\mathcal{P}^{3+1}\rangle$ as also invariant along it 4×1-momentum $\mathbf{P}_0$ (parallel to 4-velocity by Poincaré $\mathbf{c} = c\cdot\mathbf{i}_\alpha$) of a particle or a body M with its scalar cosine and 3-vector sine orthoprojections:

$$\mathbf{P}_0 = P_0 \cdot \mathbf{i}_\alpha = m_0 \cdot \mathbf{c} = P_0 \cdot \begin{bmatrix} \sinh\ \gamma \cdot \mathbf{e}_\alpha \\ \cosh\gamma \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ P \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ E/c \end{bmatrix}. \qquad (98A-II)$$

It is preserved under $\mathbf{F} = \mathbf{0} \leftrightarrow \mathbf{P}_0 = \mathbf{Const}$. The scalar value $P_0 = m_0 c = E_0/c$ is pseudo-Euclidean invariant for the particle or body M. As vectorial differential characteristic, it has the 1-st order of differentiation along a world line and tangent to it. In Ch. 10A we'll consider tensor trigonometrically all characteristics of absolute motion of M along its world line in $\langle\mathcal{P}^{3+1}\rangle$, with respect to the base $\tilde{E}_1$, up to the superior 4-th order.

These hyperbolic forms of the dynamical characteristics are obtained from Laws of the Newtonian mechanics, but with introduction of the relativistic time in $\tilde{E}_m$ for moving objects parallel to direction of motion in $\tilde{E}_1$ (as above). The hyperbolic angles of motion are bivalent 4×4-tensors Γ and $d\Gamma$ in $\tilde{E}_1$. The former is a main argument of the *measureless trigonometric tensor of motion* acting in space-time $\langle\mathcal{P}^{3+1}\rangle$ and hyperbolic geometry – see about it also in Chs. 6. It is a *pseudobiorthogonal tensor*. In the original base $\tilde{E}_1$, its definition and canonical forms due to (324), (348) and (362), (363) are following:

$$\{roth\ (\pm\Gamma)\}_{(3+1)\times(3+1)} = \cosh\Gamma \pm \sinh\Gamma = F(\gamma, \mathbf{e}_\alpha)$$

$$\begin{array}{|c|c|}\hline \cosh\gamma \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' & \pm\sinh\gamma \cdot \mathbf{e}_\alpha \\ \hline \pm\sinh\gamma \cdot \mathbf{e}'_\alpha & \cosh\gamma \\ \hline \end{array} =$$

$$\langle roth\ \Gamma\rangle : \quad roth\Gamma \cdot I^{\pm} \cdot roth\Gamma = I^{\pm} \quad (\mathbf{e}_\alpha\mathbf{e}'_\alpha = \overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}) \qquad (100A)$$

$$= \begin{array}{|c|c|}\hline I_{3\times 3} + (\cosh\gamma - 1) \cdot \mathbf{e}_\alpha\mathbf{e}'_\alpha & \pm\sinh\gamma \cdot \mathbf{e}_\alpha \\ \hline \pm\sinh\gamma \cdot \mathbf{e}'_\alpha & \cosh\gamma \\ \hline \end{array} = \begin{array}{|c|c|}\hline I_{3\times 3} + (\cosh\gamma - 1) \cdot \overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha} & \pm\sinh\gamma \cdot \mathbf{e}_\alpha \\ \hline \pm\sinh\gamma \cdot \mathbf{e}'_\alpha & \cosh\gamma \\ \hline \end{array}.$$

It is splitted projectively in 3×3-tensor orthoprojection into $\langle\mathcal{E}^3\rangle^{(1)}$, scalar cosine orthoprojection onto $\overrightarrow{ct^{(1)}}$ and two mutually transposed sine vector oblique projections. Logically, that in the limit case $\gamma \to 0$, we have $roth\ \Gamma \to I_{4\times 4}$.

Suppose that a material object M is moving progressively with respect to $\tilde{E}_1$ in $\langle\mathcal{P}^{3+1}\rangle$ at instantaneous velocity $\mathbf{v} = v\cdot\mathbf{e}_\alpha = c\cdot\mathbf{tanh}\ \gamma = c\cdot\tanh\gamma\cdot\mathbf{e}_\alpha$ or proper velocity $\mathbf{v}^* = v^*\cdot\mathbf{e}_\alpha = c\cdot\mathbf{sinh}\ \gamma = c\cdot\sinh\gamma\cdot\mathbf{e}_\alpha$ in the subspace $\langle\mathcal{E}^3\rangle^{(1)}$. On an arbitrary world line in the base $\tilde{E}_1$, we obtain the most general kinematical parameter as a tensor of an absolute 4×4-velocity $\mathcal{T}_C = c\cdot roth\ \Gamma$. As its right column, we get the vector of 4-velocity $\mathbf{c} = c\mathbf{i}$ by Poincaré with the pseudo-Euclidean module c. Recall (sect. 6.4), that at $\gamma = \omega$ we have $v^* = c$, $v = c/\sqrt{2}$. In its turn, also on the basis of physical-mathematical isomorphism (sect. 12.3), the physical dynamical tensors of momentum and energy are proportional to our tensor of motion (100A) *as their measureless trigonometric prototype*, namely with using constant coefficients c and m_0. Mainly, these following instantaneous dynamical tensors of momentum–energy $\mathcal{T}_P$ and of energy–momentum $\mathcal{T}_E$ are defined in the original base $\tilde{E}_1$ as

$$\mathcal{T}_P = P_0\cdot roth\ \Gamma = m_0 c\cdot roth\ \Gamma = m_0\cdot\mathcal{T}_C, \quad \mathcal{T}_E = P_0 c\cdot roth\ \Gamma = E_0\cdot roth\ \Gamma = m_0 c^2\cdot roth\ \Gamma.$$

If we let, that $c = \text{const}$, then $\mathcal{T}_E \sim \mathcal{T}_P$). Of course, all three tensors are compatible with the metric reflector tensor of the Minkowskian space-time $\langle \mathcal{P}^{3+1} \rangle$. Moreover, they are pseudo-Euclidean orthogonal and preserve their symmetric form under orthospherical transformation of $\tilde{E}_1$, i. e., in $\langle \tilde{E}_{1u} \rangle$. Asymmetrical tensors, obtained after two-step or multistep non-collinear motions may be represented in their polar form (19A) – see in sect. 11.3 and further in Ch. 7A. For example, consider the tensor of momenta $\mathcal{T}_P$ easily and clarity obtained from dimensionless tensor (100A). Its canonical tensor form is preserved under $\mathbf{F} = \mathbf{0} \leftrightarrow \mathcal{T}_P = \text{CONST}$. Then in the base $\tilde{E}_1$, it has this physical form:

$$\mathcal{T}_P = \begin{array}{|c|c|} \hline P \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} + P_0 \cdot \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} & \mathbf{p} \\ \hline \mathbf{p}' & E/c \\ \hline \end{array} = \begin{array}{|c|c|} \hline mc \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} + m_0 c \cdot \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} & m\mathbf{v} \\ \hline m\mathbf{v}' & mc \\ \hline \end{array}. \quad (101A)$$

The $(3+1) \times (3+1)$-tensor is splitted projectively in the 3×3-tensor orthoprojection $\{[\cosh\, \gamma \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} + \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'}] \cdot P_0\}$ into $\langle \mathcal{E}^3 \rangle^{(1)}$, the scalar cosine projection $P = P_0 \cdot \cosh\gamma$ onto the time-arrow $\overrightarrow{ct^{(1)}}$ (accordinly $E = E_0 \cdot \cosh\gamma = m_0 c^2 \cdot \cosh\gamma$), and two mutually transposed 3×1- and 1×3-vector sine projections $\mathbf{p} = P_0 \cdot \sinh\gamma \cdot \mathbf{e}_\alpha = m_0 \mathbf{v}^* = m\mathbf{v}$ and $\mathbf{p}'$. In all admissible pseudo-Cartesian bases, the values $P_0 = m_0 c$ and $E_0 = m_0 c^2$ for a massive material point are the pseudo-Euclidean scalar invariants, but $\mathbf{P}_0 = m_0 \mathbf{c}$ (98A-II) as a right column $\mathbf{P}_0$ in (101A) is a *geometric invariants* in space-time $\langle \mathcal{P}^{3+1} \rangle$ similar to a world line.

In its turn, the Lorentzian contraction of moving objects extent in the direction of this movement, fixed by Observer in the universal base $\tilde{E}_1$, has coordinate nature. It is described in 3-dimensional variant by the measureless $(3+1)\times(3+1)$-tensor of hyperbolic deformation (Ch. 4A). Due to Lorentzian *seeming* decreasing of moving body volume, *its coordinate density* seems to increase. But there is no pressing force acting on the body in the direction of movement. Inner physical force is absolute characteristic in (81A), its value is determined only in own instantaneous inertial base $\tilde{E}_m$ with the proportional inner acceleration F/m_0.

In our tensor trigonometric interpretation of STR, all the relativistic transformations of physical values may be determined more clarity and briefly with the use of these measureless trigonometric tensors and further operations of mathematical analysis over them. We used the signature of the Minkowski space-time $\langle \mathcal{P}^{3+1} \rangle$ with $\{I^{\pm}\}$ in (93A), (100A), (101A), etc. This is explained historically by the fact, that Henri Poincaré, discovering in 1905 the new relativistic space-time, introduced the imaginary hyperbolic angle as its angle of motion, and later Minkowski in 1908 [66, 65] realificated it by using his unity metric tensor $\{I^{\pm}\}$. It is in the case, a signature of this metric tensor corresponds to the original imaginary time-arrow with 4-velocity by Poincaré $\mathbf{c}$ and to the real-valued Euclidean subspace. Unfortunately, Einstein later presented this to the exact opposite, for example, in [69]. The same signature with $\{I^{\pm}\}$ will be used by us in the hypothetical so-called Looking Glass of the Theory of Relativity – beyond the horizon of events as if in another adjacent othersided world in the entire geometric and physical space-time $\langle \mathcal{P}^{3+1} \rangle$ by Minkowski. See this in detail in Ch. 10A.

* * *

Let use (79A), (81A), (86A) for deducing the *relativistic Ziolkovsky formula*, in particular, for the *photon rocket* of Eugen Sänger [112] moving due to reactive force of the light.

$$F = m_0(\tau) \cdot g(\tau) = u \cdot \frac{dm_0(\tau)}{d\tau} \;\Rightarrow\; u \cdot \frac{dm_0(\tau)}{m_0(\tau)} = g(\tau) d\tau = c\, d\gamma(\tau) \Rightarrow$$

$$\Rightarrow m_0(\tau) = m_0 \exp[-(c/u) \cdot \gamma(\tau)] = m_0 \exp\{-(c/u) \cdot \text{arsinh}\, [v^*(\tau)/c]\},$$

where m_0 and m are the initial and current mass of the rocket in the base $\tilde{E}_m$, and u is the fuel outflow velocity, $\gamma(\tau) = \text{arsinh}\, [v^*(\tau)/c]$. We deal with the hyperbolic motion! For a hypothetical photon rocket (as theoretically ideal variant), there holds $u = c$, and

$$m_0(\tau) = m_0 \exp[-\gamma(\tau)] = m_0 \exp\{-\text{arsinh}\, [v^*(\tau)/c]\} = m_0 \exp\{-\text{artanh}\, [v(t)/c]\}.$$

Compare the values of the own mass in terms of the coordinate and proper velocities of the photon rocket obtained by the Ziolkovsky formula and its relativistic variant above:

$$m_0 \exp(-v^*/c) < m_0 \exp[-\text{arsinh } (v^*/c)] = m_0 \exp[-\text{artanh } (v/c)] < m_0 \exp(-v/c),$$

and this is equivalent to the trigonometric inequalities $\sinh\gamma > \gamma > \tanh\gamma$.

Let that the hypothetical photon rocket flies to the star Proxima (i. e., *nearest*) Centauri and returns to the Earth. Then the *ideal parameters* (by taken time) of the flight are:

- the fuel outflow velocity $u = c$ for a photon rocket (as the theoretical maximum),
- constant acceleration $g = 10$ m/sec^2 as on the Earth – along hyperbola and catenary,
- the one-way distance $L = 2\chi \approx 40.3 \cdot 10^{15} m \approx 4.25$ light years.

Consider trigonometric computations for the reverse hyperbolic motion of the rocket – see at Figure 3A. This example illustrates clearly *the twins paradox*. For this flight, of course, as a hypothetical travel, with (86A), (87A), (94A) and a consequence from (95A-I), we have:

$$\chi = L/2 = R\cdot(\cosh\gamma_{max} - 1), \ \ \cosh\gamma = 1 + gx/c^2 = 1 + x/R \to (\cosh\gamma - 1) \sim x, \ (R = c^2/g);$$

$$\tau = 4(c/g)\gamma_{max}, \qquad t^{(1)} = 4(c/g)\sinh\gamma_{max}, \qquad t^{(1)}/\tau = \sinh\gamma_{max}/\gamma_{max};$$

$$v_{max} = c\cdot\tanh\gamma_{max}, \qquad v^*_{max} = c\cdot\sinh\gamma_{max};$$

$$m_0(\tau)/m_0 = \exp[4(-c/u)\gamma_{max}], \ \text{ at } u = c: \ m_0(\tau)/m_0 = \exp[-4\gamma(\tau)], \ (\gamma = c\tau/R).$$

Computations give the following results mapping below at Figure 3A:

$$\chi \approx 20.15 \cdot 10^{15} \text{ m}, \quad (L = 2\chi \approx 40.3 \cdot 10^{15} m), \quad R \approx 9 \cdot 10^{15} \text{ m}, \quad t_F \approx 305 \text{ days};$$

$$\cosh\gamma_{max} \approx 3.239, \ \ \sinh\gamma_{max} \approx 3.081, \ \ \tanh\gamma_{max} \approx 0.951, \ \ \gamma_{max} \approx 1.844$$

under acting the hyperbolic trigonometric inequality $\cosh\gamma > \sinh\gamma > \gamma > \tanh\gamma$;

$v_{max} \approx 0.951c$ and $v^*_{max} \approx 3.061c$ with the corresponding difference in both times

$$t^{(1)} \approx 3.70 \cdot 10^8 \text{ sec} \approx 11.7\text{years}, \ \tau \approx 2.21 \cdot 10^8\text{sec} \approx 7,01 \text{ years} < 2L \approx 8,50\text{light years!}$$

Various cosmic STR-evaluations were first analyzed by P. Langevin in [85]. Our STR-evaluation is the very clear trigonometric interpretation of *the twins paradox* in ideal regime of the cosmic flight with the Earth acceleration: we obtain for the 1-st twin-astronaut proper time $\tau \approx 7$ years and for the 2-nd twin on the Earth $t^{(1)} \approx 11.7$ years at time relation $t^{(1)}/\tau \approx 1.67$. Coordinate time $t^{(1)}$ on the Earth of light spreading there and back with velocity c ($2L \approx 8.50$ light years) is greater than proper time of the twin-astronaut! Relative decreasing own mass due to only expenditure of fuel, due to our relativistic formula, is $m_0(\tau)/m_0 = \exp(-4\gamma_{max}) \approx 1/1600$!!! (In Ch. 9A, by (209A), we'll show the equivalency of this kinematic time decrease with the time decrease from influence of only accelerations!)

A photon rocket with *terrestrial acceleration* reaches the proper velocity c for period less than one year, and further the velocity increases up to $3c$, but at the end of the trip the own mass of the rocket becomes insignificantly small ($m_0/1600$). Hence such cosmic flights even to nearest stars with return of astronauts onto the Earth by STR Laws are impossible for contemporary people (no for robots) as well as the empty project of voyages based on GTR through "wormholes-tunnels" in the Universe as a pseudo-scientific PR-populism, etc.!

However, the paradoxical inequality $\tau \approx 7,01$ years $< 2L \approx 8,50$ light years (gotten due to the specific initial parameters of the flight) shows, that astronauts during such reverse cosmic flight as if outstrip the light!!! Indeed, a radio-signal sent by the astronauts at the moment of their departure from the Earth to the Star Proxima Centauri theoretically after its reflection of the Star must return to the Earth in $2L = 4\chi \approx 8,50$ light years. But the astronauts return onto the Earth in $\tau \approx 7$ years $< 2L$ by their *same clock*! This unusual paradox of STR, may be interpreted as follows.

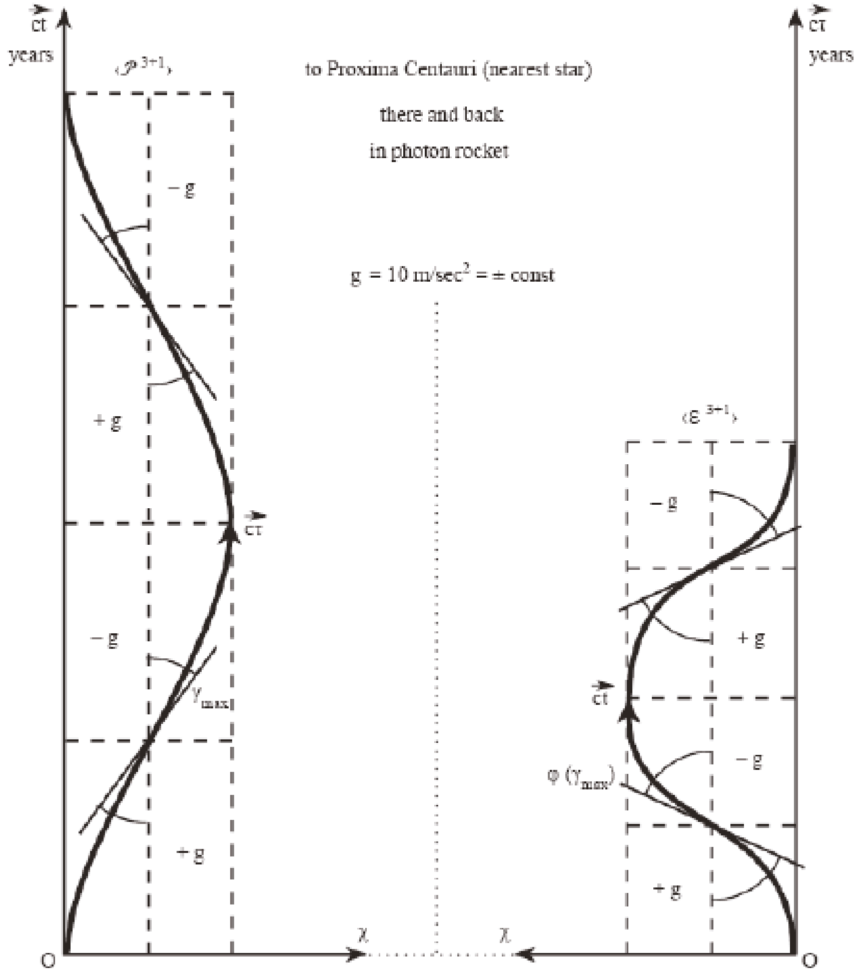

Figure 3A. Reverse hyperbolic motion of a body (as the photon rocket) in coordinates: pseudo-Cartesian (at the left on hyperbola) and quasi-Cartesian (at the right on catenary) under acting reactive force causing constant inner acceleration.

In the instantaneous space $\langle\mathcal{E}^3\rangle^{(m)}$ connected with the rocket and in the space $\langle\mathcal{E}^3\rangle^{(1)}$, light spreads at usual coordinate velocity $c = dx^{(m)}/d\tau = d\chi/d(ct^{(1)})$. However, from the point of view of the astronauts by their clock in the rocket, relative of them velocity of light in $\langle\mathcal{E}^3\rangle^{(1)}$ is $d\chi/d\tau = dx^{(1)}/d(ct^{(m)}) = \cosh\gamma \cdot c > v^* = \sinh\gamma \cdot c$, i. e., *the astronauts do not outstrip the light in* $\langle\mathcal{E}^3\rangle^{(1)}$! (It is caused by the reason, that the space $\langle\mathcal{E}^3\rangle^{(m)}$ and time $\overrightarrow{ct^{(m)}}$ with respect to ones in the base $\tilde{E}_1$ are rotated at the hyperbolic angle $\gamma = \text{arsinh}\ (v^*/c) = \text{artanh}\ (v/c)$ with dilation of time and space in the rocket (Ch. 3A). Consequently, the radio-signal returns to people of the Earth in $t^{(1)} = 2L \approx 8.50$ years, they will meet the astronauts on the Earth in $t^{(1)} \approx 11.7$ years. This paradox is interpreted also by tensor trigonometry. In general, similar kinematic effects of STR, with real difference of time in different frames of reference, are possible only under action of two great Principles of Nature. They are the Postulate of Relativity by Poincaré [63] and the Mach Principle [55] (sect. 12.3 and Ch. 9A). See presentation from a point of view of acceleration in (209A).

Chapter 6A

Isomorphic mapping of pseudo-Euclidean space of index 1 into Special quasi-Euclidean space with Beltrami pseudosphere

Space itself, without matter moving or field, has no any physical sense. It with its geometry are abstract math models, used for maximal adequate and convenient description, according to H. Poincaré [61], of laws of matter motions in coordinate forms. So, in Ch. 5A, with this approach and in the universal base $\tilde{E}_1$ of $\langle\mathcal{P}^{2+1}\rangle$ for the hyperboloid I, we introduced the uninertial *Special quasi-Euclidean space* $\langle\mathcal{Q}_C^{2+1}\rangle^{\ddagger}$ (96A) with own Especial quasi-Cartesian *cross* base $\tilde{E}_C = \{\chi, \overrightarrow{c\tau}\}$ for presentations of hyperbolic motions by the time-like catenary $\overrightarrow{ct}$. Its Euclidean subspace $\langle\mathcal{E}^2\rangle^{(1)}$ is the same and constant. In the latter, orthospherical rotations $rot\ \Theta$ are preserved, but around $\overrightarrow{c\tau}$. The hyperbolic world line as the proper time-arrow $\overrightarrow{c\tau}$ is transformed with Lambertian $\gamma(\varphi)$ in $\tilde{E}_1$ into new rectified time axis $\overrightarrow{c\tau}$, as it is permanently orthogonalized, with respect to $\langle\mathcal{E}^2\rangle^{(1)} \equiv \mathrm{CONST}$ – see at Figure 2A(1)–(4). Coordinates of points on catenary world line in cross base $\tilde{E}_{1,2} = \{\chi, \overrightarrow{c\tau}\}$ fix proper time $c\tau$ and proper distance χ in $\tilde{E}_1$. Synchronism of events for N_1 and N_m is parallelism to $\langle\mathcal{E}^3\rangle^{(1)}$.

Time-like space $\langle\mathcal{Q}_C^{2+1}\rangle^{\ddagger}$ is synthesized from the internal (light) conic cavities with only a time-like hyperbolic part of hyperboloid I, without a space-like ellipsoidal part in the external cavity) by an exchange $\overrightarrow{ct}$ and $\overrightarrow{c\tau}$ at $\langle\mathcal{E}^2\rangle^{(1)} \equiv \mathrm{CONST}$. *Space-like* space $\langle\mathcal{Q}_C^{2+1}\rangle^{\leftrightarrow}$ is synthesized from the external conic cavity with the entire hyperboloid II in $\langle\mathcal{P}^{2+1}\rangle$ for it by an exchange $\langle\mathcal{E}^2\rangle^{(1)}$ and $\langle\mathcal{E}^2\rangle^{(2)}$ at $\overrightarrow{ct} = \mathrm{const}$. So, one may implement mentally such operations at Figures 4 and 2A(4)! Both spaces have own groups of $rot\ \Theta$ in $\langle\mathcal{E}^2\rangle^{(1)}$ and $\langle\mathcal{E}^2\rangle^{(2)}$, but only one-step own admissible $rot\ \Phi(\Gamma)$ with own Euclidean *quasi-invariants*.

In 1-st variant, one-sheet hyperboloid I as a locus of parallel time-like hyperbolae $\pm\overrightarrow{c\tau}(\gamma)$ in $\tilde{E}_1 = \{\chi, \overrightarrow{ct}\}$ is transformed in a centered cylinder expressed in *cross* base $\tilde{E}_C = \{\chi, \overrightarrow{c\tau}\}$; its generatrix lines are rectified hyperbolae as the *new time axes* $\overrightarrow{c\tau}$. A circular set of axes $\overrightarrow{ct}$ expressed in $\tilde{E}_1$ is transformed with the direction outside the central axis $\overrightarrow{ct}$ in a catenoid I as a locus in $\langle\mathcal{Q}_C^{2+1}\rangle^{\ddagger}$ of *time-like catenaries* $\pm\overrightarrow{ct}(\varphi)$ (see in Ch. 5A) expressed in $\tilde{E}_C = \{\chi, \overrightarrow{c\tau}\}$ (at $ct \leftrightarrow c\tau$). The external cavities of the light cone with space-like content are concentrated *inside the new centralized proper time axis* $\overrightarrow{c\tau}$ *with annihilation*. A catenoid I is a *minimal hypersurface* in the space $\langle\mathcal{Q}_C^{2+1}\rangle^{\ddagger}$. It has cylindrical topology (as a one sheet hyperboloid I) and it is obtained with revolving a time-like catenary $\pm\overrightarrow{ct}(\varphi)$ around the new time axis $\overrightarrow{c\tau}$ at O, see Figures 2A(4). The Euclidean length of the world line $\overrightarrow{ct}(\varphi)$ is a coordinate time ct due to (95A-II). The proper time $c\tau$ is measured *by Euclidean way* along axis $\overrightarrow{c\tau}$. Transformation $\langle\mathcal{P}^{2+1}\rangle \to \langle\mathcal{Q}_C^{2+1}\rangle^{\ddagger}$, with transformation of hyperbolae in catenaries, go with replacing pseudo-Euclidean measure for time by Euclidean one in the real-valued *Special uninertial time-like quasi-Euclidean space-time* $\langle\mathcal{Q}_C^{2+1}\rangle^{\ddagger}$ of the kinematic curves as a locus of time-like arrows $\pm\overrightarrow{ct}(\varphi)$ *with any slopes*. The catenoid I is a result of *dilating* the hyperboloid I time axes with local $k = d\tau/dt = \mathrm{sech}\ \gamma$.

In 2-nd variant, a two-sheet hyperboloid II as a twain locus of space-like hyperbolae $\pm\lambda(\gamma)$ in $\tilde{E}_1 = \{\chi, \overrightarrow{ct}\}$ is transformed in a twain circular set of rectified hyperbolae λ expressed in the cross base $\tilde{E}_C = \{x_2, \overrightarrow{ct}\}$ and radiated from their two centers C_{II} (Figure 4) as the *new space axes* λ in the new Euclidean subspace $\langle\mathcal{E}^3\rangle^{(2)}$. A twain circular set of axes χ expressed in $\tilde{E}_1$ is transformed with the direction to the time axis $\overrightarrow{ct}$ in a two-sheet catenoid II as a twain locus in $\langle\mathcal{Q}_C^{2+1}\rangle^{\leftrightarrow}$ of *space-like catenaries* $\pm\chi(\varphi)$ in $\tilde{E}_C = \{\lambda, \overrightarrow{ct}\}$ (at $\chi \leftrightarrow \lambda$). The internal cavity of the light cone with the hyperboloid I are concentrated inside the new Euclidean subspace $\langle\mathcal{E}^2\rangle^{(2)}$ with annihilation. A catenoid II is a two-sheet *sag* hypersurface in the Special quasi-Euclidean space $\langle\mathcal{Q}_C^{2+1}\rangle^{\leftrightarrow}$, in addition to previous one. Its two sheets have also affine topology (as two sheets of a hyperboloid II) and it is obtained with revolving two space-like catenaries $\chi(\varphi)$ around the preserved time axis $\overrightarrow{ct}$. The catenoid II is a result of *dilating* the hyperboloid II space axes with local $k = d\lambda/d\chi = \mathrm{sech}\ \gamma$.

Further catenoids I and II can be transformed also isomorphically into tractricoids I and II by compressed transformation of their spaces with bases in new Special quasi-Euclidean spaces with their new Especial universal quasi-Cartesian bases. We'll do it by our very simple and descriptive geometric manner – on the example Catenoid I $\to$ Tractricoid I (as the *Beltrami pseudosphere*) – see at Figure 2A(4).

Namely, the involute of a catenary $\mathcal{C}_R = ct(c\tau)$ (at $c\tau = R\gamma$ in Ch. 5A) is the *Minding tractrix* $\mathcal{L}_R$ [43]. As we saw in Ch. 5A, the *Euclidean length* of a catenary till M is equal to $ct = R\sinh\gamma \equiv \tan\varphi(\gamma)$, see (95A-II); this length is the same for the tangent to catenary at M (it is rectified $\overrightarrow{ct}$). The tangent MM' is normal to the tractrix and it is its radius of curvature $R_T = R\tan\varphi$. It is a vector-distance $\overrightarrow{ct}$ between both curves, translating its current length onto the tractrix as $d\mathcal{L}_R = R\tan\varphi\, d\varphi$ (both curves are perpendicular each other at points M and M'). *At evolute-involute Euclidean metric's transfer, its space and time slopes are exchanged* ! Revolving double catenary around the new time axis $\overrightarrow{c\tau_R}$ produces the catenoid I. Revolving it with double Minding tractrix produces the tractricoid I as the *Beltrami pseudosphere*, compressed inside the catenoid I. In result we obtain, that $\langle \mathcal{Q}_T^{2+1}\rangle^\ddagger \subset \langle \mathcal{Q}_C^{2+1}\rangle^\ddagger$. Its generating *double tractrix* is a *continuous curve, but with a middle cusp point. It is expressed correctly only in compressed* $\langle \mathcal{Q}_T^{1+1}\rangle^\ddagger$ *and in own Especial quasi-Cartesian base* $\tilde{E}_T = \{\chi_R, \overrightarrow{c\tau_R}\}$ with Euclidean axis $x = \chi_R$ in interval $-R\div+R$ and the reper axis $y = \overrightarrow{c\tau_R}\}$. Such Minding tractrix (in addition to Huygens one) and pseudosphere were discovered by Ferdinand Minding in 1838 [43], the latter as a real-valued surface of the constant negative Gaussian curvature.

This pseudosphere was applied by Eugenio Beltrami for first, though very partial interpretation of the Lobachevsky plane [44] – in the region of only hyperbolic geodesics motions as $d\gamma$ and $d\varphi(\gamma)$, due to our tensor trigonometry. However the *tensor-vector-scalar (tvs)-forms* of their 1-st metric forms are different (at $n > 1, q = 1$) without possibility of their even local isomorphism, as for space and time like spatial curves too. These tvs-forms are identical for the Minkowski hyperboloid I and the tractricoid I, but as one step, in their enveloping binary spaces (i. e. at $n > 1$) – see below. Both in $\langle \mathcal{P}^{n+1}\rangle$ and $\langle \mathcal{Q}_T^{n+1}\rangle^\ddagger$, with single geodesic hyperbola and Minding tractrix plus n purely circular extremals, with equal and constant negative Gaussian curvature and identical cylindrical topology at any point on them, they are isomorphic, but due to the Minding Theorem for real-valued 2D surfaces [15, p. 240]. In this Chapter we'll prove strictly their global isometry, but only *as one-step*!

The central circular zone – an *equator* of the hyperboloid I and of the double pseudosphere (where $\gamma = 0 \leftrightarrow \varphi(\gamma) = 0$ at the points C_I at Figure 4) corresponds to the infinitely far conventional border of the whole projective hyperplane with upper and lower parts in the flat cotangent model of the hyperboloid I (Ch. 12). Figures cannot pass through this equator of the pseudosphere under regular motions, but they pass it as bended under $180°$, then metric and topology are preserved. Figures on the hyperboloid I pass freely through this equator without the broken (as also through the border in its cylindrical tangent projective model).

Let's explain how the Minding tractrix two coordinates are expressed in sequential bases under its generation from time-like pro-hyperbola and next catenary. In result, the tractrix is interpreted in its Especial base $\tilde{E}_T = \{\chi_R, \overrightarrow{c\tau_R}\}$ and with respect to initial $\tilde{E}_1 = \{\chi, \overrightarrow{ct}\}$ of pro-hyperbola and $\tilde{E}_C = \{\chi, \overrightarrow{c\tau}\}$ of next time-like catenary. The time axis $\overrightarrow{c\tau_R}$, asymptotic for the generatrix tractrix of the Beltrami pseudosphere, is the axis of its proper revolution. The space axes for these tractrix, catenary and hyperbola have the common vector of the directional cosines $\mathbf{e}_\alpha$, the bases $\tilde{E}_C$ and $\tilde{E}_T$ have the common center O_1 as zero point of these connected catenary $\overrightarrow{\mathcal{C}_R}(c\tau) = \overrightarrow{ct}$ and tractrix $\overrightarrow{\mathcal{L}_R}(c\tau_R)$. The point O_1 is a cusp for the double tractrix, therefore it belongs to the curve. It is the mapping of a zero point C_I of the pro-hyperbola $\overrightarrow{c\tau}$ in $\tilde{E}_1$ of space-time $\langle \mathcal{P}^{3+1}\rangle$. See all at Figures 4 and 2A(4). Under STR $c\tau > 0$ and, in upper and lower parts of the tractrix, we have velocities $v > 0$ and $v < 0$, $d\gamma > 0 \to g = \text{const} > 0$; and at the point O_1: $\gamma = 0, \chi_R = 0, c\tau = c\tau_R = 0$.

Taking into account (86A), (87A), (94A), in $\langle \mathcal{Q}_T^{1+1} \rangle^{\ddagger}$, the tractrix *radius of curvature* is $ct = R \cdot \sinh\gamma$ ($c\tau = R\gamma$) and its *compressed two coordinates* are bonded with such ones of the time-like pro-hyperbola and the next time-like catenary in $\tilde{E}_1$ and $\tilde{E}_C$ as follows:

$$\left. \begin{array}{l} \chi_R = \sin\varphi(\gamma)ct - \chi \equiv \tanh\gamma \cdot ct - \chi = \text{sech}\ \gamma \cdot \chi = k_1 \cdot \chi < \chi, \\ c\tau_R = c\tau - \cos\varphi(\gamma)ct \equiv c\tau - \text{sech}\ \gamma \cdot ct = (1 - \tanh\gamma/\gamma)c\tau = k_2 \cdot c\tau < c\tau. \end{array} \right\} \quad (102A)$$

Thus, $\gamma = 0 \to \chi_R = 0,\ c\tau_R = 0$, and further the coefficients of compression monotonically change from 1 to 0 (k_1) and from 0 to 1 (k_2) as the point M is here moving from O_1 to O. They influence on coordinates mapping $\chi \to \chi_R$, $c\tau \to c\tau_R$ and transform the previous two curves into the *reversed continuous tractrix* $\overrightarrow{\mathcal{L}_R}(c\tau_R)$ perpendicular in the current points M and M' (Figure 2A(4)) to the time-like catenary $\chi_R(c\tau_R)$! Due to (86A) and (87A) for hyperbolic motion, equations (102A) may be also represented in the pure trigonometric form (with *coefficient of similarity* R) as the defining function from only the angle γ ($0 \le |\gamma| \le \infty$).

Let's reduce of the tractrx relations (102A) into its hyperbolic type equations in the base $\tilde{E}_T = \{\chi_R, \overrightarrow{c\tau_R}\}$ of Special quasi-Euclidean plane $\langle \mathcal{Q}_T^{1+1} \rangle^{\ddagger}$ in parametric form from γ or $c\tau$

$$\left. \begin{array}{l} \chi_R = R \cdot x = R(1 - \text{sech}\ \gamma) = R(1 - \text{sech}\ \frac{c\tau}{R}), \\ c\tau_R = R \cdot y = R(\gamma - \tanh\gamma) = R(\frac{c\tau}{R} - \tanh\frac{c\tau}{R}) > 0. \end{array} \right\} \Rightarrow \boxed{d\mathcal{L}_R = R\tanh\gamma\ d\gamma} \quad (103A)$$

Corollary 1. *Condition $R = 1$ come to the unity tractrix as unique trigonometric object. All such tractrices $\chi_R(c\tau_R)$ from (103A) are homothetic to each other with the coefficient R off unity one as well as homothetic curves: circles, equilateral hyperbolae, catenaries etc..*

- parametric equations of *the unity double tractrix in parameters* γ, φ, ξ with (360-II) are

$$\left. \begin{array}{l} \pm z = 1 - x = \text{sech}\gamma \equiv \cos\varphi = \sin(\pi/2 - \varphi) = \sin\xi, \quad (0 < |z| \le 1), \\ \pm y = \gamma - \tanh\gamma \equiv \gamma(\varphi) - \sin\varphi = \gamma(\xi) - \cos\xi = \ln\cot(\xi/2) - \cos\xi; \\ (\text{with inequality } \gamma > \tanh\gamma;\ 0 \le |\varphi(\gamma)| \le \pi/2,\ \pi/2 \ge |\xi(\gamma)| \ge 0). \end{array} \right\} \quad (103A - I)$$

- direct equation of *the unity double tractrix in the spatial coordinate* z is

$$\pm y = \pm y(|z|) = \text{arsech}(z) - \sqrt{1 - z^2}. \quad (103A - II)$$

In addition, $z_R = R \cdot z = r$ – the *local radius of revolution for the Beltrami pseudosphere.*

Compare parametric equations of the Minding tractrix with ones of spherical cycloid:

$$\left. \begin{array}{r} z_R = R \cdot z = r = R\cos\varphi, \\ \pm y_R = \pm R \cdot y = R(\varphi - \sin\varphi), \end{array} \right\} \ d\mathcal{L}_R = 2R \cdot \sin(\varphi/2)\ d\varphi,\ \mathcal{L}_R = R \cdot \mathcal{L}(\varphi) = 4R[1 - \cos(\varphi/2)].$$

Corollary 2. *A tractrix is hyperbolic analog of a spherical cycloid with one cycle. All cycloids $y_R(z_R)$ are homothetic with coefficient R, if $R = 1$ the cycloid is unique trigonometric object.*

From space-like catenary by evolute-involute transpher we get the Huygens tractrix in $\langle \mathcal{Q}_T^{2+1} \rangle^{\leftrightarrow}$. By rotation of the Minding tractrix around its y_R we get the one sheet "horn shaped" tractricoid I. By rotation of the Huygens tractrix around its y_R we get the "flying saucer shaped" tractricoid II. They are connected also by rotation at $\Pi/2$ (see Ch. 5A)! We express them by angles $\gamma(\varphi)$ and $\upsilon(\xi)$, bonded by (360-IY), Ch. 6. With specific analogy (331), (334), we translate them to spherical forms. Now we can give more generally and together the parametric hyperbolic equations of the Minding tractrix and the historically first Huygens tractrix with exchange of their space and time coordinates in $\langle \mathcal{Q}_T^{2+1} \rangle^{\ddagger}$ and $\langle \mathcal{Q}_T^{2+1} \rangle^{\leftrightarrow}$ for the simplest construction of tractricoids I and II with parameter R:

$$\left. \begin{array}{l} \mathbf{x}_R = R \cdot \mathbf{d} = R \cdot [1 - \text{sech}\ \gamma(\upsilon)] \cdot \mathbf{e}_\alpha = R \cdot [1 - \tanh\upsilon(\xi)] \cdot \mathbf{e}_\alpha, \\ y_R = R \cdot h = R \cdot [\gamma(\upsilon) - \tanh\gamma(\upsilon)] = R \cdot [\ln\coth\upsilon(\xi)/2 - \text{sech}\ \upsilon(\xi)]. \end{array} \right\} \quad (103A - III)$$

$$\left. \begin{array}{l} \mathbf{x}_R = R \cdot \mathbf{d} = R \cdot [\gamma(\upsilon) - \tanh\gamma(\upsilon)] \cdot \mathbf{e}_\alpha = R \cdot [\ln\coth\upsilon(\xi)/2 - \text{sech}\ \upsilon(\xi)] \cdot \mathbf{e}_\alpha, \\ y_R = R \cdot h = R \cdot [1 - \text{sech}\ \gamma(\upsilon)] = R \cdot [1 - \tanh\upsilon(\xi)]. \end{array} \right\} \quad (103A - IY)$$

All tractricoids I and tractricoids II with mutually inverse generating tractrices (as both catenoids with generating catenaries) are homothetic with the coefficient of similarity R (to unity ones) in own enveloping Special quasi-Euclidean spaces $\langle Q_T^{2+1}\rangle^{\ddagger}$ and $\langle Q_T^{2+1}\rangle^{\leftrightarrow}$. Both branches of complete Minding and Huygens tractrices are meeting at their cusp points, but daring researchers may use s- and u-shape tractrices as *regular curves* without such points.

Feature: *In process of orthogonal transfer of the parametric evolute into its parametric involute, the principal angle-argument for the first curve is interchanged in complementary one for the second curve (velocities along evolute into supervelocities along involute).*

The exchange $\chi_R \leftrightarrow \mathcal{L}_R$ gives logically $d\mathcal{L}_R/dc\tau_R = c\cdot\tanh\gamma = \sin\varphi(\gamma) = v/c$ due to (103A), but now along the Huygens tractrix. This is similar to the exchange $ct \leftrightarrow c\tau$ at producing time-like catenary in Ch. 5A. Then the Huygens tractrix is involute of a space-like catenary! It is a generatrix for the tractricoid II with topology of the hyperboloid II. It is mapping in the last 4-th Special enveloping quasi-Euclidean space $\langle Q_T^{2+1}\rangle^{\leftrightarrow} \subset \langle Q_C^{2+1}\rangle^{\leftrightarrow}$. The tractricoid II is gotten by revolving the Huygens tractrix around own shortened ordinate time-axis $\pm\overrightarrow{y}_R$ in interval $-R\div+R$. The axis has a pointed cusp top and is directed to center O under asymptotic Euclidean plane.

Further we must take into account, that *initial angle γ with complementary v and their spherical analogs changed above own nature into contrary, by a reason of the evolute-involute metric's transfer*! So, now v is the motion angle with respect to the time-arrow, and γ is complementary to it. Both spherical–hyperbolic analogies in (331) are conserved between γ and φ, γ and ξ in $\tilde{E}_T = \{\chi_R, \overrightarrow{c\tau_R}\}$ beginning from zero point O_1 (i. e., at $\chi_R = 0,\ c\tau_R = 0:\ \gamma = 0,\ \varphi(\gamma) = 0,\ v(\gamma) = \infty,\ \xi(\gamma) = \pi/2$). $\tilde{E}_T$ is universal again for *the same 8 specific functions* $\gamma(\varphi)$, $\varphi(\gamma)$, $\gamma(\xi)$, $\xi(\gamma)$, $v(\xi)$, $\xi(v)$, $v(\varphi)$, $\varphi(v)$ with formulae of simplest differential relations of types (332-III): $d\varphi = \mathrm{sech}\ \gamma\ d\gamma,\ d\gamma = \sec\varphi\ d\varphi$.

We have for kinematic Minding tractrix real and specific connections in tensor forms (Ch. 6):

$$\overline{roth}\ [\Gamma, \Upsilon] \qquad = \qquad roth\ [\Upsilon, \Gamma] \qquad (103A-Y)$$

$$\left|\ \frac{\coth[\gamma,v]\cdot\overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}'}{\mathrm{csch}\ [\gamma,v]\cdot\mathbf{e}'_\alpha}\ \middle|\ \frac{\mathrm{csch}\ [\gamma,v]\cdot\mathbf{e}_\alpha}{\coth[\gamma,v]}\ \right| \begin{matrix}\cdots\\ \cdots\end{matrix} \left|\ \frac{\cosh[v,\gamma]\cdot\overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}'}{\sinh[v,\gamma]\cdot\mathbf{e}'_\alpha}\ \middle|\ \frac{\sinh[v,\gamma]\cdot\mathbf{e}_\alpha}{\cosh[v,\gamma]}\ \right| \equiv$$

$$\overline{def}\ [\Phi, \Xi] \qquad = \qquad def\ [\Xi, \Phi] \qquad (103A-YI)$$

$$\equiv \left|\ \frac{\csc[\varphi,\xi]\cdot\overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}'}{\cot\ [\varphi,\xi]\cdot\mathbf{e}'_\alpha}\ \middle|\ \frac{\cot\ [\varphi,\xi]\cdot\mathbf{e}_\alpha}{\csc[\varphi,\xi]}\ \right| \begin{matrix}\cdots\\ \cdots\end{matrix} \left|\ \frac{\sec[\xi,\varphi]\cdot\overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}'}{\tan[\xi,\varphi]\cdot\mathbf{e}'_\alpha}\ \middle|\ \frac{\tan[\xi,\varphi]\cdot\mathbf{e}_\alpha}{\sec[\xi,\varphi]}\ \right|.$$

That is why, the Minding tractrix gives kinematics of hyperbolic motion, but with the angle v and distance with the angle γ, bonded as one-to-one by relations (360) and with spherical ones in $\tilde{E}_T$ of $\langle Q_T^{2+1}\rangle^{\ddagger}$. Evaluate the kinematics along Minding tractrix (103A), in comparison with one for pro-hyperbola and catenary in Ch. 5A, under new hyperbolic invariant here in $\langle Q_T^{2+1}\rangle^{\ddagger}$:

$$(dc\tau_R)^2 = (d\mathcal{L}_R)^2 - (d\chi_R)^2 \Leftrightarrow\ 1 = \left[\frac{d\mathcal{L}_R}{dc\tau_R}\right]^2 - \left[\frac{d\chi_R}{dc\tau_R}\right]^2 = \coth^2\gamma - \mathrm{csch}^2\gamma \Rightarrow \qquad (103A-YII)$$

$$\left.\begin{aligned} s^* &= \frac{d\chi_R}{d\tau_R} = c\cdot\mathrm{csch}\ \gamma \equiv c\cdot\cot\varphi(\gamma) = \frac{c^2}{v^*} = c\cdot\sinh\ v \equiv c\cdot\tan\xi(v),\\ s &= \frac{d\mathcal{L}_R}{d\tau_R} = c\cdot\coth\ \gamma \equiv c\cdot\csc\varphi(\gamma) = \frac{c^2}{v} = c\cdot\cosh\ v \equiv c\cdot\sec\ \xi(v). \end{aligned}\right\} \to s^2 - (s^*)^2 = c^2$$

As seen from the first expression, here we use as if analogy with STR time invariant. However, in the point of bifurcation at $\gamma = v = \omega(\pi/4)$ (see in sect. 6.4), the space and time slopes of the kinematic tractrix are exchanged. We have the right triangle of supervelocities s^* and s (in term of the angle γ) in other vector space – on the Minkowski hyperboloid I of radius c. In addition, this gives there the Identity for usual velocities: $1/c^2 = 1/v^2 - 1/(v^*)^2$ (see it preliminary in Ch. 5A). So, $s^* = c\cdot\mathrm{csch}\ \gamma$ decreases from ∞ up to 0. If it is expressed through the angle v, then $s^* = c\cdot\sinh v$ increases from 0 up to ∞ as proper velocity in STR.

The Minding tractrix, in process of uniformly accelerated motion along it due to its description in $\tilde{E}_T = \{\chi_R, \overrightarrow{c\tau_R}\}$, asymptotically tends to axis $\overrightarrow{c\tau_R}$. At Figure 2A(4) χ_F is the focus of catenary and tractrix, then $c\tau_{R(F)} + \chi_{R(F)} = kR = c\tau_F - \chi_F$, as here catenary and tractrix have $\varphi(\omega) = \pi/4$ – see at Figure 2A(4). At the tractrix focus χ_F, related to $\gamma_F = \omega = \mathrm{arsinh}\ 1 \approx 0.881$, we have:

$$z_F = \sqrt{2}/2 \approx 0.707,\ h_F = \omega - \sqrt{2}/2 \approx 0.174,\ \mathcal{L}_F = \ln 2/2;\ \ (ds/dh)_F = 1,\ w_F^* = c.$$

In addition, at Figure 2(4) we have the values: $k = d_F - h_F = 1 - \text{sech}\,\gamma_F + \gamma_F - \tanh\gamma_F \approx 0.467$. And from (106A), (105A) and (87A), the useful limit formulae may be easily inferred:
$\lim_{\gamma\to\infty}\chi_R = \lim_{\gamma\to\infty}(c\tau - c\tau_R) = R,\ \lim_{\gamma\to\infty}(\mathcal{L}_R - c\tau_R) = R(1-\ln 2)$, where $c\tau > \mathcal{L}_R > c\tau_R$.

Using connections of hyperbolic motion parameters in (86A), (87A), (91A), (103A) we get for Minding and Huyhens tractrices the 1-st differential arc and the length off zero point O_1, expressed in own Especial quasi-Cartesian bases $\tilde{E}_T$ with orthogonal presentation on tractices own quasiplanes:

$$(d\mathcal{L}_R)^2 = (dc\tau_R)^2 + (d\chi_R)^2 = (R\tanh\gamma\ d\gamma)^2 \to R_T(d\gamma) = R\tanh\gamma \to R_T(d\varphi) = R\tan\varphi(\gamma) \Rightarrow$$

$$\left.\begin{array}{l} d\mathcal{L}_R = R\tanh\gamma\ d\gamma \equiv R\tan\varphi\ d\varphi \equiv dx_R^* = v\ d\tau = \tanh\gamma\ dc\tau \equiv \sin\varphi\ dc\tau, \\ \mathcal{L}_R = R\cdot\mathcal{L} = R\cdot\ln\cosh\gamma \equiv R\cdot\ln\sec\varphi \equiv x^* < c\tau = R\gamma < ct = R\cdot\sinh\gamma. \end{array}\right\} \quad (104A)$$

Here R_T is radius of the tractrix curvature, $-R_T = R_1$ is radius of first principal curvature of the tractricoid I. At $\gamma \to 0$, it is $\mathcal{L}_R \to R\gamma^2/2 = g\tau^2/2$; $g = F/m_0 = c^2/R$ is inner acceleration (81A).

From (103A), with two specific analogies (331), in addition to invariant of time-like pro-hyperbola (93A) and quasi-invariant of catenary (96A), we obtain in $\langle\mathcal{Q}_T^{1+1}\rangle^\ddagger$ the quasi-invariant of the Minding tractrix with its curvature $K_T = -1/ct = -(R\sinh\gamma)^{-1} = -\text{csch}\ \gamma/R \equiv -\cot\varphi/R$ in the Especial quasi-Cartesian base $\tilde{E}_T = \{\chi_R, \overrightarrow{c\tau_R}\}$ along the curve also from zero points C_I and O_1 in hyperbolic and true spherical variants, accordingly with respect to χ_R as $\varphi(\gamma)$ and $\overrightarrow{c\tau_R}$ as $\xi(v)$:

$$(R-\chi_R)^2 + (R\gamma - c\tau_R)^2 = R^2 = R^2\cdot(\text{sech}^2\gamma + \tanh^2\gamma) = R^2\cdot(\tanh^2 v + \text{sech}^2 v) \equiv$$
$$\equiv R^2\cdot[\cos^2\varphi(\gamma) + \sin^2\varphi(\gamma)] = R^2\cdot[\sin^2\xi(v) + \cos^2\xi(v)], \quad (|\chi_R| \le R, \varphi = \pm\pi/2). \quad (105A)$$

The equation is an invariant to orthospherical rotations in $\langle\mathcal{E}^2\rangle \subset \langle\mathcal{Q}_T^{2+1}\rangle^\ddagger$ with the same reflector tensor. Along the tractrix, it is *one-step tangent-secant quasi-invariant of the time-like pro-hyperbola* in $\langle\mathcal{P}^{1+1}\rangle$. In $\langle\mathcal{Q}_T^{1+1}\rangle^\ddagger \subset \langle\mathcal{Q}_T^{2+1}\rangle^\ddagger$ it is one-step sine-cosine quasi-invariant with $\varphi(\gamma)$, expressed by equation of the same tangent circle as to catenary (96A) in point O_I, as situated on a torus around and tangent to the catenoid I and normal to the Tractricoid I – Figure 2A(4). Along this circle the same spherical angle $\varphi(\gamma)$ by analogy (331), but of this tractrix, is summarized! Analogy (331) breaks at $\varphi = \pm\pi/2$ in C_{II}. Both hyperbolic angles are bonded along this tractrix, due to (360-IY).

By rotations of Minding and Huygens tractrices around axes y we get the tractricoid I and II. In their enveloping binary spaces $\langle\mathcal{Q}_T^{2+1}\rangle^\ddagger$ and $\langle\mathcal{Q}_T^{2+1}\rangle^\leftrightarrow$, we have also the Euclidean metric and orthogonal differentiation in universal bases $\tilde{E}_T$ with angles $\varphi(\gamma)$ and $\xi(v) = \pi/2 - \varphi(\gamma)$. Therefore, we can give the hyperbolic and spherical equations of the Minding tractrices and the historically first Huygens tractrices *with exchange of their space-time coordinates visually by seemning rotation at the right angle* $\Pi/2$, with common direction of $y_1 = \overrightarrow{c\tau_R}$ and $y_2 = \overrightarrow{z_R}$ (as in Ch. 5A for catenaries and catenoids), for our simplest presentations of tractricoids with their 1-st metric forms in two variants of parameterization in γ and φ, under bonds of angles in (360), with 3D base $\tilde{E}_T$ of $\langle\mathcal{Q}_T^{2+1}\rangle^\ddagger$. (Due to these equations, the values of both geodesic tractrices radius and length are the same on the tractricoids I and II.) Thus, for the tractricoid I we get 1-st metric form in *vs* presentations:

$$\left.\begin{array}{l} R\cdot\mathbf{z}_{(I)} = R\cdot z_{(I)}\cdot\mathbf{e}_\alpha = R\cdot\text{sech}\ \gamma\cdot\mathbf{e}_\alpha \equiv R\cdot\cos\varphi\cdot\mathbf{e}_\alpha, \\ R\cdot y_{(I)} = c\tau_R = R\cdot(\gamma - \tanh\gamma) \equiv R\cdot[\ln\cot(\pi/2-\varphi)/2] - \sin\varphi]. \end{array}\right\} \quad (106A)$$

$$\left.\begin{array}{l} -R\ d\mathbf{x}_{(I)} = R\ d\mathbf{z}_{(I)} = R\ d[z_{(I)}\cdot\mathbf{e}_\alpha] = R\ d(\text{sech}\ \gamma\cdot\mathbf{e}_\alpha) = R\cdot(-\text{sech}\ \gamma\cdot\tanh\gamma\ d\gamma\cdot\mathbf{e}_\alpha + \text{sech}\ \gamma\ d\alpha\cdot\mathbf{e}_\mu), \\ R\ dy_{(I)} = d(c\tau_R) = R\ d(\gamma - \tanh\gamma) = R\cdot\tanh^2\gamma\ d\gamma. \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} -R\ d\mathbf{x}_{(I)} = R\ d\mathbf{z}_{(I)} = R\ d[z_{(I)}\cdot\mathbf{e}_\alpha) = R\ d(\cos\varphi\cdot\mathbf{e}_\alpha] = R\cdot(-\sin\varphi\ d\varphi\cdot\mathbf{e}_\alpha + \cos\varphi\ d\alpha\cdot\mathbf{e}_\mu), \\ R\ dy_{(I)} = d(c\tau_R) = R\ d[(\gamma(\varphi) - \sin\varphi] = R\cdot(\sec\varphi\ d\varphi - \cos\varphi\ d\varphi) = R\cdot\sin^2\varphi\cdot\sec\varphi\ d\varphi. \end{array}\right\} \Rightarrow$$

$$dl_1^2(\gamma) = [-R_T(d\gamma)]^2 d\gamma^2 + [Rn(\gamma)]^2 d\alpha^2 = R^2(\tanh^2\gamma\ d\gamma^2 + \text{sech}^2\gamma\ d\alpha^2) = R^2\{[d\mathcal{L}_{(I)}(\gamma)]^2 + \text{sech}^2\gamma\ d\alpha^2\} \equiv$$
$$\equiv dl_1^2(\varphi) = [-R_T(d\varphi)]^2 d\varphi^2 + [Rn(\varphi)]^2 d\alpha^2 = R^2(\tan^2\varphi\, d\varphi^2 + \cos^2\varphi\ d\alpha^2) = R^2\{[d\mathcal{L}_{(I)}(\varphi)]^2 + \cos^2\varphi\ d\alpha^2\}$$

We got above real spherical radius of the tractrix in $\langle\mathcal{Q}_T^{2+1}\rangle^\ddagger$ as also radius of the tractricoid I first principal curvatute $R_1 = -R_T(d\varphi) = -R\tan\varphi \equiv -R\sinh\gamma$, with respect to a parallel part of $d\varphi$.

Radius of the second principal curvature of the surface is calculted with the Meusnier Theorem from radius $Rn(\varphi) = R\cos\varphi$ of rotation $d\alpha_1$ under cosine slope ξ to $\mathbf{r}_1$ at Meusnier angle $\xi = \pi/2 - \varphi$. Then $R_2(\varphi) = Rn(\varphi)/\cos\xi = R\cot\varphi \equiv R\ \text{csch}\ \gamma = R\sinh v$, with respect to a normal part of $d\varphi$. With (106A), we reveal two principal rotations at point M on the tractricoid I in $\langle\mathcal{Q}_T^{2+1}\rangle^\ddagger$ as follows

$$R_1(\varphi)\ \overline{\overline{d\varphi}} = -R\tan\varphi\ \overline{\overline{d\varphi}}\ \equiv\ R_1(\gamma)\ \overline{\overline{d\gamma}} = -R\tanh\gamma\ \overline{\overline{d\gamma}} \equiv -R\sinh\gamma\ \overline{\overline{d\varphi}}, \quad (106A-I)$$

$$R_2(\varphi)\ \overset{\perp}{d\varphi} = +R\cot\varphi\ \overset{\perp}{d\varphi} \equiv +R\ \text{csch}\ \gamma\ \overset{\perp}{d\varphi} = +R\sinh v\ \overset{\perp}{d\varphi} = R\cos\varphi\ d\alpha. \quad (106A-II)$$

Then the Gaussian and middle curvatures of the tractricoid I as the Beltrami pseudosphere are $K_G = -1/R^2$, $\overline{K} = \mp 1/2R^{-1}(-\cot\varphi + \tan\varphi) \equiv \mp 1/2R^{-1}(-\sinh v + \sinh\gamma)$ $[\overline{K}(\pi/4) = \overline{K}(\omega) = 0]$.

For the nD Beltrami pseudosphere in the cylindrical coordinates with $\{c\tau_R, z_{1R}, ..., z_{nR}\}$, in vector spatial equation (103A-III), the secant part splits into orthoprojections onto n Euclidean axes of nD Cartesian subbase of $\tilde{E}_T$ in $\langle\mathcal{Q}_T^{n+1}\rangle^\ddagger$ proportionally to directional cosines. We use, beginning from Ch. 5A, here and further such simplest presentations of the metric forms with $\mathbf{e}_\alpha$.

The best trigonometric descriptivity of the spherical presentation with the angle φ is seen at calculation of unusual finite volume and area for the tractricoid I:

$$\left.\begin{aligned} &V = 2\int_0^{\frac{\pi}{2}}[\pi r^2(\varphi)]\,[\sin\varphi\cdot Rd\mathcal{L}(\varphi)] = 2\pi R^3\int_0^{\frac{\pi}{2}}[\cos^2\varphi]\,[\sin\varphi\cdot\tan\varphi\, d\varphi] = \\ &= 2\pi R^3\int_0^{\frac{\pi}{2}}\sin^2\varphi\cos\varphi\, d\varphi = 2\pi R^3\int_0^{\frac{\pi}{2}}\sin^2\varphi]\, d(\sin\varphi) = 2\pi R^3\int_0^{\frac{\pi}{2}}\, d[(\sin^3\varphi/3)] = \tfrac{2}{3}\pi R^3, \\ &S = 2\int_0^{\frac{\pi}{2}}[2\pi r(\varphi)]\,[Rd\mathcal{L}(\varphi)] = 4\pi R^2\int_0^{\frac{\pi}{2}}\cos\varphi\cdot\tan\varphi\, d\varphi = -4\pi R^2\int_1^0\, d(\cos\varphi) = 4\pi R^2. \end{aligned}\right\}\quad (107A)$$

Although the results from (106A) for the tractricoid I were known else from the classic works by Ferdinand Minding [43], however our tensor trigonometric approach gives as well seen the most simplest and descriptive manner of their validation, useful, for example, in the education process.

Single geodesic Minding tractrix with n purely circular extremals exist *at each point* of the *nD Beltrami pseudosphere*, as it has no Poles, similar single time-like hyperbola *at each point* on the Minkowski nD hyperboloid I without Poles. Both objects have identical and constant Gaussian curvature $K_G = -1/R^2$, cylindrical topology and the common *tvs-structure* of their 1-st metric forms. Between these objects, there is isomorphic relation in direction of their ordinate axes $\overrightarrow{y}$ using equal values of γ and α at the Figure 2A(3, 4). Then, according to the Minding Theorem [43], they must be isometric, but only as one-step and only in the universal base for their enveloping spaces, namely, from the side of the tractricoid I. Indeed, all its same metric properties were established above on the basis of only one-step specific spherical–hyperbolic analogy (331). On the hyperboloid I, parallel and normal parts of $d\gamma$ have constant radii of principal curvatures $R_1 = -R$ and $R_2 = +R$ with $R_1R_2 = -R^2 = \text{const}$. On the tractricoid I, they are not constant, but they change so, that their product is rested also constant $R_1R_2 = -R^2 = \text{const}$ at each point of hypersurface. At principal motions on the tractricoid I and the hyperboloid I along the geodesic Minding tractrix and the geodesic hyperbola from equivalent zero points, the angle γ changes from zero till infinity. Generally, at principal motions on both objects from arbitrary, but equivalent points M, we have identical values of their Gaussian curvature. In $\langle\mathcal{Q}_T^{2+1}\rangle^\ddagger$, a parallel part of rotation as $d\varphi$ has the parallel principal curvature $K_1 = -(R\tan\varphi)^{-1}$, a simultaneous normal part of rotation as $d\varphi$ gives the normal principal curvature $K_2 = +(R\cot\varphi)^{-1}$. In $\langle\mathcal{P}^{2+1}\rangle$, a parallel part of rotation as $d\gamma$ has the parallel curvature $K_1 = -R^{-1}$, a simultaneous normal part of rotation as $d\gamma$ gives the normal curvature $K_2 = +R^{-1}$. All they are united in the constant Gaussian curvature $K_G = -R^{-2}$ for the tractricoid I in $\langle\mathcal{Q}_T^{2+1}\rangle^\ddagger$ and for the hyperboloid I in $\langle\mathcal{P}^{2+1}\rangle$ with constant parallel and normal principal curvatures $K_1 = -R^{-1}$ and $K_2 = +R^{-1}$. In their universal base, the Gaussian curvature is identical in Euclidean and pseudo-Euclidean metrics, even in *tvs* presentations (Ch. 10A). In the beginning of principal motions on these two objects, we chose arbitrary equivalent zero points on these objects, which are bonded one-to-one at equal γ and α. *Therefore, we proved: these geometric objects are entirely one-step isometric in their common universal base!*

In Chs. 5A, 6A we revealed geometric meaning of tangent-secant quasi-invariant of progenitor time-like hyperbola translated during transformation from $\langle\mathcal{P}^{n+1}\rangle$ in Special $\langle\mathcal{Q}^{n+1}\rangle$ into catenary and tractrix at Figure 2A(3, 4). In two Special spaces, the quasi-invariant generates a circle tangent at zero point O_1 to catenary (96A) and normal at zero and cusp point O_1 to tractrix (105A). We added the Enveloping Torus bonded at equator the Triad of Hyperboloid I, Catenoid I and Tractricoid I after revolving three generating curves around axis $\overrightarrow{c\tau_R}$ with the similarity coefficient R for additive summation of the principal spherical angle $\varphi(\gamma)$ at contrary specific analogies in (331A).

* * *

From the Pole O of a top part of two-sheets tractricoid II, as its zero, but singular cusp point, in general, n geodesic Huygens tractrices issue. This Pole cannot change own place due to also one-step admissible motions along Huygens tractrices on it similar Minding ones. This differs limited motions on tractricoid II from non-limited ones on hyperboloid II. At evolute-involute transpher of space-like catenary into time-like Huygens tractrix in its quasi-Euclidean space, we get that on tractricoid II, $\varphi(\gamma)$ is the motion angle and also the Meusnier angle between normal to this tractrix and radius $\mathbf{r}$.

For the tractricoid II we get its 1-st metric form with also more informative *vs* presentation:

$$\left.\begin{array}{l} R\cdot \mathbf{z}_{(II)} = R\cdot(\gamma-\tanh\gamma)\cdot \mathbf{e}_\alpha \equiv R\cdot[\ln\cot(\pi/2-\varphi)/2-\sin\varphi]\cdot\mathbf{e}_\alpha,\\ R\cdot y_{(II)} = R\cdot \mathrm{sech}\ \gamma \equiv R\cdot\cos\varphi. \end{array}\right\}\Rightarrow \qquad (108A)$$

$$\left.\begin{array}{l} R\ d\mathbf{z}_{(II)} = R\ d(\gamma-\tanh\gamma)\cdot\mathbf{e}_\alpha = R\cdot[\tanh^2\gamma\ d\gamma\cdot\mathbf{e}_\alpha + (\gamma-\tanh\gamma)\ d\alpha\cdot\mathbf{e}_\nu],\\ R\ dy_{(II)} = R\ d\ \mathrm{sech}\ \gamma = -R\cdot\mathrm{sech}\ \gamma\cdot\tanh\gamma\ d\gamma. \end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{l} Rd\mathbf{z}_{(II)} = Rd[\gamma(\varphi)\cdot\mathbf{e}_\alpha] = Rd[\ln\cot(\pi/2-\varphi)/2\cdot\mathbf{e}_\alpha] = R[\sec\varphi\ d\varphi\cdot\mathbf{e}_\alpha + [\ln\cot(\pi/2-\varphi)/2]\ d\alpha\cdot\mathbf{e}_\nu],\\ Rdy_{(II)} = Rd\cos\varphi = -R\cdot\sin\varphi\ d\varphi. \end{array}\right\}\Rightarrow$$

$$dl_2^2(\gamma) = R^2[\tanh^2\gamma\ d\gamma^2 + (\gamma-\tanh\gamma)^2\ d\alpha^2] = R^2\ \{[d\mathcal{L}_{(II)}(\gamma)]^2 + (\gamma-\tanh\gamma)^2\ d\alpha^2\} \equiv$$

$$\equiv dl_2^2(\varphi) = R^2\{\tan^2\varphi\, d\varphi^2 + [\ln\cot(\pi/2-\varphi)/2-\sin\varphi]^2\ d\alpha^2\} = R^2\{[d\mathcal{L}_{(II)}(\varphi)]^2 + [\ln\cot(\pi/2-\varphi)/2-\sin\varphi]^2\ d\alpha^2\}.$$

* * *

Return to discussing above perfect surfaces (from the set of such with constant Gaussian curvature). Indeed, in its turn, Minkowski hyperboloids I and II have own progenitor as the **hyperspheroid** of radius R in $\langle\mathcal{Q}^{3+1}\rangle$ (see in Figure 4 and Ch. 8A) with scalar or vector presentations in two forms, bonded through complementary angles of motions φ and $\xi = \pi/2-\varphi$ (as the Meusnier angle too):

$$\left.\begin{array}{l} \mathbf{x}_R = R\cdot\mathbf{d} = R\cdot\sin\varphi_i\cdot\mathbf{e}_\alpha = R\cdot\cos\xi_i\cdot\mathbf{e}_\alpha,\\ y_R = R\cdot h = R\cdot\cos\varphi_i = R\cdot\sin\xi_i. \end{array}\right\}(\text{under } I^{\pm} \text{ at } \varphi=+i\gamma \Rightarrow \text{ hyperboloid II}) \Rightarrow \ (109A-II)$$

$$\left.\begin{array}{l} \mathbf{x}_R = R\cdot\mathbf{d} = R\cdot\cos\varphi_i\cdot\mathbf{e}_\alpha = R\cdot\sin\xi_i\cdot\mathbf{e}_\alpha,\\ y_R = R\cdot h = -R\cdot\sin\varphi_i = -R\cdot\cos\xi_i. \end{array}\right\}(\text{under } I^{\mp} \text{ at } \varphi=-i\gamma \Rightarrow \text{ hyperboloid I}) \Rightarrow \ (109A-I)$$

$$d(l_R/R)]^2_{(II)} = d\varphi_p^2 = \cos^2\varphi_p\ d\varphi_p^2 + \sin^2\varphi_p\ d\varphi_p^2 = d\varphi_i^2 + \sin^2\varphi_i\ d\alpha_1^2 = \left(\overline{\overline{d\varphi}}\right)^2_Q + \left(\overset{\perp}{d\varphi}\right)^2_E = d\xi_p^2 = d\xi_i^2 + \cos^2\xi_i\ d\alpha_1^2.$$

$$d(l_R/R)]^2_{(I)} = d\varphi_q^2 = \sin^2\varphi_q\ d\varphi_q^2 + \cos^2\varphi_q\ d\varphi_q^2 = d\varphi_i^2 + \cos^2\varphi_i\ d\alpha_2^2 = \left(\overline{\overline{d\varphi}}\right)^2_Q + \left(\overset{\perp}{d\varphi}\right)^2_E = d\xi_q^2 = d\xi_i^2 + \sin^2\xi_i\ d\alpha_2^2.$$

Geometrically, we can choose any variant (109A –I) or (109A-II) as separate one in own $\langle\mathcal{Q}^{2+1}\rangle$*!* These two *purely angular metric forms*, with own summary motions $d\varphi_p$ and $d\varphi_q$, are compatible in $\langle\mathcal{Q}^{3+1}\rangle$ and given in *normal presentations* as the two *Absolute Euclidean Pythagorean Theorems* – see strictly to the end of Ch. 10A. And its spherical geometry is similar up to the scale coefficient R to the tensor trigonometry of $\langle\mathcal{Q}^{3+1}\rangle$, as all these must be namely for any perfect hypersurfaces! In the base $\tilde{E}_1$, the angle of motion φ is counted from the frame axis $\overrightarrow{y}^{(1)}$ as angular change along some meridian between two Poles. In the left options, we apply the common principal angle of motion φ_i, counted off the hyperspheroid North Pole, as ones do usually for hyperboloids II and I – see such in (132A) and (133A) in Ch. 7A. Any orthospherical differential angles $d\alpha$ are counted along some parallels from the certain choosing zero meridian. However in independent option (109A-I), one may, as is more customary, accept as the principal angle of motion the complementary angle ξ, which is counted from the Euclidean hyperplane, beginning off the Equator at zero value, as contrary angular change along some meridian. *See more in tensor-vector-scalar (tvs) forms in Ch. 10A.*

Binary spaces $\langle\mathcal{P}^{2+1}\rangle$, $\langle\mathcal{Q}^{2+1}\rangle$, $\langle\mathcal{Q}_C^{2+1}\rangle^\updownarrow$, $\langle\mathcal{Q}_C^{2+1}\rangle^\leftrightarrow$ $\langle\mathcal{Q}_T^{2+1}\rangle^\updownarrow$, $\langle\mathcal{Q}_T^{2+1}\rangle^\leftrightarrow$ with reflector tensor $I^\pm$ have the common subgroup of orthospherical rotations $\langle rot\ \Theta\rangle$. In contrast to $\langle\mathcal{Q}^{2+1}\rangle$ with its hyperspheroid and complete rotations group, in $\langle\mathcal{Q}_T^{2+1}\rangle^\updownarrow$ with its Beltrami pseudosphere, the united set of $\langle rot\ \Phi(\Gamma)\rangle$ and $\langle rot\ \Theta\rangle$ is not a group, because only its normal orthospherical part is angular component. Along tractrices we have only non-angular differential $d\mathcal{L}_R(\varphi)$, not relating to rotations! And it is **on this group idea**, we divided in [16] a full set of hypersurfaces of the constant Gaussian curvature into subsets of **perfect** and **imperfect** surfaces with their enveloping spaces, namely, with angular complete 1-st metric forms or not angular ones. The perfect hypersurfaces have own complete groups of motions on integral and differential levels, caused by the fact that such geometric objects have one determined them radius R besides constant Gaussian curvature. Then, with our simplest tensor-differential trigonometric approach, we gave in [16] three *1-st purely angular metric forms* for well-known three surfaces of constant curvature as perfect ones and only for them their *Absolute Pythagorean theorems*, where their orthogonal or pseudo–orthogonal angular differentials are summarized in the complete angular differential! So, the tractricoid I is not a perfect surface and has only one-step integral and differential quasi-invariants under its constant Gaussian curvature. The Einsteinian curved GTR space-time is imperfect, without motion group and even without some sugroups. Poincaré complex space-time $\langle\mathcal{Q}^{3+1}\rangle_c$ and Minkowski space-time $\langle\mathcal{P}^{3+1}\rangle_c$ are perfect with the Lorentzian homogeneous group. Lobachevsky–Bolyai hyperplane and Minkowski hyperboloids are perfect hypersurfaces and, with the hyperboloids II, they are polysteps isometric. For perfect surfaces, complete angular differentials are *polysteps invariants* of their motions groups.

Chapter 7A

Trigonometric models of two-steps, polysteps, and integral non-collinear motions in STR and two hyperbolic geometries

We continue studying two-steps and polysteps principal motions (rotations) $\langle roth\ \Gamma\rangle$ – see in Chs. 11 and 5A. They are analyzed with wide using tensor trigonometry in two directions:
1) The rotations in $\langle \mathcal{P}^{3+1}\rangle \equiv \langle \mathcal{E}^2\rangle \boxtimes \overrightarrow{y} \equiv$ CONST (motions in hyperbolic subspaces), which correspond to the physical flat and spatial movements in STR with their vectors of the directional cosines; and on the embedding into it Minkowskian hyperboloid II (top sheet) at $n = 3$ or in the equivalent to II the Lobachevsky–Bolyai hyperbolic space – see in Ch. 12.
2) The rotations in $\langle \mathcal{P}^{n+1}\rangle \equiv \langle \mathcal{E}^n\rangle \boxtimes \overrightarrow{y} \equiv$ CONST, and motions in the embedding into it nD Minkowskian hyperboloid II as equivalent to nD Lobachevsky–Bolyai hyperbolic space.
We'll pay attention more in detail to Minkowskian hyperboloid I in final part of last Ch. 10A!

Such operations are admitted in $\langle \mathcal{P}^{n+1}\rangle \equiv \langle \mathcal{E}^n\rangle \boxtimes \overrightarrow{y} \equiv$ CONST with right bases here:
1) rotations of the two types, as principal hyperbolic $roth\ \Gamma$ and orthospherical $rot\ \Theta$;
2) parallel translations preserving the space structure with the reflector-tensor $I^{\pm}$.

Hyperbolic and orthospherical rotations have their real canonical forms in $\tilde{E}_1 = \{I\}$. That is why, in polar and summing formulae they are given *initially* in $\tilde{E}_1$, but really they may be translated from $\tilde{E}_1$ into the bases of action $\tilde{E}_k$, due to the Rule of multisteps transformations (Ch. 11). Hyperbolic tensor of motion $roth\ \Gamma$ (100A), on the basis of its pro-tensors (324) and (362), is defined due to conditions (348): $roth\Gamma \cdot I^{\pm} \cdot roth\Gamma = I^{\pm}$. The orthospherical tensor has in $\langle \mathcal{P}^{3+1}\rangle$ and $\langle \mathcal{Q}^{3+1}\rangle$ canonical form (497). Their structures in $\langle \mathcal{P}^{n+1}\rangle$ or space-time $\langle \mathcal{P}^{3+1}\rangle \equiv \langle \mathcal{E}^3 \boxtimes \overrightarrow{ct}\rangle$ correspond to the metric reflector tensor (100A):

$$\{roth\Gamma\}_{4\times 4} = F(\gamma, \mathbf{e}_\alpha) = \cosh\Gamma + \sinh\Gamma \qquad \{rot\ \Theta\}_{4\times 4} \qquad I^{\pm}$$

$$\left[\begin{array}{c|c} \cosh\gamma_i \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' & \sinh\gamma_i \cdot \mathbf{e}_\alpha \\ \hline \sinh\gamma_i \cdot \mathbf{e}'_\alpha & \cosh\gamma_i \end{array}\right] \cdots\cdots \left[\begin{array}{c|c} \{rot\ \Theta\}_{3\times 3} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \end{array}\right] \cdots\cdots \left[\begin{array}{c|c} I_{3\times 3} & \mathbf{0} \\ \hline \mathbf{0}' & -1 \end{array}\right], \qquad (110A)$$

$$(\overleftarrow{\mathbf{e}_\alpha \mathbf{e}'_\alpha} = \mathbf{e}_\alpha \mathbf{e}'_\alpha).$$

In STR, $\{rot\ \Theta\}_{4\times 4}$ may cause induced orthospherical rotations in result summing two or more of the principal rotation $roth\ \Gamma$ as a "Lorentzian boost" (in non-Euclidean geometries, this causes angular shifting in figures). In differential form this causes the induced Thomas precession in time [93]. $\{rot\ \Theta\}$ may be independent also inside the Euclidean subspace (named sometimes by Wigner rotation [94]). See all these in detail in this Chapter.

The *motion tensor* $roth\ \Gamma$ with $\mathbf{e}_\alpha$ in $\tilde{E}_1$ and in another universal base $\tilde{E}_{1u} = rot\ \Theta \cdot \tilde{E}_1$ with $\mathbf{e} = rot\ \Theta_{3\times 3} \cdot \mathbf{e}_\alpha$ has canonical form (362) – see in Ch. 6. The time-arrows $\overrightarrow{ct}^{(k)}$ are used usually as the frame axes for counting the hyperbolic angle γ. At first, we consider two-steps hyperbolic rotations realized as if in $\langle \mathcal{P}^{2+1}\rangle \equiv$ CONST – see above, in order to infer the general law of summing two-steps motions (rotations) or velocities in tensor, vector and scalar forms. The new pseudo-Cartesian base can be represented in $\tilde{E}_1 = \{I\}$ by two ways: with ordering (485) of matrices and in the polar forms (491):

$$\tilde{E}_3 = roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdot \tilde{E}_1 = (roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdot roth^{-1}\Gamma_{12})_{\tilde{E}_2} \cdot roth\ \Gamma_{12} \cdot \tilde{E}_1 =$$

$$= roth\ \Gamma_{13} \cdot rot\ \Theta_{13} \cdot \tilde{E}_1 = (roth\ \Gamma_{13} \cdot rot\ \Theta_{13} \cdot roth^{-1}\Gamma_{13})_{\tilde{E}_{1h}} \cdot roth\ \Gamma_{13} \cdot \tilde{E}_1 = \qquad (111A)$$

$$= rot\ \Theta_{13} \cdot roth\ \overset{\angle}{\Gamma}_{13} \cdot \tilde{E}_1 = (rot\ \Theta_{13} \cdot roth\ \overset{\angle}{\Gamma}_{13} \cdot rot'\Theta_{13})_{\tilde{E}_{1u}} \cdot rot\ \Theta_{13} \cdot \tilde{E}_1 = T_{13} \cdot \tilde{E}_1 = \{T_{13}\}.$$

For subsequent correct derivations, we attach especial importance to a sequence of operations in any multisteps transformations. This issue has already been covered in detail in Ch. 11.

First pairs of matrices in each three rows of (111A) are given initially in the base $\tilde{E}_1 = \{I\}$ in their canonical forms. The second matrix from these pairs is being translated in each rows in the given base of its real action $\tilde{E}_k$. This relates to the two-steps rotation in the first row and to both polar decompositions of the summary matrix T_{13} in the second and third rows with right and contrary ordering of hyperbolic and orthospherical rotations, due to general formulae (485)–(488) and (491) from Ch. 11. So, in the first variant of polar decomposition, $rot\ \Theta_{13}$ has the center of its application in the final point of the rotation $roth\ \Gamma_{13}$.

Corollary. *Generally, two-steps non-collinear hyperbolic rotations* $roth\ \Gamma$ *in* $\langle \mathcal{P}^{n+1} \rangle$ *or on the hyperboloid II can be represented as hyperbolic and induced orthospherical rotations.* Hyperbolic rotations $roth\ \Gamma_{1j}$ are executed in $\langle \mathcal{P}^{n+1} \rangle$ relatively of the frame axis $\overrightarrow{ct}^{(1)}$. Orthospherical rotations are executed in $\langle \mathcal{E}^3 \rangle^{(1h)}$ *for an object or a base* around the axis $\overrightarrow{\mathbf{e}_N}$.

In accordance with (352), the bases $\langle \tilde{E}_{1u} \rangle = \langle rot\ \Theta \cdot \tilde{E}_1 \rangle$ are universal too (in STR, they are called the rest bases). Due to (111A), there holds

$$roth\ \overset{\angle}{\Gamma}_{13} = rot\ (-\Theta_{13}) \cdot roth\ \Gamma_{13} \cdot rot\ \Theta_{13} = rot'\Theta_{13} \cdot roth\ \Gamma_{13} \cdot rot\ \Theta_{13}. \qquad (112A)$$

For $\overset{\angle}{\Gamma}_{13}$, the vector of directional cosines in (363) is shifted with respect to that of Γ_{13} to backwards at Θ_{13}. Moreover, in $\langle \mathcal{P}^{3+1} \rangle$, for hyperbolic non-collinear two-steps rotations, the arising angle of secondary orthospherical shift is realized in its sign contrary to the sign of angle ε between the rotations, i. e., $\theta_{13} < 0$ at $\varepsilon > 0$ due to the Signs Rule from sect 12.2:

$$\mathbf{e}_{\overset{\angle}{\sigma}} = \{rot\ (-\Theta_{13})\}_{3\times 3} \cdot \mathbf{e}_\sigma \ \text{(under rule } \varepsilon > 0 \to \theta_{13} < 0\ !) \ \Rightarrow\ \cos\theta_{13} = \mathbf{e}'_{\overset{\angle}{\sigma}} \cdot \mathbf{e}_\sigma. \qquad (113A)$$

In accordance with (474), (475) and by (111A) and (325), there holds

$$roth\ \Gamma_{13} = \sqrt{TT'} = \sqrt{roth\ \Gamma_{12} \cdot roth\ (2\Gamma_{23}) \cdot roth\ \Gamma_{12}} = \sqrt{roth\ (2\Gamma_{13})}. \qquad (114A)$$

$$rot\ \Theta_{13} = roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdot roth\ \overset{\angle}{\Gamma}_{31} = roth\ \Gamma_{31} \cdot roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} = roth^{-1}\Gamma_{13} \cdot T_{13}. \quad (115A)$$

Formula (115A) represents $rot\ \Theta_{13}$ as the angular defect Θ_{13} of the closed cycle of rotations $roth\ \Gamma_{ij}$ in the hyperbolic triangle 123 with addition of (114A). It is executed from the first point 1 to the final point 3 in the bases of particular rotations along of the triangle legs! If rotations $roth\ \Gamma_{ij}$ are collinear, then the triangle degenerates into the segment γ_{13}.

Further, we shall often use the operation of permutation of particular motions with change of their order into contrary one (for some more simple calculations). In the original universal base $\tilde{E}_1 = \{I\}$, permutation in (111A) of two motions (velocities) leads to a new pseudo-Cartesian base $\tilde{E}'_3 = \{T'_{13}\} = \{T_{13}\}'$:

$$\tilde{E}'_3 = roth\ \Gamma_{23} \cdot roth\ \Gamma_{12} \cdot \tilde{E}_1 = T'_{13} \cdot \tilde{E}_1 =$$

$$= roth\ \overset{\angle}{\Gamma}_{13} \cdot rot\ (-\Theta_{13}) \cdot \tilde{E}_1 = rot\ (-\Theta_{13}) \cdot roth\ \Gamma_{13} \cdot \tilde{E}_1. \qquad (116A)$$

Thus there are two points of view at matrix (112A): as in (111A) and as in (116A)!

In addition to (114A) and (115A), if matrices in $\tilde{E}_1$ are ordered inversely, then

$$roth\ \overset{\angle}{\Gamma}_{13} = \sqrt{T'T} = \sqrt{roth\ \Gamma_{23} \cdot roth\ (2\Gamma_{12}) \cdot roth\ \Gamma_{23}} = \sqrt{roth\ (2\overset{\angle}{\Gamma}_{13})}. \qquad (117A)$$

$$rot\ (-\Theta_{13}) = roth\ \overset{\angle}{\Gamma}_{13} \cdot roth\ \Gamma_{32} \cdot roth\ \Gamma_{21} = roth\ \Gamma_{32} \cdot roth\ \Gamma_{21} \cdot roth\ \Gamma_{13} = T'_{13} \cdot roth^{-1}\Gamma_{13}. \quad (118A)$$

It represents $rot\ (-\Theta)$ for inverse closed cycle (115A) of $roth\ \Gamma_{ij}$ with addition of (117A).

In STR, in general, in binary basis spaces $\langle \mathcal{P}^{3+1} \rangle$, $\langle \mathcal{Q}^{2+1} \rangle$, and even on their *perfect hyperspaces*, the orthospherical angles and motions are used very widely as independent and secondary. They act always within some Euclidean subspaces. Therefore they have Euclidean nature. Their notations can be different, but usually they are Θ, θ, $d\theta$ or $d\alpha$, $d\beta$, and so one! So, the orthospherical rotation may be both the usual independent Euclidean rotational of vectors, for example, of the unity ones $\mathbf{e}_\alpha$ in $\langle \mathcal{E}^3 \rangle$ and the induced rotational angular shift – spherical or hyperbolic of the non-Euclidean or relativistic nature as in the Thomas precession [93].

First the *induced* shift θ in STR was discussed by É. Borel in 1913 [90] and L. Silberstein in 1914 [92], as a consequence of principal Lorentzian transformations non-commutativity. In 1913, L. Föppl and P. Daniell – theorists from Göttingen have inferred physical formula for it as a possible induced precession $d\theta/dt$ [91]. In 1926 L. Thomas gave relativistic STR-interpretation [93] of the experimental coefficient $1/2$ to the additional electron spin due to such an induced precession. This event was first convincing and obvious confirmations of STR *with its group type transformations*, because the experimental coefficient $1/2$ had no other interpretation! In 1928 the *Thomas precession* have got general interpretation in the STR-invariant Quantum wave equation of Paul Dirac [101] in the Minkowski space-time.

The angles Γ_{13} and $\overset{\angle}{\Gamma}_{13}$ differ only by their vectors of directional cosines. Due to (491) or (112A), the scalar summary hyperbolic angle does not depend on ordering of summands (direct or inverse). The case when the directional cosines of motions are either equal or additively opposite to each other, corresponds to collinear motions.

Let $\mathbf{e}_\alpha = \{\cos\alpha_k,\ k = 1, 2, 3\}$ be the vector of directional cosines for Γ_{12}, $\mathbf{sinh}\ \gamma_{12}$, $\mathbf{tanh}\ \gamma_{12}$, and $\mathbf{v}_{12}$ in the Cartesian subbase $\tilde{E}_1^{(3)}$; $\mathbf{e}_\beta = \{\cos\beta_k,\ k = 1, 2, 3\}$ be the vector of directional cosines for Γ_{23}, $\mathbf{sinh}\ \gamma_{23}$, $\mathbf{tanh}\ \gamma_{23}$, and $\mathbf{v}_{23}$ in the Cartesian subbase $\tilde{E}_2^{(3)}$. Define as the *conditional characteristic*, the *orthospherical* angle ε between directions $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$ as if they are in the same subspace $\langle \mathcal{E}^3 \rangle$ by the following formal value of its cosine:

$$\cos\varepsilon = \begin{bmatrix} \cos\beta_1 \\ \cos\beta_2 \\ \cos\beta_3 \end{bmatrix}' \cdot \begin{bmatrix} \cos\alpha_1 \\ \cos\alpha_2 \\ \cos\alpha_3 \end{bmatrix} = \mathbf{e}'_\beta \mathbf{e}_\alpha = \mathbf{e}'_\alpha \mathbf{e}_\beta, \qquad 0 \le |\varepsilon| \le \pi \qquad (119A).$$

Here $\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = \cos^2\beta_1 + \cos^2\beta_2 + \cos^2\beta_3 = 1$. If the partial cosines are pairly equal, then $\cos\varepsilon = 1$. If they are pairly additively opposite, then $\cos\varepsilon = -1$. Thus, in these cases, $\mathbf{v}_{12}$ and $\mathbf{v}_{23}$ are conventionally collinear, with the same or opposite directions. If $\cos\varepsilon = 0$, then $\mathbf{v}_{12}$ and $\mathbf{v}_{23}$ are conventionally orthogonal. In general, they form the conventional angle ε (as $\mathbf{v}_{12}$ and $\mathbf{v}_{23}$ is in different Euclidean spaces).

We suppose the invariant $R = 1$ in tensor trigonometric approach to STR and geometries.

Further, evaluate the final hyperbolic matrix $roth\ \Gamma_{13}$ with the use of (114A), in that number, the eigen angle γ_{13} in the original base E_1 and directional cosines $\cos\sigma_k$, $k = 1, 2, 3$, of $roth\ \Gamma_{13}$ in the Cartesian subbase $\tilde{E}_1^{(3)}$. For rotations (motions) in the inverse order, the scalar angle of summary rotation (motion) $roth\ \overset{\angle}{\Gamma}_{13}$ is the same γ_{13} according to (112A). The directional cosines of $roth\ \overset{\angle}{\Gamma}_{13}$ are $\cos\overset{\angle}{\sigma}_k$, $k = 1, 2, 3$. By (113A), we obtain

$$\cos\theta_{13} = \begin{bmatrix} \cos\overset{\angle}{\sigma}_1 \\ \cos\overset{\angle}{\sigma}_2 \\ \cos\overset{\angle}{\sigma}_3 \end{bmatrix}' \cdot \begin{bmatrix} \cos\sigma_1 \\ \cos\sigma_2 \\ \cos\sigma_3 \end{bmatrix} = \mathbf{e}'_{\overset{\angle}{\sigma}} \cdot \mathbf{e}_\sigma = \cos\theta_{13} = \mathbf{e}'_\sigma \cdot \mathbf{e}_{\overset{\angle}{\sigma}}, \qquad (120A),$$

where $\sin\theta_{13} < 0$, if $\varepsilon > 0$. See **Rule** *for the sign of* θ_{13} in sect 12.2 of Part I.

For transformations in the direct and inverse variants of two-steps motions in angular Lambert measure ($\gamma = \lambda/R$), both they are connected by substitutions of partial angles as:

$$\gamma_{12} \leftrightarrow \gamma_{23}, \qquad \alpha_k \leftrightarrow \beta_k, \quad (\text{but } \gamma_{13} = \text{const}). \tag{121A}$$

Note right away, that the very wonderful in STR and non-Euclidean geoetries is next: at summing motions (rotations) we can combine formally their directional unity vectors $\mathbf{e}_\alpha$ in $\langle\mathcal{E}^3\rangle^{(1)}$ and $\mathbf{e}_\beta^{(m)}$ in $\langle\mathcal{E}^3\rangle^{(m)}$, as if in $\langle\mathcal{E}^2\rangle^{(1)} \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle^{(1)}$ or as if in $\langle\mathcal{E}^2\rangle^{(m)} \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle^{(m)}$!

In (111A), block-to-block multiplication of matrices with structure (363) is unwieldy. Though in last Ch. 10A, we'll realize it by simplest universal manner! Below we use for two-steps motions quite simple way. At first, let us evaluate the matrix product in (114A) as:

$B = \{roth\ \Gamma_{12} \cdot roth\ (2\Gamma_{23})\} = \{b_{ij}\}$.

For tensor trigonometric analysis of two-steps hyperbolic motions or two-steps relativistic velocities in STR, it is enough to use $\mathbf{3 \times 3}$ modal matrices from (111A). But for generality we use 4×4 (or $(n+1) \times (n+1)$) matrices! Only fourth row of B is used in next computations. The matrices $roth\ \Gamma$ must be used in any of canonical forms (362), (363). Then we obtain:

$$\begin{aligned}
b_{41} &= [\sinh\gamma_{12} \cdot \cosh(2\gamma_{23}) \cdot \cos\varepsilon + \cosh\gamma_{12} \cdot \sinh(2\gamma_{23})] \cdot \cos\beta_1 + \\
&\qquad + \sinh\gamma_{12} \cdot (\cos\alpha_1 - \cos\varepsilon \cdot \cos\beta_1), \\
b_{42} &= [\sinh\gamma_{12} \cdot \cosh(2\gamma_{23}) \cdot \cos\varepsilon + \cosh\gamma_{12} \cdot \sinh(2\gamma_{23})] \cdot \cos\beta_2 + \\
&\qquad + \sinh\gamma_{12} \cdot (\cos\alpha_2 - \cos\varepsilon \cdot \cos\beta_2), \\
b_{43} &= [\sinh\gamma_{12} \cdot \cosh(2\gamma_{23}) \cdot \cos\varepsilon + \cosh\gamma_{12} \cdot \sinh(2\gamma_{23})] \cdot \cos\beta_3 + \\
&\qquad + \sinh\gamma_{12} \cdot (\cos\alpha_3 - \cos\varepsilon \cdot \cos\beta_3), \\
b_{44} &= \sinh\gamma_{12} \cdot \sinh(2\gamma_{23}) \cdot \cos\varepsilon + \cosh\gamma_{12} \cdot \cosh(2\gamma_{23}).
\end{aligned}$$

At the beginning, we evaluate the diagonal corner element s_{44} of the symmetric matrix $S = roth^2\ \Gamma_{13} = roth\ (2\Gamma_{13})$ multiplying the 4-th row of B and the 4-th column of $roth\ \Gamma_{12}$:

$$s_{44} = \cosh(2\gamma_{13}) = \cos(2i\gamma_{13}) = \cos^2 i\gamma_{13} - \sin^2 i\gamma_{13} = \cosh^2\gamma_{13} + \sinh^2\gamma_{13} = 2\cosh^2\gamma_{13} - 1 =$$

$$= \cosh(2\gamma_{12}) \cdot \cosh(2\gamma_{23}) + \cos\varepsilon \cdot \sinh(2\gamma_{12}) \cdot \sinh(2\gamma_{23}) - 2\sin^2\varepsilon \cdot \sinh^2\gamma_{12} \cdot \sinh^2\gamma_{23} =$$

$$= 2(\cosh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\varepsilon \cdot \sinh\gamma_{12} \cdot \sinh\gamma_{23})^2 - 1.$$

We get 1-st *commutative scalar* cosine formula for summing two rotations in $\langle\mathcal{P}^{n+1}\rangle$ or two hyperbolic motions on hyperboloid-II and in Lobachevsky–Bolyai non-Euclidean geometry:

$$\cosh\gamma_{13} = \cosh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\varepsilon \cdot \sinh\gamma_{12} \cdot \sinh\gamma_{23} = \tag{122A − I}$$

$$= \cos i\gamma_{12} \cdot \cos i\gamma_{23} - \cos\varepsilon \cdot \sin i\gamma_{12} \cdot \sin i\gamma_{23} = \tag{122A − II}$$

$$\cosh\gamma_{13} = \cosh\gamma_{12} \cdot \cosh\gamma_{23} - \cos\ A_{123} \cdot \sinh\gamma_{12} \cdot \sinh\gamma_{23} = \tag{122A − III}$$

We use in (122A-I) and in the subsequent formulae for summation of two-steps hyperbolic and spherical (Ch. 8A) motions or identical rotations the so-called external orthospherical angle ε between segments **12** and **23**, similar as was adopted for relativistic summing two velocities by Einstein in 1905 [67]. Although in non-Euclidean geometries for the triangle **123**, the geometers use the internal angle A_{123} between segments **12** and **23**, beginning from the Euler spherical geometry. They are complementary and bonded as $\varepsilon = \pi - A \rightarrow d\varepsilon = -dA$ with differences of signs in (122-I), (122-II) and (122-III). However in the Euler flat scalar trigonometry, for rotational summation of two angles, in fact the external angle between them is used. Therefore, such an external angle is universal one! We illustrate at Figure 4A the internal angle as **123** at top **2** and the external angle as **2′23** at top **2** without distortions.

The acute or obtuse or right or zero angle ε between hyperbolic segments 12 and 23, due to (119A) and to property of their directive unity vectors, has a pure *orthospherical* nature as well as the angle θ in (120A) and all angles at the tops of geometric figures on the non-Euclidean "perfect surfaces" (i. e., with constant radius-parameter R), for example, on the hyperspheroid and Minkowskian hyperboloids II and I. As the geometric property, ε is the external angle for two Euclidean orthoprojections of the hyperbolic curvilinear segments 12 and 23 in the base $\tilde{E}_1 = \{I\}$. *How the sum of the external angles of a hyperbolic triangle is distorted – see below in analysis of the Lambert angular defect.* For relativistic motions in $\langle \mathcal{P}^{3+1} \rangle$, there holds $\gamma_{ij} > 0$, as in STR the same angle between velocities corresponds to the inequality $\Delta ct > 0$ (motions to future). It relates to upper parts of both hyperboloids.

For the motions angles as segments $\gamma = \lambda / R$ on hyperboloid-II and Lobachevsky–Bolyai non-Euclidean plane, their lengths by Lambert measure follow to the *Rule of a parallelogram*

$$|\gamma_{12} - \gamma_{23}| \le \gamma_{13} \le \gamma_{12} + \gamma_{23}, \quad (\varepsilon \in [0; \pi]), \tag{123A}$$

similar to one in Euclidean geometry. For angles of motion or their trigonometric projections in Euclidean subspaces, their directional cosines range is $[-1; +1]$. Due to unequalities $\gamma > 0$ and (123A), distance in hyperbolic geometry by the measure γ is a norm too.

Due to (122A) and following scalar formulae, only scalar values of summary γ_{13} or v_{13} does not depend on summands ordering. Besides, due to (111A), the *complete law of summation* for two or more modal transformations of bases (as here geometric motions) or immediately for segments or velocities summation must include the induced orthospherical rotation *rot* Θ_{13}. And *namely the tensor trigonometric approach give us such complete law*. We'll consider the mentioned immediate summation of segments to the end of Ch. 10A.

The scalar value of sine is evaluated from (122A-II and I) exactly and simply, including two commutative variants, as the *mirror Pythagorean sums* provide that $\gamma_{12} \leftrightarrow \gamma_{23}$:

$$\cos^2 i\gamma_{13} = \cos^2 i\gamma_{12} \cdot \cos^2 i\gamma_{23} - 2\cos\varepsilon \cdot \cos i\gamma_{12} \cdot \cos i\gamma_{23} \cdot \sin i\gamma_{12} \cdot \sin i\gamma_{23} + \cos^2\varepsilon \cdot \sin^2 i\gamma_{12} \cdot \sin^2 i\gamma_{23} \Rightarrow$$

$$\cosh^2\gamma_{13} = \cosh^2\gamma_{12} \cdot \cosh^2\gamma_{23} + 2\cos\varepsilon \cdot \cosh\gamma_{12} \cdot \cosh\gamma_{23} \cdot \sinh\gamma_{12} \cdot \sinh\gamma_{23} + \cos^2\varepsilon \cdot \sinh^2\gamma_{12} \cdot \sinh^2\gamma_{23} \Rightarrow$$

$$\sin^2 i\gamma_{13} = 1 - \cos^2 i\gamma_{13} \Rightarrow \sinh^2\gamma_{13} = \cosh^2\gamma_{13} - 1 \quad \Rightarrow \quad \sinh^2\gamma_{13} = (v^*_{13}/c)^2 =$$

$$= \sinh^2\gamma_{12} + \sinh^2\gamma_{23} + (1+\cos^2\varepsilon) \cdot \sinh^2\gamma_{12} \cdot \sinh^2\gamma_{23} + 2\cos\varepsilon \cdot \cosh\gamma_{12} \cdot \sinh\gamma_{12} \cdot \cosh\gamma_{23} \cdot \sinh\gamma_{23} =$$

$$= (\cosh\gamma_{23} \cdot \sinh\gamma_{12} + \cos\varepsilon \cdot \cosh\gamma_{12} \cdot \sinh\gamma_{23})^2 + (\sin\varepsilon \cdot \sinh\gamma_{23})^2 =$$

$$= (\cosh\gamma_{12} \cdot \sinh\gamma_{23} + \cos\varepsilon \cdot \cosh\gamma_{23} \cdot \sinh\gamma_{12})^2 + (\sin\varepsilon \cdot \sinh\gamma_{12})^2 = \sinh^2\gamma_{13}. \tag{124A}$$

The scalar value of tangent is evaluated trigonometrically with the combine use of (122A) and (124A) also commutatively, as the *mirror Pythagorean sums* provide that $\gamma_{12} \leftrightarrow \gamma_{23}$:

$$\tanh^2\gamma_{13} = (v_{13}/c)^2 = \tag{125A}$$

$$= \left[\frac{\tanh\gamma_{12} + \cos\varepsilon \cdot \tanh\gamma_{23}}{1 + \cos\varepsilon \cdot \tanh\gamma_{12} \cdot \tanh\gamma_{23}}\right]^2 + \left[\frac{\sin\varepsilon \cdot \operatorname{sech}\,\gamma_{12} \cdot \tanh\gamma_{23}}{1 + \cos\varepsilon \cdot \tanh\gamma_{12} \cdot \tanh\gamma_{23}}\right]^2 =$$

$$= \left[\frac{\tanh\gamma_{23} + \cos\varepsilon \cdot \tanh\gamma_{12}}{1 + \cos\varepsilon \cdot \tanh\gamma_{12} \cdot \tanh\gamma_{23}}\right]^2 + \left[\frac{\sin\varepsilon \cdot \operatorname{sech}\,\gamma_{23} \cdot \tanh\gamma_{12}}{1 + \cos\varepsilon \cdot \tanh\gamma_{12} \cdot \tanh\gamma_{23}}\right]^2.$$

Middle in (124A) and first in (125A) variants are most preferred in following applications.

From (125A), with $\tanh\gamma = v/c$ and after it reducing, the Poincaré–Einstein relativistic Law of two non-collinear velocities summation follows [63, 67; 76, p. 34]. Below it is given in the clear *tangent form*, but without matrix with angle of orthospherical shift (see further):

$$\tanh\gamma_{13} = \frac{\sqrt{\tanh^2\gamma_{12} + \tanh^2\gamma_{23} + 2\cos\varepsilon \cdot \tanh\gamma_{12} \cdot \tanh\gamma_{23} - \sin^2\varepsilon \cdot \tanh^2\gamma_{12} \cdot \tanh^2\gamma_{23}}}{1 + \cos\varepsilon \cdot \tanh\gamma_{12} \cdot \tanh\gamma_{23}}. \tag{126A}$$

Scalar reverse secant commutative variant of two non-collinear segments summation is expressed from (122A) in terms of *relativistic factors* [76, p. 35], which we give here and below in pure trigonometric form:

$$\mathrm{sech}\gamma_{13} = \sqrt{1-\tanh^2\gamma_{13}} = \frac{\mathrm{sech}\gamma_{12}\cdot\mathrm{sech}\gamma_{23}}{1+\cos\varepsilon\cdot\tanh\gamma_{12}\cdot\tanh\gamma_{23}}. \qquad (127A)$$

Let's pay attention to the fact that all expressions above for two-steps summing motions or velocities with paired trigonometric functions were gotten by us through multiplication of tensors of 1-st and 2-nd motions in (111A), and the hyperbolic part with these functions was revealed by polar decomposition of this product. Its residue is the induced here especial spherical shift, which we'll represented later.

If $\cos\varepsilon = \pm 1$, formulae (122A), (124A), (125A) give the additive rules (69A)-(72A). Generally, in (124A) and (125A), we see two (sine and tangent) *Big Pythagorean Theorems*, they will be interpreted later on the space-like hyperboloid II with the Lobachevsky–Bolyai geometry. But if $\cos\varepsilon = 0$, then for two conventionally orthogonal hyperbolic segments we get two (sine and cosine) *Small Pythagorean Theorems*:

$$\cosh\gamma_{13} = \cosh\gamma_{12}\cdot\cosh\gamma_{23} \;\Leftrightarrow\; \mathrm{sech}\;\gamma_{13} = \mathrm{sech}\;\gamma_{12}\cdot\mathrm{sech}\;\gamma_{23}. \qquad (128A-I)$$

$$\sinh^2\gamma_{13} = (\cosh\gamma_{23}\cdot\sinh\gamma_{12})^2 + \sinh^2\gamma_{23} = (\cosh\gamma_{12}\cdot\sinh\gamma_{23})^2 + \sinh^2\gamma_{12}. \qquad (129A-I)$$

$$\tanh^2\gamma_{13} = (\mathrm{sech}\;\gamma_{23}\cdot\tanh\gamma_{12})^2 + \tanh^2\gamma_{23} = (\mathrm{sech}\;\gamma_{12}\cdot\tanh\gamma_{23})^2 + \tanh^2\gamma_{12}. \qquad (130A-I)$$

In 3D Euclidean space, not more than three vectors can be orthogonal. Perform sequentially two operations of three conventionally orthogonal segments summing, we obtain three-steps scalar commutative formulae:

$$\cosh\gamma_{14} = \cosh\gamma_{12}\cdot\cosh\gamma_{23}\cdot\cosh\gamma_{34} \Leftrightarrow \mathrm{sech}\gamma_{14} = \mathrm{sech}\;\gamma_{12}\cdot\mathrm{sech}\;\gamma_{23}\cdot\mathrm{sech}\;\gamma_{34}. \qquad (128A-II)$$

$$\begin{gathered}\sinh^2\gamma_{14} = \sinh^2\gamma_{12} + \sinh^2\gamma_{23} + \sinh^2\gamma_{34} + \\ +\sinh^2\gamma_{12}\cdot\sinh^2\gamma_{23} + \sinh^2\gamma_{12}\cdot\sinh^2\gamma_{34} + \sinh^2\gamma_{23}\cdot\sinh^2\gamma_{34} + \sinh^2\gamma_{12}\cdot\sinh^2\gamma_{23}\cdot\sinh^2\gamma_{34} = \\ = (\cosh\gamma_{34}\cdot\cosh\gamma_{23}\cdot\sinh\gamma_{12})^2 + (\cosh\gamma_{34}\cdot\sinh\gamma_{23})^2 + \sinh^2\gamma_{34} = \\ = (\cosh\gamma_{12}\cdot\cosh\gamma_{23}\cdot\sinh\gamma_{34})^2 + (\cosh\gamma_{12}\cdot\sinh\gamma_{23})^2 + \sinh^2\gamma_{12}. \qquad (129A-II)\end{gathered}$$

$$\begin{gathered}\tanh^2\gamma_{14} = \tanh^2\gamma_{12} + \tanh^2\gamma_{23} + \tanh^2\gamma_{34} - \\ -(\tanh^2\gamma_{12}\cdot\tanh^2\gamma_{23} + \tanh^2\gamma_{12}\cdot\tanh^2\gamma_{34} + \tanh^2\gamma_{23}\cdot\tanh^2\gamma_{34}) + \tanh^2\gamma_{12}\cdot\tanh^2\gamma_{23}\cdot\tanh^2\gamma_{34} = \\ = (\mathrm{sech}\;\gamma_{34}\cdot\mathrm{sech}\;\gamma_{23}\cdot\tanh\gamma_{12})^2 + (\mathrm{sech}\;\gamma_{34}\cdot\tanh\gamma_{23})^2 + \tanh^2\gamma_{34} \qquad (130A-II)\end{gathered}$$

These formulae for functions of the summary angle in $\langle\mathcal{P}^{n+1}\rangle$ may be always presented in the quadric form as a sum of n quadrates by $n!$ identical variants. (We give only one last example in (129A) and in (130A) in the direct order of the motions from six variants.) If in these summation formulae at least one of the particular angles is infinite ($\gamma_{ij} = \infty$, $\tanh\gamma_{ij} = 1$ or $v_{ij} = c$), then the final angle is infinite too. This result corresponds to the Einstein's Velocity Postulate (15A), but only as the consequence of STR in $\langle\mathcal{P}^{3+1}\rangle$.

As generalization of multiplicative cosine variants (128A) for a lot of the conventionally orthogonal hyperbolic segments $\gamma_{(k)}$ in the n-dimensional Lobachevsky–Bolyai space or on the n-dimensional Minkowski hyperboloid II in $\langle\mathcal{P}^{n+1}\rangle$, this simplest multiplicatively commutative scalar cosine formula is realized in the base $\bar{E}_1 = \{I\}$ as follows:

$$\cosh\gamma = \prod_{k=1}^{t}\cosh\gamma_{(k)}, \quad \gamma = \lambda/R = \mathrm{arcosh}\left(\prod_{k=1}^{t}\cosh\gamma_{(k)}\right); \quad \varepsilon_{(k)} = \pm\pi/2\;. \qquad (131A-I)$$

Scalar summary γ does not depend on ordering of conventionally orthogonal partial angles, with the relativistic law of $t \le n$ orthogonal velocities summation. If all these t segments are infinitesimal, then the *Infinitesimal polysteps Pythagorean Theorem* holds, even for non-conventionally orthogonal infinitesimal hyperbolic segments on the hyperboloid II:

$$\gamma^2 = \overline{\gamma^2_{(k)}}\;(\gamma_{(k)} = \lambda_{(k)}/R \to 0), \quad (\varepsilon_{ij} = \pm\pi/2\;\text{ for n space axes }\overrightarrow{x}^{(k)}),\, k = 1,2,...,n. \qquad (131A-II)$$

Let in (131A-I) differential angles $d\gamma$ and $d\gamma_{(k)}$ instead each angle itself. Take into account decomposition $\cosh\gamma \to 1 + \gamma^2/2 + \cdots$. Realize it as substitution in (131A). Now, let us use the unusual representation $\cosh d\gamma = 1 + (d\gamma)^2/2$ for differentials of the 1-st order. Then, we get the following commutative formulae for independent orthogonal hyperbolic differential ($k \le n$) and proportional inner accelerations ($k \le 3$)

$$d\gamma = \sqrt{d\gamma_1^2 + ... + d\gamma_k^2} \;\Rightarrow\; g = c(d\gamma/d\tau) = \sqrt{g_1^2 + ... + g_k^2};\; (\varepsilon_{ij} = \pm\pi/2),\; k = 1,2,...,n. \qquad (131A-III)$$

with the *non-relativistic law* of inner accelerations $g_{(k)}$ summation on the hyperboloid II of accelerations (see it also in the beginning of Ch. 9A). These two Pythagorean theorems in STR for such proportional angular 1-st differentials and inner accelerations (all applied in $\bar{E}_M$ at its zero point M) are valid also from sine quadrics (129A-I and II), transforming them into quadrics for $d\gamma_k$ and inner accelerations $g_{(k)}$.

Let us turn to the Minkowski hyperboloids II and I – see initially about them in Ch. 12 with their projections for visualization onto common pseudoplane at Figure 4. They are main geometric objects with radius-parameter R (but of radii iR and $\pm R$) in the pseudo-Euclidean space $\langle \mathcal{P}^{3+1} \rangle$ by Minkowski or as his space-time (Ch. 12). For them, radius-parameter R is also their coefficient of similarity to the *trigonometric hyperboloids II and I* at $R = 1$, because both hyperboloids are *perfect curvilinear 3D hypersurfaces* in $\langle \mathcal{P}^{3+1} \rangle$ (or 2D in $\langle \mathcal{P}^{2+1} \rangle$) with the admissible polysteps motions on them from the continuous Lorentz group, equivalent to the group of rotations in enveloping $\langle \mathcal{P}^{3+1} \rangle$.

Now we discuss their geometric peculiarities and metric forms gotten in vector presentations by descriptive method used us for catenoids and tractricoids in previous two Chapters, as both hyperboloids are connected also by visual rotation at angle $\Pi/2$ in projections onto the pseudoplane. In the STR with trigonometric hyperboloids, due to the Poincaré brilliant idea [63], we may use his complex space-time $\langle \mathcal{Q}^{3+1} \rangle_c$ *with Euclidean metric tensor* $\{I\}$, *imaginary time-arrow and angles* $i\gamma$ *in tensor* $\boldsymbol{roth}\ \boldsymbol{i}\Gamma^{(m)} = F(i\gamma, \mathbf{e}_\alpha)$ (100A). **In metric forms' signs, we trust to ones in** $\langle \mathcal{Q}^{3+1} \rangle_c$**!**

The **hyperboloid II** of two coupled sheets having with the metric tensor $\{+I\}$ of $\langle \mathcal{Q}^{3+1} \rangle_c$ two antipodal Lobachevsky–Bolyai geometries with radius $\mp R$ (Ch. 12) is seeming as *symmetric cups* – see else in (149A). Its time-like principal pseudonormal, space-like tangent and sine binormal are:

$$\mathbf{p}_{(II)} = \mathbf{r}_{(II)} = \begin{bmatrix} \sin i\gamma_j \cdot \mathbf{e}_\alpha \\ \cos i\gamma_j \end{bmatrix} = -[\mathbf{p}_{(I)}]'_\alpha = \mathbf{i}_{(I)} = \mathbf{i}, \quad \mathbf{i}_{(II)} = +[\mathbf{i}_{(I)}]'_\alpha = \mathbf{p}_{(I)} = \mathbf{p}, \quad \mathbf{b}_{(II)} = \begin{bmatrix} \mathbf{e}_\nu \\ 0 \end{bmatrix}.$$

From Pole point C_{II} on II in $\langle \mathcal{Q}^{n+1} \rangle_c$ at Figure 4, we have *space-like* geodesic meridians $\mathcal{H}_R = R\gamma_j$ with $d\mathcal{H}_R = Rd\gamma_j$ (under $d\alpha = 0$), radii of curvature iR and of revolution $Rn_2 = r = iR \sin i\gamma_j$. The 1-st angular metric form of II with the *Euclidean Absolute Pythagorean theorem* reduces mixed motion with $d\alpha_1$ on II as a perfect surface to hyperboloidal angular arc along hypotenuse $Rd\gamma_{\mathbf{p}}$:

$$\left.\begin{aligned} \mathbf{x}_{(II)} &= x_{(II)} \cdot \mathbf{e}_\alpha = iR \cdot \sin i\gamma_j \cdot \mathbf{e}_\alpha, \\ y_{(II)} &= ct = iR \cdot \cos i\gamma. \end{aligned}\right\} \text{[here from its top Pole – see in detail in (225A–228A), Ch. 10A]}$$

$$\left.\begin{aligned} d\mathbf{x}_{(II)} &= d(x_{(II)} \cdot \mathbf{e}_\alpha) = iR\, d(\sin i\gamma_j \cdot \mathbf{e}_\alpha) = iR \cdot (\cos i\gamma_j\ di\gamma_j \cdot \mathbf{e}_\alpha + \sin i\gamma_j\ d\alpha_1 \cdot \mathbf{e}_\nu), \\ dy_{(II)} &= d(ct) = iR\, d\cos i\gamma_j = R \cdot \sin i\gamma_j\ d\gamma_j. \end{aligned}\right\} \Rightarrow$$

$$[d\lambda_{(II)}]^2 = (iR)^2 di\gamma_j^2 + [Rn_2(i\gamma_j)]^2 d\alpha_1^2 = (iR^2)\, di\gamma_j^2 + (iR^2)\sin^2 i\gamma_j\, d\alpha_1^2 = R^2 d\gamma_{\mathbf{p}}^2 \equiv ||d\mathbf{x}_{(II)}||^2 + ||dy_{(II)}||^2; \Rightarrow$$

$$[d\lambda_{(II)}/R]^2 = d\gamma_{\mathbf{p}}^2 = (\sin i\gamma_j\ d\gamma_j)_Y^2 + [(\cos i\gamma_j\ d\gamma_j)^2 + (i \sin i\gamma_j\ d\alpha_1)^2]_X = \cos^2 i\gamma_j\ d\gamma_j^2 + \sin^2 i\gamma_j\ (d\gamma_j^2 - d\alpha_1^2) \Rightarrow$$

$$d\gamma_{\mathbf{p}}^2 = [i\ di\gamma_{\mathbf{p}}]^2 = \cosh^2\gamma_{\mathbf{p}}\ d\gamma_{\mathbf{p}}{}^2 - \sinh^2\gamma_{\mathbf{p}}\ d\gamma_{\mathbf{p}}{}^2 = d\gamma_j^2 + \sinh^2\gamma_j\ d\alpha_1^2 = \left(\overline{\overline{d\gamma_{\mathbf{p}}}}\right)_P^2 + \left(\overset{\perp}{d\gamma_{\mathbf{p}}}\right)_E^2 > 0. \quad (132A)$$

Hyperbolic space-like meridian arc $Rd\gamma_j$ along tangent $\mathbf{i}_{(II)}$ with mutual γ_j is accompanied by orthogonal arc $R \sinh\gamma_j\ d\alpha \cdot \mathbf{e}_\nu$ at binormal $\mathbf{b}_\nu$, caused by our motion tensor $\boldsymbol{rot}\ \boldsymbol{i}\Gamma_j = F(i\gamma_j, \mathbf{e}_\alpha)$. At arbitrary point M of II, instead infinitesimal angles $\gamma_{(k)}$ as in (131A-II), we can by (131A-I) with exactness up to 2-nd order introduce n independent space-like differentials $d\gamma_{(k)}$ as Euclidean orthoprojections of the total differential $d\gamma$ applied in $\tilde{E}_M$ at its zero point M. All they are situated on the tangent n-dimensional *Euclidean hyperspace* $\mathcal{E}_M^n$ with slope in external cavity of isotropic cone.

In STR, II is the hyperboloid of velocities at $R = c$ and their accelerations g at $K_R = g/c^2$. Its pseudonormal $ic \cdot \mathbf{p}_{(II)}$ is *4-velocity by Poincaré* [63, 64] of absolute matter motion in the 4D space-time $\langle \mathcal{Q}^{3+1} \rangle_c$. Its tangent and sine projections into $\langle \mathcal{E}^3 \rangle^{(1)}$ are velocities $\mathbf{v}$ and $\mathbf{v}^*$ (Ch. 5A).

The **hyperboloid I** of one sheet with the nD hyperbolic–elliptical non-Euclidean geometry of radius $\mp R$ (Ch. 12) is seeming as *hourglass* – see else in (146A). It is gotten by differentiating II under $\mathbf{e}_\alpha = \mathbf{const}$ or by rotating II at $\Pi/2$ in their projections on a pseudoplane as in Chs 5A, 6A for progenitors. Its time-like tangent, space-like principal pseudonormal and cosine binormal are:

$$\mathbf{p}_{(I)} = \mathbf{r}_{(I)} = \begin{bmatrix} \cos i\gamma_j \cdot \mathbf{e}_\alpha \\ -\sin i\gamma_j \end{bmatrix} = +[\mathbf{p}_{(II)}]'_\alpha = \mathbf{i}_{(II)} = \mathbf{p}, \quad \mathbf{i}_{(I)} = -[\mathbf{i}_{(II)}]'_\alpha = \mathbf{p}_{(II)} = \mathbf{i}, \quad \mathbf{b}_{(I)} = \begin{bmatrix} \mathbf{e}_\mu \\ 0 \end{bmatrix}.$$

On its upper half in the base $\tilde{E}_M$ at M, in result of its cutting by the rotated around OM centered pseudoplane in the angular interval of *visual inclination* from $|\pi/2|$ along $\overrightarrow{ct}$ till zero parallel to $\langle \mathcal{E}^3 \rangle$. We have in these cuts of I: time-like hyperbola and *hyperboloidal geodesics* in $|\pi/2| \geq \varphi_r > |\pi/4|$, horolines at $\varphi_r = |\pi/4|$, space-like circular and *ellipsoidal extremals* in $|\pi/4| > \varphi_r \geq 0$. This hyperboloid I is also a hyperboloid of supervelocities at $R_K = c$ and their accelerations j at $R = c^2/j$ in the so-called *Looking Glass of Theory of Relativity* – see in Ch. 10A. It is caused by the fact, that the mathematical roles of 4-velocities and 4-accelerations on II and I are contrary, because their pseudonormals and tangents are connected by one act of differentiation in $di\gamma_j$, as we see above.

From Equator point C_I on I in $\langle Q^{n+1}\rangle_c$ at Figure 4 (at $\gamma_0 = 0,\ \alpha_0 = \alpha$), under $\varphi_r = |\pi/2|$, we have one *time-like* pure hyperbolic geodesic meridian $\mathcal{H}_R = -Ri\gamma_j$ with $d\mathcal{H}_R = -Rdi\gamma_j$, radii of curvature $-R$ and of revolution $Rn_1 = r = +R\cosh i\gamma$. The 1-st angular metric form of I with the *pseudo-Euclidean Absolute Pythagorean theorem* reduces mixed motion with $d\alpha_2$ on I as a perfect surface to angular ones (hyperboloidal or horoline or ellipsoidal arc) along hypotenuse $Rdi\gamma_q$:

$$\left.\begin{array}{l} \mathbf{x}_{(I)} = x_{(I)}\cdot\mathbf{e}_\alpha = +R\cdot\cos i\gamma_j\cdot\mathbf{e}_\alpha, \\ y_{(I)} = ct = -R\cdot\sin i\gamma_j. \end{array}\right\} \text{[here from its Equator – see in detail in (235A–238A), Ch. 10A]}$$

$$\left.\begin{array}{l} d\mathbf{x}_{(I)} = d(x_{(I)}\cdot\mathbf{e}_\alpha) = +R\, d(\cos i\gamma_j\cdot\mathbf{e}_\alpha) = +R\cdot(-\sin i\gamma_j\ di\gamma_j\cdot\mathbf{e}_\alpha + \cos i\gamma_j\ d\alpha_2\cdot\mathbf{e}_\mu), \\ dy_{(I)} = d(ct) = -R\, d\sin i\gamma_j = -R\cdot\cos i\gamma_j\ di\gamma_j. \end{array}\right\} \Rightarrow$$

$$[d\lambda_{(I)}]^2 = (-R)^2 di\gamma_j^2 + [Rn_1(i\gamma_j)]^2 d\alpha^2 = R^2 di\gamma_j^2 + R^2\cos^2 i\gamma_j\ d\alpha_2^2 = R^2 di\gamma_q^2 \equiv \|d\mathbf{x}_{(I)}\|^2 + \|dy_{(I)}\|^2 \Rightarrow$$

$$[d\lambda_{(I)}/R]^2 = -d\gamma_q^2 = (\cos i\gamma_j\ di\gamma_j)^2_Y + [(\sin i\gamma_j\ di\gamma_j)^2 + (\cos i\gamma_j\ d\alpha_2)^2]_X = \sin^2 i\gamma_j\,(-d\gamma_j^2) + \cos^2 i\gamma_j\,(-d\gamma_j^2 + d\alpha_2^2) \Rightarrow$$

$$-d\gamma_q^2 = -\cosh^2\gamma_q\ d\gamma_q{}^2 + \sinh^2\gamma_q\ d\gamma_q{}^2 = -d\gamma_j^2 + \cosh^2\gamma_j\ d\alpha_2^2 = -\left(\overline{\overline{d\gamma_q}}\right)^2_P + \left(\overset{\perp}{d\gamma_q}\right)^2_E < 0, \quad (133A-H)$$

$$+d\gamma_q^2 = -\sinh^2\gamma_q\ d\gamma_q{}^2 + \cosh^2\gamma_q\ d\gamma_q{}^2 = -d\gamma_j^2 + \cosh^2\gamma_j\ d\alpha_2^2 = -\left(\overline{\overline{d\gamma_q}}\right)^2_P + \left(\overset{\perp}{d\gamma_q}\right)^2_E > 0. \quad (133A-S)$$

The hyperbolic time-like meridian arc $R\,di\gamma_j$ along tangent $\mathbf{i}_{(I)}$ with primary γ_j is accompanied by the Euclidean arc $R\cosh\gamma_j\ d\alpha\cdot\mathbf{e}_\mu$ at binormal $\mathbf{b}_\mu$, caused by motion tensor $rot\ i\Gamma_j = F(i\gamma_j, \mathbf{e}_\alpha)$.

Thus, (133A-H) is integrated in *hyperboloidal curves*, (133A-S) is integrated in *ellipsoidal curves*. That is, if $[d\lambda_{(I)}/R]^2 < 0$, summary $d\lambda_{(I)}/R$ is the hyperboloidal geodesic *time-like motion* on I; if $[d\lambda_{(I)}/R]^2 > 0$, summary $d\lambda_{(I)}/R$ is the ellipsoidal extremal *space-like motion* on I. But at $d\lambda_{(I)}/R = 0$, we have the straight geodesic gorolines in the isotropic cone dividing from two sides its internal and external cavities. All hyperbolic and hyperboloidal geodesics fill the internal cone, all circular and ellipsoidal extremals fill the external cone. In any point M of I, these two types of curves are intersected. All these three curves on I are relating to its hyperbolic-elliptical non-Euclidean geometry of radii $\mp R$ as on such a perfect hypersurface. What is more, in any point of I, there are only one pure hyperbolic geodesic and (n-1) pure circular extremals, similar to the geodesic Minding tractrix and circular extremals on the tractricoid I from Ch. 6A. It is the last leads to cylindrical topology of the hyperboloid I, limiting the freedom of figures motions on it by its space-like circular extremals of the length $2\pi R$. Thus, combination of these straight gorolines and circular extremals was realized by the great engineer Vladimir Shukhov in 1919 in the Moscow's Shukhov radio tower and further in other his objects with new elegant and economical one sheet hyperboloid architecture.

We can choose any variant from two (109A–I) and (109A-II) as separate one in own $\langle Q^{2+1}\rangle_c$!

From final real-valued hyperbolic presentations of metric forms (132A) and (133A) on Minkowski hyperboloids II and I by their translation into real-valued $\langle\mathcal{P}^{3+1}\rangle$, we can see that it was possible to use immediately the enveloping Minkowski pseudo-Euclidean binary space $\langle\mathcal{P}^{3+1}\rangle$ with its metric reflector tensor $\{\pm I\}$ (17A-I), i. e., conserving the classical real-valued Euclidean 3D subspace $\langle\mathcal{E}^3\rangle$. Hence, such $\langle\mathcal{P}^{3+1}\rangle$ is also a common enveloping binary space for both Minkowski hyperboloids. We revealed this result on the basis of Poincaré ideas that the time-arrow $\overrightarrow{ct}$ as a frame axis $\overrightarrow{y}$ and all hyperbolic angles are imaginary by nature. Einstein later presented $\langle\mathcal{P}^{3+1}\rangle$ to the exact opposite – with contrary metric tensor $\{\mp I\}$ (17A-II) and anti-Euclidean subspace $\langle i\mathcal{E}^3\rangle$. We'll see similar consistence and non-consistence in choosing metric tensor in our tensor trigonometric theory of arbitrary world lines together in $\langle Q^{3+1}\rangle_c$ and $\langle\mathcal{P}^{3+1}\rangle$ in last Ch. 10A with its concomitant hyperboloids II and I in the same space-time.

Let evaluate with tensor trigonometry the directional cosines of final rotation $roth\Gamma_{13} = \sqrt{S}$ in (114A), and those of the vectors $\mathbf{sinh}\gamma_{13}$, $\mathbf{tanh}\gamma_{13}$, and $\mathbf{v}_{13}$ in the Cartesian subbase $\tilde{E}_1^{(3)}$, taking advantage of their equality for matrices $roth\ \Gamma_{13}$ and $roth\ (2\Gamma_{13})$. (We use the arithmetic, as also trigonometric here, square root $\sqrt{S}$, because in it the angle Γ_{13} is bisected, see this in sect. 6.3!) Compute the 3 remained non-diagonal $(4,k)$-th elements of the 4-th row of the matrix $S = \{s_{ij}\}$. Thus we need to multiply the 4-th row of $B = \{b_{ij}\}$ and the k-th column of $roth\ \Gamma_{12}$, $k = 1, 2, 3$:

$$s_{4k} = s_{k4} = \sinh(2\gamma_{13})\cdot\cos\sigma_k = 2\cosh\gamma_{13}\cdot\sinh\gamma_{13}\cdot\cos\sigma_k =$$
$$= 2\cosh\gamma_{13}\cdot[(\sinh\gamma_{12}\cdot\cosh\gamma_{23} + \cos\varepsilon\cdot\sinh\gamma_{23}\cdot\cosh\gamma_{12})\cdot\cos\alpha_k + \sinh\gamma_{23}\cdot(cos\beta_k - \cos\varepsilon\cdot\cos\alpha_k)].$$

This allows us to infer all *vectorial* trigonometric formulae for two-steps motions in the hyperbolic non-Euclidean geometry or two-steps hyperbolic rotations (100A) in $\langle\mathcal{P}^{3+1}\rangle$. The vectorial formulae with directional cosines also propagate into hyperbolic motions summation on the hyperboloid II.

They depend, but only as vectors and not as scalars (!), on ordering of two summands γ_{12} and γ_{23}. So, *vector sines* in contrary variants of ordering two motions, expressed in Cartesian subbase, are:

$$\left.\begin{array}{l}
(1)\ \ \mathbf{sinh}\gamma_{13} = \sinh\gamma_{13}\cdot\mathbf{e}_\sigma = \mathbf{v}^{*}_{13}/c = (v^{*}_{13}/c)\cdot\mathbf{e}_\sigma = \\
= (\cosh\gamma_{23}\cdot\sinh\gamma_{12} + \cos\varepsilon\cdot\cosh\gamma_{12}\cdot\sinh\gamma_{23})\cdot\mathbf{e}_\alpha + \sin\varepsilon\cdot\sinh\gamma_{23}\cdot\mathbf{e}_\nu = \\
= [\cosh\gamma_{23}\cdot\sinh\gamma_{12} + \cos\varepsilon\cdot(\cosh\gamma_{12}-1)\cdot\sinh\gamma_{23}]\cdot\mathbf{e}_\alpha + \sinh\gamma_{23}\cdot\mathbf{e}_\beta; \\
(2)\ \ \mathbf{sinh}\overset{\angle}{\gamma}_{13} = \sinh\gamma_{13}\cdot\mathbf{e}_{\overset{\angle}{\sigma}} = \overset{\angle}{\mathbf{v}}{}^{*}_{13}/c = (\overset{\angle}{v}{}^{*}_{13}/c)\cdot\mathbf{e}_{\overset{\angle}{\sigma}} = \\
= (\cosh\gamma_{12}\cdot\sinh\gamma_{23} + \cos\varepsilon\cdot\cosh\gamma_{23}\cdot\sinh\gamma_{12})\cdot\mathbf{e}_\beta + \sin\varepsilon\cdot\sinh\gamma_{12}\cdot\mathbf{e}_{\overset{\angle}{\nu}} = \\
= [\cosh\gamma_{12}\cdot\sinh\gamma_{23} + \cos\varepsilon\cdot(\cosh\gamma_{23}-1)\cdot\sinh\gamma_{12}]\cdot\mathbf{e}_\beta + \sinh\gamma_{12}\cdot\mathbf{e}_\alpha; \\
(3)\ \ \sinh\gamma_{13}\cdot\cos\sigma_k = (\cosh\gamma_{23}\cdot\sinh\gamma_{12} + \cos\varepsilon\cdot\cosh\gamma_{12}\cdot\sinh\gamma_{23})\cdot\cos\alpha_k + \\
+ \sinh\gamma_{23}\cdot(\cos\beta_k - \cos\varepsilon\cdot\cos\alpha_k),\ k=1,2,3;\ \ \mathbf{e}_\sigma = \{\cos\sigma_k\}\ \text{(for direct order)}.
\end{array}\right\} \qquad (135A)$$

From here, under conditions $\gamma_{12} = \gamma$ and $\gamma_{23} = d\gamma$, we obtain the same metric form (132A) of the Minkowski hyperbolid II, but in its vector form – see more in (235A-238A), Ch. 10A.

For the next, it is useful to express the vector $\mathbf{e}_\nu$ of orthogonal increment of motion:

$$\mathbf{e}_\nu = \left\{\frac{\cos\beta_k - \cos\varepsilon\cdot\cos\alpha_k}{\sin\varepsilon}\right\}_{k=1,2,3} = \frac{\mathbf{e}_\beta - \cos\varepsilon\cdot\mathbf{e}_\alpha}{\sin\varepsilon} = \frac{\overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}\cdot\mathbf{e}_\beta}{||\overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}\cdot\mathbf{e}_\beta||} \qquad (136A)$$

The vector $\mathbf{e}_\nu$ (and $\mathbf{e}_{\overset{\angle}{\nu}}$ for inversely ordered summary motions at $\mathbf{e}_\alpha \leftrightarrow \mathbf{e}_\beta$ – see further) is used in biorthogonal decompositions of principal motion increment into tangential and normal parts, for physical velocities (see at Figure 4A), inner accelerations, curvatures, etc..

They are executed through biorthogonal representation of the 2-nd vector in the sum:

$$\mathbf{e}_\beta = \cos\varepsilon\cdot\mathbf{e}_\alpha + \sin\varepsilon\cdot\mathbf{e}_\nu,\quad \mathbf{e}'_\nu\cdot\mathbf{e}_\alpha = 0,\quad \mathbf{e}'_\nu\cdot\mathbf{e}_\beta = \sin\varepsilon\ \ (\varepsilon\in[0;\pi]). \qquad (137A).$$

Our approach is seen descriptively in the tangent presentations at Figure 4A, var. 1 and 2. Thus, from vectorial formulae (135A) and scalar formula (122A) similar vector relations for tangents in ordering γ_{12}, γ_{23} (and vice versa for γ_{23}, γ_{12} – see in (135A)) are inferred as:

$$\mathbf{tanh}\gamma_{13} = \tanh\gamma_{13}\cdot\mathbf{e}_\sigma = \mathbf{v}_{13}/c = (v_{13}/c)\cdot\mathbf{e}_\sigma = \frac{\mathbf{sinh}\ \gamma_{13}}{\cosh\gamma_{13}} = \qquad (138A)$$

$$= \frac{\tanh\gamma_{12} + \cos\varepsilon\cdot\tanh\gamma_{23}}{1+\cos\varepsilon\tanh\gamma_{12}\cdot\tanh\gamma_{23}}\cdot\mathbf{e}_\alpha + \frac{\sin\varepsilon\cdot\operatorname{sech}\ \gamma_{12}\cdot\tanh\gamma_{23}}{1+\cos\varepsilon\tanh\gamma_{12}\cdot\tanh\gamma_{23}}\cdot\mathbf{e}_\nu =$$

$$= \frac{\tanh\gamma_{12} + \cos\varepsilon\cdot(1-\operatorname{sech}\ \gamma_{12})\cdot\tanh\gamma_{23}}{1+\cos\varepsilon\tanh\gamma_{12}\cdot\tanh\gamma_{23}}\cdot\mathbf{e}_\alpha + \frac{\operatorname{sech}\ \gamma_{12}\cdot\tanh\gamma_{23}}{1+\cos\varepsilon\tanh\gamma_{12}\cdot\tanh\gamma_{23}}\cdot\mathbf{e}_\beta.$$

Sine and tangent formulae, in squared and vectorial variants (124A), (135A) and (125A), (138A), have in $\tilde{E}_1^{(3)}$ such interpretation. The second segment γ_{23} on a hyperboloid II is decomposed into a pair of segments such that their projections into $\langle\mathcal{E}^3\rangle^{(1)}$ are directed along $\mathbf{e}_\alpha$ and $\mathbf{e}_\nu$. We get these big and small hyperbolic right triangles on a hyperboloid II: $\gamma_{13} = (\gamma_{12} + \overset{=}{\gamma}_{23}) \boxplus \overset{\perp}{\gamma}_{23}$ and $\gamma_{23} = \overset{=}{\gamma}_{23} \boxplus \overset{\perp}{\gamma}_{23}$, – with such spherically orthogonal sums and corresponding to them sine or tangent right triangles in $\langle\mathcal{E}^3\rangle^{(1)}$. (*Segments γ are 4-dimensional, their space projections are 3-dimensional!*) Let us perform hyperbolic sine projecting γ_{13} and γ_{23} (in its spherically orthogonal decomposition) into $\langle\mathcal{E}^3\rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(1)}$. The result is two orthogonalized projections of γ_{23} and γ_{13} into $\langle\mathcal{E}^3\rangle^{(1)}$:

$$\mathbf{sinh}\ \gamma_{23} = \overset{=}{\mathbf{sinh}}\ \gamma_{23} + \overset{\perp}{\mathbf{sinh}}\ \gamma_{23}\ \ \rightarrow\ \ \mathbf{sinh}\gamma_{13} = (\mathbf{sinh}\ \gamma_{12} + \overset{=}{\mathbf{sinh}}\ \gamma_{23}) + \overset{\perp}{\mathbf{sinh}}\ \gamma_{23}.$$

We have in $\tilde{E}_1^{(3)}$ in squared variant the **Big Pythagorean Theorem** corresponding to (124A), and the **Small Pythagorean Theorem** for second segment corresponding to orthogonal case (129A):

$$\sinh^2\gamma_{13} = \sinh^2(\gamma_{12} + \overset{=}{\gamma}_{23}) + \sinh^2\overset{\perp}{\gamma}_{23}\ ,\quad \sinh^2\gamma_{23} = \sinh^2\overset{=}{\gamma}_{23} + \sinh^2\overset{\perp}{\gamma}_{23}\ .$$

In these formulae, $\sinh\overset{=}{\gamma}_{13} = \cos\varepsilon\cdot\sinh\gamma_{13}$, $\sinh\overset{\perp}{\gamma}_{23} = \sinh\overset{\perp}{\gamma}_{13} = \sin\varepsilon\cdot\sinh\gamma_{13}$. Their *cosines*, are, due to (122A), the scalar projections into $\overrightarrow{ct}$ parallel to $\langle\mathcal{E}^3\rangle$. The analogical proportional relations act in tangent variant, according to (125A) and (138A) – see it below and further at Figure 4A. **Therefore, we can strictly formulate both Pythagorean theorems.**

* * *

The Big Pythagorean Theorem. *Sum of two segments or motions is presented in biorthogonal form, commutative in Euclidean geometry and non-commutative in all non-Euclidean geometries.* It acts in the quasi- and pseudo-Euclidean spaces with index $q = 1$ how in sine vectorial decompositions (135A) as result of summing two rotations, and also on the perfect hypersurfaces in them, including hyperspheroid and hyperboloid II, as a result of summing two motions from the start point in $\tilde{E}_1^{(n)}$ with correction of the 2-nd segment in $\tilde{E}_2^{(n)}$, i. e., in their non-Euclidean geometries.

Tangent formulae (125A) and (138A) are interpreted by analogous way, but with the use of tangent cross projecting. The angle γ_{23} is decomposed as before and then all these parallel and normal components are subjected to *cross projecting* (see in Ch. 4A) into $\langle\mathcal{E}^3\rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(2)}$. It should be taken into account by correction with additional coefficient sechγ_{12} (only by formal analogy with Lorentzian contraction). Their *tangent summation*, with these analogous Big and Small Pythagorean Theorems (125A) and (138A), are identical to *tangent model* at Figure 4A as in the Klein homogeneous coordinates. Big and Small theorems are relative and act in $\tilde{E}_1^{(n)}$ and $\tilde{E}_2^{(n)}$.

Furthermore, this important property of the summations into $\mathbf{sinh}\ \gamma_{13}$, $\mathbf{tanh}\ \gamma_{13}$ unites to a certain extent the Euclidean geometry with non-Euclidean *hyperbolic* and *spherical* geometries! Distinction is the following. In Euclidean geometry the vectors $\mathbf{a}_{12} = a_{12}\cdot\mathbf{e}_\alpha$ and $\mathbf{a}_{23} = a_{23}\cdot\mathbf{e}_\beta$ are summarized commutatively, i. e., in their direct and inverse orders with the same result $\mathbf{a}_{13} = a_{13}\cdot\mathbf{e}_\sigma$. Two variants of the biorthogonal non-Euclidean summation (direct and inverse) are noncommutative from the different sign of the angle of orthospherical rotation ($\mp\theta_{13}$) after the summing process.

The Big Pythagorean Theorem is valid for two variants of orthoprojections. In both the cases, modules of hypotenuses are equal, but the summary vectors $\mathbf{a}_{13}$ are distinct by the orthospherical rotation as in (120A). Thus, formulae (124A), (135A) and (125A), (138A), can be presented in two biorthogonal forms with decompositions either of γ_{12} with respect to $\mathbf{e}_\alpha$ or of γ_{23} with respect to $\mathbf{e}_\beta$.

Next summand non-collinear to previous is in other Euclidean subspace. Namely this theorem, in particular, allowed Poincaré as in three projections and Einstein as entirely to infer the relativistic Law of summing two non-collinear velocities in vector and scalar forms under conditions $\{\cos\alpha_1 = 1, \cos\alpha_2 = \cos\alpha_3 = 0\} \to \cos\varepsilon = \cos\beta_1$. Thanks to this geometric theorem, orthoprojections of velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ along the axes x_1, x_2, x_3 were summarized as if Euclidean orthogonal each others in the final physical formula [63], [67].

Further in vector formula (138A), put $\mathbf{tanh}\,\gamma_{12}\cdot\cos\alpha_1 = \pm v/c \approx 10^{-4}$, $\cos\alpha_1 = \pm 1$. Then $\cos\varepsilon = \pm\cos\beta_1$, see (119A), and $\mathbf{tanh}\,\gamma_{23} = c/c = 1$, that is why $\mathbf{tanh}\,\gamma_{13} = 1$ too. Here $v \approx 30$ km/sec is the orbital velocity of the Earth moving around the Sun. Hence,

$$\mathbf{tanh}\ \gamma_{13} = \mathbf{e}_\sigma = \frac{[\mathbf{tanh}\,\gamma_{12} \pm \cos\beta_1\cdot(1 - \mathrm{sech}\ \gamma_{12})]\cdot\mathbf{e}_\alpha + \mathrm{sech}\ \gamma_{12}\cdot\mathbf{e}_\beta}{1 \pm \cos\beta_1\cdot\mathbf{tanh}\,\gamma_{12}} =$$

$$= \frac{1}{1 \pm \cos\beta_1\cdot\mathbf{tanh}\,\gamma_{12}}\cdot\begin{bmatrix}\pm\mathbf{tanh}\,\gamma_{12} + \cos\beta_1\\ \mathrm{sech}\ \gamma_{12}\cdot\cos\beta_2\\ \mathrm{sech}\ \gamma_{12}\cdot\cos\beta_3\end{bmatrix} = \begin{bmatrix}\cos\sigma_1\\ \cos\sigma_2\\ \cos\sigma_3\end{bmatrix},\ (\mathbf{tanh}\,\gamma_{13} = 1),$$

where $\beta_1, \beta_2, \beta_3$ and $\sigma_1, \sigma_2, \sigma_3$ are the true and seemed angles, under which the Star is observed. From this, the complete list of *relativistic formulae for aberration* follows:

$$\tan\beta'_{12} = \frac{\cos\sigma_2}{\cos\sigma_1} = \frac{\mathrm{sech}\ \gamma_{12}\cdot\cos\beta_2}{\pm\mathbf{tanh}\,\gamma_{12} + \cos\beta_1}, \qquad \tan\beta'_{13} = \frac{\cos\sigma_3}{\cos\sigma_1} = \frac{\mathrm{sech}\ \gamma_{12}\cdot\cos\beta_3}{\pm\mathbf{tanh}\,\gamma_{12} + \cos\beta_1},$$

$$\cos\delta^{\pm} = (\mathbf{e}_\sigma^+)'\cdot\mathbf{e}_\sigma^- = \frac{\mathrm{sech}^2\gamma_{12} - \sin^2\beta_1\cdot\mathbf{tanh}^2\,\gamma_{12}}{1 - \cos^2\beta_1\cdot\mathbf{tanh}^2\,\gamma_{12}},\quad R_a = \frac{\delta^{\pm}}{2}\quad (\text{as how } \gamma_{12} \pm \gamma_{23}).$$

If the Star is observed in the simplest variant under $\beta_1 = \pi/2$, then for maximal δ^m:

$$\cos\delta^m = \mathrm{sech}^2\gamma_{12} - \mathbf{tanh}^2\,\gamma_{12} \equiv \cos 2\varphi(\gamma_{12}),\ \sin\delta^m = 2\,\mathbf{tanh}\,\gamma_{12}\cdot\mathrm{sech}\gamma_{12} \equiv \sin 2\varphi(\gamma_{12}).$$

If $\beta_3 = \pi/2$, then $\cos\beta_2 = \sin\beta_1$. In this special case, we obtain *trigonometric variant* of Einstein's formula for the orthogonally observed aberration [53, p. 36–39]:

$$\tan\beta'_{12} = \frac{\sin\beta_1 \cdot \text{sech}\,\gamma_{12}}{\cos\beta_1 \pm \tanh\gamma_{12}}$$

$$\left(\sin\beta'_{12} = \frac{\sin\beta_1 \cdot \text{sech}\,\gamma_{12}}{1 \pm \cos\beta_1 \cdot \tanh\gamma_{12}}, \qquad \cos\beta'_{12} = \frac{\cos\beta_1 \pm \tanh\gamma_{12}}{1 \pm \cos\beta_1 \cdot \tanh\gamma_{12}}\right).$$

For the orthogonally observed aberration, we have the simplest *Einsteinian variant* at:

$$\sigma_1 = \beta'_{12} = \beta'_1 = \pi/2 - \sigma_2, \quad \cos\sigma_2 = \sin\sigma_1 = \sin\beta'_1 \quad \sigma_3 = \beta_3 = \pi/2.$$

Then either $\beta'_1 < \beta_1$ (if the sign $+$ is chosen), or $\beta'_1 > \beta_1$ (if the sign $-$ is chosen); and the angles β_1 and β'_1 are permuted iff the signs $\pm$ and $\mp$ are permuted. All these formulae immediately follow from indicated above general formula for $\mathbf{tanh}\gamma_{13} = \mathbf{e}_\sigma$.

For J. Bradley formula (1727), A. Einstein introduced relativistic time-correcting factor sech γ_{12} (here it is in secant form (127A)) and used Lorentzian transformation instead of Galilean ones [67]. This small correction makes the formula of aberration identical in two inertial frames of reference associated either with the Earth, or with the Star: $\mathbf{e}_\alpha$ and $\mathbf{e}_\sigma$ are permuted iff signs $\pm$ and $\mp$ are permuted. The maximal *angular radius of aberration* is achieved if $\beta_1 = \pi/2$, and it is $R_a = \delta^m/2 \approx 10^{-4}$ rad. Note, that the *angle of orthospherical rotation* θ_{13} will be calculated below. Some Soviet academic authors did not distinguish in aberration the angles $\delta^\pm$ for $\gamma_{12} \pm \gamma_{23}$ and θ_{13} for $\gamma_{12} + \gamma_{23}, \gamma_{23} + \gamma_{12}$?! See for θ_{13} further.

* * *

According to (135A) and (136A), the vectors $\mathbf{e}_\sigma$ and $\mathbf{e}_\eta$ are linear combinations of $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$. Hence all the four unit vectors are in the same Euclidean plane $\langle\mathcal{E}^2\rangle \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle$. Similar arguments for inverse ordering of motions give similar results, but *the first* directed vector is $\mathbf{e}_\beta$ and *the second* one is $\mathbf{e}_\alpha$. The new vector of orthogonal increment (for the inverse order of the full motion) is expressed similar (136A), (137A) by permutation:

$$\mathbf{e}_{\underset{\nu}{\measuredangle}} = \left\{\frac{\cos\alpha_k - \cos\varepsilon \cdot \cos\beta_k}{\sin\varepsilon}\right\} = \frac{\mathbf{e}_\alpha - \cos\varepsilon \cdot \mathbf{e}_\beta}{\sin\varepsilon}, \tag{139A}$$

$$\mathbf{e}_\alpha = \cos\varepsilon \cdot \mathbf{e}_\beta + \sin\varepsilon \cdot \mathbf{e}_{\underset{\nu}{\measuredangle}}, \qquad \mathbf{e}'_\beta \cdot \mathbf{e}_{\underset{\nu}{\measuredangle}} = 0, \quad \mathbf{e}'_\alpha \cdot \mathbf{e}_{\underset{\nu}{\measuredangle}} = \sin\varepsilon, \quad \mathbf{e}'_\nu \cdot \mathbf{e}_{\underset{\nu}{\measuredangle}} = -\cos\varepsilon. \tag{140A}$$

The vectors $\mathbf{tanh}\,\overset{\measuredangle}{\gamma}_{13}$, $\mathbf{sinh}\,\overset{\measuredangle}{\gamma}_{13}$, and $\overset{\measuredangle}{\mathbf{v}}_{13}$ are directed in the subbase $\bar{E}_1^{(3)}$ along $\mathbf{e}_{\underset{\sigma}{\measuredangle}}$, and their modules do not change. The vectors $\mathbf{e}_\sigma$, $\mathbf{e}_{\underset{\sigma}{\measuredangle}}$, $\mathbf{e}_\nu$ and $\mathbf{e}_{\underset{\nu}{\measuredangle}}$ are linear combinations of $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$, hence they lie in the same plane $\langle\mathcal{E}^2\rangle \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle$. The rotations (113A) and (112A) act in the common trigonometric plane of the matrix *rot* Θ_{13}, hence this plane is identical to $\langle\mathcal{E}^2\rangle$ too. The Euclidean plane includes *formally* all these six introduced and considered unity vectors of diagonal cosines: $\mathbf{e}_\alpha$, $\mathbf{e}_\beta$, $\mathbf{e}_\sigma$, $\mathbf{e}_{\underset{\sigma}{\measuredangle}}$, $\mathbf{e}_\nu$, $\mathbf{e}_{\underset{\nu}{\measuredangle}}$.

(In general cases, for internal and external multiplications of unity vectors there holds:

$$\mathbf{e}'_1 \cdot \mathbf{e}_2 = \cos\theta_{12}, \quad \mathbf{e}_1 \cdot \mathbf{e}'_2 = \cos\theta_{12} \cdot \overleftarrow{\mathbf{e}_1 \cdot \mathbf{e}'_2} = \sec\ \theta_{12} \cdot \overleftarrow{\mathbf{e}_1 \cdot \mathbf{e}'_1} \cdot \overleftarrow{\mathbf{e}_2 \cdot \mathbf{e}'_2}.$$

They may be also useful. The last formulae are the special cases of (196) in Ch. 5.

The matrix *rot* Θ_{13} can be calculated not only from multiplicative formula (115A). In $\langle\mathcal{P}^{3+1}\rangle$, it may be directly calculated in canonical form (497) due to (499). Indeed, the normal unity axis $\overrightarrow{\mathbf{e}_N}$ of this orthospherical rotation is found in terms of vector product for unity vectors of first γ_{12} ($\mathbf{e}_\alpha$) and second γ_{23} ($\mathbf{e}_\beta$) motions in (135A), with (137A) as:

$$\left.\begin{aligned}\overrightarrow{\mathbf{r}_N}(\theta) &= \mathbf{e}_{\underset{\sigma}{\measuredangle}} \otimes \mathbf{e}_\sigma = -\sin\theta \cdot \overrightarrow{\mathbf{e}_N}, \text{where } \overrightarrow{\mathbf{e}_N} = \mathbf{e}_\alpha \otimes \mathbf{e}_\nu \\ \overrightarrow{\mathbf{r}_N}(\varepsilon) &= \mathbf{e}_\alpha \otimes \mathbf{e}_\beta = +\sin\varepsilon \cdot \mathbf{e}_\alpha \otimes \mathbf{e}_\nu = +\sin\varepsilon \cdot \overrightarrow{\mathbf{e}_N}\end{aligned}\right\} \Rightarrow \overrightarrow{\mathbf{r}_N}(\theta) = -\left[\frac{\sin\theta}{\sin\varepsilon} \cdot \overrightarrow{\mathbf{r}_N}(\varepsilon).\right] \tag{141A}$$

The orthospherical rotation or shift $\mp\theta$ is realized in the base $\tilde{E}_{1h} = roth\ \Gamma \cdot \tilde{E}_1$ (see (111A). In general, in $\langle \mathcal{P}^{3+1} \rangle$, it has a current normal axis $\overrightarrow{\mathbf{e}_N}^{(3)}$ in $\langle \mathcal{E}^3 \rangle^{(1h)}$ and acts in the plane $\langle \mathcal{E}^2 \rangle^{(1h)}$ under hyperbolic inclination γ_{13} to $\langle \mathcal{E}^2 \rangle^{(1)} \equiv \langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle^{(1)}$ and with signs due to (113A), (119A). These values of $\overrightarrow{\mathbf{r}_N}(\theta)$ and $\cos\theta_{13}$ give us the matrix $rot\ \Theta_{13}$ in canonical form (497) if $n = 3$. Due to (499), (113A), (120A), we get additional variants *for shifting* θ_{13}:

$$\cos\theta_{13} = \mathbf{e}'_\sigma \cdot \mathbf{e}_{\overset{\angle}{\sigma}} = tr\ rot\ \Theta/2 - 1 = (tr[rot\ \Theta]_{3\times 3} - 1)/2, |\sin\theta_{13}| = |\overrightarrow{\mathbf{r}_N}(\theta_{13})|. \quad (142A)$$

Speaking strictly, angular shift θ must supplement the hyperbolic laws of summing motions (velocities) (135A)-(138A). So, this shift is the cause of non-commutativity of these laws.

Due to the sign's **Rule** (see in (113A) from sect. 12.2) in hyperbolic geometry and STR, $\boxed{sgn\ \theta_{13} = -sgn\ \varepsilon\ !}$: if $\varepsilon > 0$, then $\theta_{13} < 0$, and if $\varepsilon < 0$, then $\theta_{13} > 0$, i. e., the leg 13 is shifted orthospherically so to the angle $A_{123} = \pi - \varepsilon$ always that to decrease the sum of angles in the hyperbolic triangle (see more further).

The vectors $\mathbf{e}_{\overset{\angle}{\sigma}}, \mathbf{e}_\sigma, \overrightarrow{\mathbf{e}_N}$ as well as the vectors $\mathbf{e}_\alpha, \mathbf{e}_\beta, \overrightarrow{\mathbf{e}_N}$ form the right triple due to (113A), this corresponds to counting scalar angles as counter-clockwise ones in the *right-handed* bases, and the oriented vector $\overrightarrow{\mathbf{e}_N}$ determines the right screw of rotations. The triple $\mathbf{e}_{\overset{\angle}{\sigma}}, \mathbf{e}_\sigma, \overrightarrow{\mathbf{r}_N}(\theta)$ is universal for analysis of polysteps motions.

All the six vectors $\mathbf{e}_\alpha$, $\mathbf{e}_\beta$, $\mathbf{e}_\nu$, $\mathbf{e}_\sigma$, $\overset{\angle}{\mathbf{e}}_\nu$, $\overset{\angle}{\mathbf{e}}_\sigma$ are *formally* inside an angle of magnitude π in the plane $\langle \mathcal{E}^2 \rangle \equiv \langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle$. From (136A), (139A), taking into account (122A), we obtain their following trigonometric properties:

$$\mathbf{e}'_\alpha \cdot \mathbf{e}_\beta = \mathbf{e}'_\beta \cdot \mathbf{e}_\alpha = \cos\varepsilon, \quad \mathbf{e}'_\alpha \cdot \mathbf{e}_\nu = \mathbf{e}'_\beta \cdot \mathbf{e}_{\overset{\angle}{\nu}} = 0,$$

$$\mathbf{e}'_\beta \cdot \mathbf{e}_\nu = \mathbf{e}'_\alpha \cdot \mathbf{e}_{\overset{\angle}{\nu}} = +\sin\varepsilon = +\sin(\pi - \varepsilon), \quad \mathbf{e}'_\nu \cdot \mathbf{e}_{\overset{\angle}{\nu}} = -\cos\varepsilon = +\cos(\pi - \varepsilon).$$

The value of $\cos\theta_{13}$ is computed with the use of (120A), and in addition vectorial variant of (135A) and its reverse analog! With respect to the original base $\tilde{E}_1$ we have

$$\cos\theta_{13} = \mathbf{e}'_\sigma \cdot \mathbf{e}_{\overset{\angle}{\sigma}} = \frac{A + \cos\varepsilon \cdot B + \cos^2\varepsilon \cdot C + \cos^3\varepsilon \cdot D}{\sinh^2\gamma_{13}} > 0; \quad (143A)$$

$$A = (\cosh\gamma_{12} \cdot \cosh\gamma_{23} - 1)(\cosh\gamma_{12} + \cosh\gamma_{23}) > 0,$$

$$B = \sinh\gamma_{12} \cdot \sinh\gamma_{23} \cdot (\cosh\gamma_{12} \cdot \cosh\gamma_{23} + \cosh\gamma_{12} + \cosh\gamma_{23} - 1) > 0,$$

$$C = \sinh^2\gamma_{12} \cdot \cosh\gamma_{23} \cdot (\cosh\gamma_{23} - 1) + \sinh^2\gamma_{23} \cdot \gamma_{12} \cdot (\cosh\gamma_{12} - 1) > 0,$$

$$D = \sinh\gamma_{12} \cdot \sinh\gamma_{23} \cdot (\cosh\gamma_{12} - 1) \cdot (\cosh\gamma_{23} - 1) > 0.$$

If $\cos\varepsilon = +1$, then $A + B + C + D = \sinh^2\gamma_{13} = \sinh^2(\gamma_{12} + \gamma_{23})$ with $\theta_{13} = 0$.
If $\cos\varepsilon = -1$, then $A - B + C - D = \sinh^2\gamma_{13} = \sinh^2(\gamma_{12} - \gamma_{23})$ with $\theta_{13} = 0$.

Theoretically extremal relativistic shift $\theta_{13} = \mp\pi/2$ takes place if conventionally orthogonal velocities are equal to the speed of light c! Moreover, function (143A) in $\cos\varepsilon$ has 3 extrema: maximal value $\cos\theta_{13} = 1$ if $\cos\varepsilon = \pm 1$, minimal value $\cos\theta_{13} = A/\sin^2\gamma_{13}$ if $\cos\varepsilon = 0$.

The latter corresponds to conventionally orthogonal two-step motions with quadratic scalar sine and tangent formulae (129A - I) and (130A - I). Below we consider in details the sine variant. At first, transform scalar sine quadratic formula (129A - I) into the form:

$$\sinh^2\gamma_{13} = (\cosh\gamma_{12} \cdot \cosh\gamma_{23})^2 - 1 = (\cosh\gamma_{12} \cdot \cosh\gamma_{23} + 1)(\cosh\gamma_{12} \cdot \cosh\gamma_{23} - 1).$$

The *absolute value* of $\cos\theta_{13}$ is minimal iff $|\theta_{13}|$ is maximal, this is equivalent to conventional orthogonality of $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$. For the sum of two hyperbolic motions, provided that $\varepsilon = \pm\pi/2$ ($\sin\varepsilon = \pm 1$), from (143A) and (135A), (138A) we obtain:

$$\cos\theta_{13} = \frac{A}{\sinh^2\gamma_{13}} = \frac{\cosh\gamma_{12} + \cosh\gamma_{23}}{\cosh\gamma_{12}\cdot\cosh\gamma_{23} + 1} > 0 \ \Rightarrow\ \sin\theta_{13} = -\frac{\sinh\gamma_{12}\cdot\sinh\gamma_{23}}{\cosh\gamma_{12}\cdot\cosh\gamma_{23} + 1}.$$

So, if $\gamma_{23} \to 0$, then $\theta_{13} \to 0$. *This is the reason for appearing induced precession in time.*

$$\left.\begin{array}{l}\tan\theta_{13} = -\frac{\sinh\ \gamma_{12}\cdot\sinh\ \gamma_{23}}{\cosh\gamma_{12}+\cosh\gamma_{23}} = -\frac{\tanh\ \gamma_{12}\cdot\tanh\ \gamma_{23}}{\mathrm{sech}\,\gamma_{12}+\mathrm{sech}\,\gamma_{23}};\\ \tanh\gamma_{13}\cdot\mathbf{e}_\sigma = \tanh\gamma_{12}\cdot\mathbf{e}_\alpha + \tanh\gamma_{23}\cdot\mathrm{sech}\ \gamma_{12}\cdot\mathbf{e}_\beta,\\ \tanh\gamma_{13}\cdot\mathbf{e}_{\underset{\sigma}{\angle}} = \tanh\gamma_{23}\cdot\mathbf{e}_\beta + \tanh\gamma_{12}\cdot\mathrm{sech}\ \gamma_{23}\cdot\mathbf{e}_\alpha.\end{array}\right\}\quad (\cos\varepsilon = 0)$$

The hyperbolic sine formula above was obtained by Arnold Sommerfeld in 1931 [95] as result of summing two orthogonal velocities in STR as if on a hypothetic then *sphere of imaginary radius* with its angular argument $\varphi = i\gamma$. This gave namely pure scalar trigonometric interpretation of coefficient $1/2$ in the Thomas precession [93] under condition that $\gamma_{ij} \to 0$ ($v_{ij} \to 0$) in this sine formula.

Three particular formulae above for the angle of orthospherical shift θ in cosine, sine and tangent variants supplement the pure hyperbolic formulae for summing two conventionally orthogonal motions (velocities) in cosine (128A), sine (129A) and tangent (130A) variants with maximal orthospherical shifting – for completeness of the results of orthogonal motions summation! In general, this angle is concomitant for the non-collinear two- and polysteps principal motions in pseudo-Euclidean, quasi-Euclidean and non-Euclidean geometries. It has own real meaning, in that number, for applications in physics and quantum mechanics.

If one of the velocities is $\pm c$, for example, it is $\tanh\gamma_{23} = \pm 1$, then $\cos\theta_{13} = \mathrm{sech}\gamma_{12}$, $\sin\theta_{13} = \mp\tanh\gamma_{12}$, $\quad \mathbf{e}_\sigma = \pm\tanh\gamma_{12}\cdot\mathbf{e}_\alpha + \mathrm{sech}\ \gamma_{12}\cdot\mathbf{e}_\beta$, $(|\mathbf{e}_\sigma| = 1)$; $\mathbf{e}_{\underset{\sigma}{\angle}} = \pm\mathbf{e}_\beta$.

Such a case corresponds to the orthogonal variant of aberration with the pseudo-Euclidean right triangle of aberration here clarity on the hyperboloid II of radius "ic" – see above.

First leg is the angle γ_{12} generated due to motion of the Earth relatively to the as if "immovable" Star. Second leg γ_{23} under the right angle ε (in its Euclidean orthoprojection) is generated due to motion of the light ray from the Star to the Earth. The hypotenuse is sum γ_{13} directed along $\mathbf{e}_\sigma$. Vector $\mathbf{e}_\alpha$ inverses direction each half a year, that is why $\cos\alpha_1 = \pm 1$ and $\cos\varepsilon = \pm\cos\beta_1$. The *angular Lambertian defect of this geodesic right triangle 123 of aberration* (see above) as θ_{13} is determined now with the use of permutation of these two legs by the formula (142A):

$$\cos\theta_{13} = \mathbf{e}'_\sigma\cdot\mathbf{e}_{\underset{\sigma}{\angle}} = 1 - \frac{(1 - \mathrm{sech}\ \gamma_{12})\cdot\sin^2\beta_1}{1 \pm \cos\beta_1\cdot\tanh\gamma_{12}}.$$

We finished consideration, with our tensor trigonometric approach, mainly of the finite motions on the Minkowski hyperboloids II isometric to motions on the Lobachevsky–Bolyai hyperbolic plane (Ch. 12) with identical to them finite rotations in the Minkowski space. We'll continue this by direct way in tensor forms in Ch. 10A, but (!) for both hyperbolids.

If to put $\gamma_{12} = \gamma$, $\gamma_{23} = d\gamma$, then for their non-collinear summation, with exactness up to first differentials, from the same formula (143A), we get 1-st differential of the angular shift $d\theta$ in scalar sine-tangent forms ($\sin\varepsilon = \sin A$) – see it further also in 3-vector forms:

$$\sin d\theta = d\theta = -\sin\varepsilon\cdot\frac{\sinh\gamma\ d\gamma}{\cosh\gamma + 1} = -\sin\varepsilon\cdot\frac{(\cosh\gamma - 1)\ d\gamma}{\sinh\gamma} = -\sin\varepsilon\cdot\tanh\ (\gamma/2)\ d\gamma.$$

* * *

Cosine formula (143A) can be applicable for other important evaluations. As before, in infinitesimal considerations we take advantage of the useful formula for the cosine of first angular differential (with exactness up to second power of the angular differential).
$\boxed{\cosh d\gamma = 1 + (d\gamma)^2/2}$ and $\boxed{\cos d\theta = 1 - (d\theta)^2/2}$ in hyperbolic and spherical forms.

In (135A(1)), with direct and inverse order, put in $\langle\mathcal{E}^3\rangle$: $\gamma_{12} = \gamma_t$ and $\gamma_{23} = d\gamma_p$ as the 1-st 3-vector hyperbolic differential with $\mathbf{e}_\beta$ also tangent to hyperboloid II. With the use of the sine formula above, we obtain the differential orthospherical shift $d\theta$, i. e., as in (141A), but at $\sin\theta \to d\theta$. Further, using (141A) with the Sign's Rule from sect. 12.2 (Part II) as $\boxed{sgn\ d\theta = -sgn\ \varepsilon}$ (at $n \le 3$) and hyperbolic trigonometry, we add to scalar cosine product (142A) the vectorial sine product in $\langle\mathcal{E}^3\rangle \subset \langle\mathcal{P}^{3+1}\rangle \equiv \langle\mathcal{E}^3\rangle \boxtimes \overrightarrow{y}$ and reveal the induced orthospherical shift $d\theta$ of $\mathbf{e}_\sigma$, negative to ε, but also around 3-rd space-like normal axis $\overrightarrow{\mathbf{e}_N} = \mathbf{e}_\alpha \otimes \mathbf{e}_\nu$, complementary till $\langle\mathcal{E}^3\rangle$ (in that number, with two relativistic factors):

$$\overrightarrow{\mathbf{r}_N}(\theta) = -d\theta\cdot\overrightarrow{\mathbf{e}_N} = \mathbf{e}_{\underset{\sigma}{\angle}} \otimes \mathbf{e}_\sigma = \mathbf{tanh}\ (\gamma/2)\otimes\mathbf{d}\gamma = \frac{\sinh\gamma\cdot\mathbf{e}_\alpha}{\cosh\gamma+1}\otimes d\gamma\cdot\mathbf{e}_\beta = \frac{\sinh\gamma}{\cosh\gamma+1}\,d\gamma\cdot\overrightarrow{\mathbf{r}_N}(\varepsilon) =$$

$$= \sin\varepsilon\cdot\frac{\sinh\gamma}{\cosh\gamma+1}\,d\gamma\cdot\overrightarrow{\mathbf{e}_N} = \sin\varepsilon\cdot\frac{\tanh\gamma}{1+\mathrm{sech}\gamma}\,d\gamma\cdot\overrightarrow{\mathbf{e}_N} = \tanh\frac{\gamma}{2}\cdot\sin\varepsilon\ d\gamma\cdot\overrightarrow{\mathbf{e}_N} = \tanh\frac{\gamma}{2}\,\overset{\perp}{d\gamma}\cdot\overrightarrow{\mathbf{e}_N}.\quad(144A-I)$$

Here the angle γ or Γ is expressed in the original base $\tilde{E}_1$. In STR, it is the universal base with relatively immovable Observer N_1 in the space-time $\langle\mathcal{P}^{3+1}\rangle$; the differential $d\gamma\cdot\mathbf{e}_\beta$ is expressed in the base $\tilde{E}_m = roth\ \Gamma\cdot\tilde{E}_1$. In $d\theta$ tangent variant, we see again the correcting coefficient 1/2 of the normal Thomas precession [93] *gotten before from experimental data*. For two hyperbolic arcs in (144A) at point M, the *third* unity normal axis $\overrightarrow{\mathbf{e}_N}$ exists only at $n = 3$ as ortho complementary in $\langle\mathcal{E}^3\rangle$ to the vectors $\mathbf{e}_\alpha$ and $\mathbf{e}_\nu$. In own moving bases, they have the cosine hyperbolic slope. See following developments of (144A) in (171A)–(173A).

* * *

The especial case is *non-conventionally* orthogonal summation of motions when angles as 1-st differentials are *infinitesimal*. Let in (144A) as in (131A-II) infinitesimal values of angles. On the hyperboloid II with $K_G = -1/R^2$ in $\langle\mathcal{P}^{2+1}\rangle$, for the hyperbolic right triangle **123** on it with 3 angles $A_k = \pi - \varepsilon_k$ at its 3 tops, at $\gamma_{12} \to 0, \gamma_{23} \to 0$ and $\varepsilon_{13} = \pi/2$, we obtain:

$$\gamma_{13} = \sqrt{\gamma_{12}^2 + \gamma_{23}^2}\ ; \quad -\theta_{13} = \frac{\gamma_{12}\cdot\gamma_{23}}{2} = \frac{a_{12}\cdot a_{23}}{2R^2} = -K_G\cdot S_{123} = -\delta_{123} \to 0.$$

Here, with angles γ and S_{123}, we get infinitesimal formulae of the plane Euclidean geometry. This confirms the *infinitesimally Euclidean metric* on the top Minkowskian hyperboloid II. *For it, we may bond the shift and angular Lambert defect:* $\theta_{13} = \delta_{123} = 2\pi - (\varepsilon_1+\varepsilon_2+\varepsilon_3) < 0$. On it in $\langle\mathcal{P}^{3+1}\rangle$ from (144A) for triangle **123** formed by $d\gamma_{12}$ and $d\gamma_{23}$ with angles A and ε ($\sin\varepsilon = \sin A$), we infer differential formula for the vector-element of its area (see [21, p. 526]):

$$-d\theta_{13}\cdot\overrightarrow{\mathbf{e}_N} = \sin\varepsilon\cdot\frac{(d\gamma_{12})\cdot(d\gamma_{23})}{2}\cdot\overrightarrow{\mathbf{e}_N} = \sin\varepsilon\cdot\frac{(d\lambda_{12})\cdot(d\lambda_{23})}{2R^2}\cdot\overrightarrow{\mathbf{e}_N} = \frac{dS_{123}}{R^2}\cdot\overrightarrow{\mathbf{e}_N} = -K_G\ dS_{123}\cdot\overrightarrow{\mathbf{e}_N}.$$

The Signs' Rule acts here in hyperbolic case: if $\varepsilon > 0$, then $\theta_{13} < 0$; if $\varepsilon < 0$, then $\theta_{13} > 0$. We get the interdependent differentials: $d\theta_{13}$ and of the vector-area of the triangle S_{123}! Due to Lambert hyperbolic result [36] or, in general, to the Gauss–Bonnet Theorem [21, p. 533], the area of geodesic triangle 123 (on a perfect surface of negative constant Gaussian curvature $K_G = -1/R^2$) and the angular defect of the triangle $d\delta_{123} = 2\pi - (\varepsilon_1+\varepsilon_2+\varepsilon_3)$ are bonded as $d\theta_{13} = d\delta_{123} = -dS_{123}/R^2 = K_G\ dS_{123} < 0$ $(\theta = 0, S = 0)$. We get differential–integral Identity of the orthospherical shift and the Lambert angular defect in geodesic hyperbolic triangles on the hyperboloid II and on the Lobachevsky–Bolyai plane at $K_G = \text{const} < 0$:

$$d\theta_{13} = d\delta_{123} = -\frac{dS_{123}}{R^2} = K_G\ dS_{123} \ \Rightarrow\ \theta_{13} = \delta_{123} = -\frac{S_{123}}{R^2} = K_G\cdot S_{123}.\quad(144A-II)$$

These formulae mean: *the angle θ_{13} of orthospherical shifting and Lambert's angular defect δ_{123} in a hyperbolic triangle are equal* ! The assertion is true also for other figures as convex polygons (formed from triangles). This is inferred through their decomposition into triangles. (If such triangle is on a hyperspheroid in $\langle Q^{n+1}\rangle$, the similar formula for orthospherical shifting θ contains the sign $\pm$, see generally in (173A) and further in Ch. 8A.) Note, that the orthospherical shifting is more general notion, than the angular deviation for geodesic two-dimensional figures, and it acts also in tensor variants. Orthospherical tensor angle of rotation Θ_{13}, due to matrix formula (115A), is identical to tensor angular defect of a geodesic triangle (or other convex polygons) on the hyperboloid II. Angular deviations take place due to dependence of parallel displacement on surfaces with curvature on its way.

Conclusion. *Orthospherical induced shifting Θ gives the clear mathematical explanation to Lambertian angular defect of figures in hyperbolic geometry and Thomas precession in STR!*

* * *

In Ch. 5A, through trigonometric relation (79A) in the instantaneous Cartesian subbase $\tilde{E}^{(3)}_m$ in the Euclidean sub-space $\langle\mathcal{E}^3\rangle^{(m)}$, we introduced the *inner* 3*-acceleration* $\mathbf{g}$, directed along the instantaneous axis $x^{(m)}$. (An inner acceleration is always with zero time-projection in $\tilde{E}^{(4)}_m$.) And at collinear two-steps or integral motions, $\mathbf{g} = \mathbf{g}_\alpha$ is collinear with velocity $\mathbf{v}_\alpha$. But at non-collinear integral motions with the current velocity $\mathbf{v}_\alpha$ and the current inner acceleration $\mathbf{g} = \mathbf{g}_\beta$ in the instantaneous Cartesian sub-base $\tilde{E}^{(3)}_m$, we can decompose this current inner acceleration with the hyperbolic differential causing it into the parallel and normal parts by the Pythagorean Theorem using (137A) in $\tilde{E}^{(3)}_m$ of $\langle\mathcal{E}^3\rangle^{(m)}$, with respect to the direction of velocity $\mathbf{e}_\alpha$ in $\tilde{E}^{(4)}_1$, as follows:

$$\left.\begin{array}{l} d\gamma_\beta\cdot\mathbf{e}_\beta = \cos\varepsilon\; d\gamma_\beta\cdot\mathbf{e}_\alpha + \sin\varepsilon\; d\gamma_\beta\cdot\; \mathbf{e}_\nu = \overline{\overline{d\gamma_\beta}}\cdot\mathbf{e}_\alpha + \overset{\perp}{d\gamma_\beta}\cdot\mathbf{e}_\nu \Rightarrow \\ \Rightarrow \mathbf{g}_\beta = g_\beta\cdot\mathbf{e}_\beta = \cos\varepsilon\cdot g_\beta\cdot\mathbf{e}_\alpha + \sin\varepsilon\cdot g_\beta\cdot\mathbf{e}_\nu = \overline{\overline{g_\beta}}\cdot\mathbf{e}_\alpha + \overset{\perp}{g_\beta}\cdot\mathbf{e}_\nu \Rightarrow \\ \Rightarrow (d\gamma_\beta)^2 = \left(\overline{\overline{d\gamma_\beta}}\right)^2 + \left(\overset{\perp}{d\gamma_\beta}\right)^2, \quad g^2_\beta = \left(\overline{\overline{g_\beta}}\right)^2 + \left(\overset{\perp}{g_\beta}\right)^2. \end{array}\right\} \qquad (145A)$$

It is the *Local Absolute Euclidean Pythagorean theorem* for spherically orthogonal decomposition in the Cartesian subbase $\tilde{E}^{(3)}_m$ of the brutto differential $d\gamma\cdot\mathbf{e}_\beta$ and the inner 3-acceleration $g_\beta\cdot\mathbf{e}_\beta$, with respect to the directional vector $\mathbf{e}_\alpha$ of the hyperbolic angle of motion γ. The parallel part accelerates motion along the curve, the normal part rotates the direction of motion with its curve.

* * *

Relativistic formulae of the **Doppler effect** for the oscillations frequency of light [76, p. 39], from the hyperbolic tensor trigonometric point of view, have simple interpretation. It is necessary in the classical formulae to change spherical tangent $\tan\varphi_R = v/c$ for hyperbolic one $\tanh\gamma = v/c$ as was did with tangent relation for velocity in STR, and to introduce the relativistic secant factor (127A) for the proper time either of moving source of a light or moving Observer of a light source. In STR only a relative velocity v has importance! With the tangent-tangent analogy, we obtain:

$$\nu^{(2)}\cdot c\tau = \nu^{(1)}\cdot\Delta ct^{(1)} = \nu^{(1)}\cdot ct^{(1)}\cdot(1-\cos\alpha\cdot\tanh\gamma) \Rightarrow \;\nu^{(1)} = \nu^{(2)}\cdot\operatorname{sech}\gamma\;/(1-\cos\alpha\cdot\tanh\gamma),$$

where $\nu^{(2)}$ is the oscillations of light frequency from the source, $\nu^{(1)}$ is frequency felt by Observer N_1, α is the angle between a light ray and a velocity vector, sech γ is the relativistic factor, $t^{(1)}$ and τ are the equivalent time intervals in both these systems. There are four specific variants:

A. *Longitudinal meeting effect*: $\alpha = 0,\; \cos\alpha = +1$, i. e., the source becomes nearer. Then the "blue shift" of light frequency is observed.

B. *Longitudinal opposite effect*: $\alpha = \pi,\; \cos\alpha = -1$, i. e., the source becomes farer. Then the "red shift" of light frequency is observed.

C. *Transversal effect*: $\alpha = \pm\pi/2, \cos\alpha = 0$. Then Observer N_1 fixes the "red shift" too, but it is less than in case B due to Einsteinian time dilation in the moving source.

D. *The Doppler effect is absent* if $\cos\alpha = (1-\operatorname{sech}\gamma)/\tanh(\pm\gamma) \approx \tanh(\pm\gamma)/2$.

We get the extremal Doppler effects for light and other radiation at $\tanh\gamma = 1, \cos\alpha = \pm 1$.

And the **Hubble Law** can be expressed in the ancestral form through the relative change of the photons frequency as $-\Delta\nu/\nu = \tanh\gamma = v/c = Hl/c = Ht$ – see more in Ch. 9A.

* * *

Consider both trigonometric hyperboloids with the unity radius-parameter R.

The hyperboloid II (see Figure 4) has $R = \pm i$. Radius may be 4-velocity $\vec{\mathbf{c}} = c \cdot \mathbf{i}$. Represent the 4×1-radius-vector of the unity hyperboloid II as its principal pseudonormal $\mathbf{i} = \mathbf{r}_{(II)} = \mathbf{p}_{(II)}$ and the principal tangent $\mathbf{i}_{(I)}$ to hyperboloid I and to a world line in $\tilde{E}_1$

$$\mathbf{i} = \mathbf{r}_{(II)} = \mathbf{p}_{(II)} = \mathbf{i}_{12} = \begin{bmatrix} \sinh\ \gamma_{12} \cdot \mathbf{e}_\alpha \\ \cosh \gamma_{12} \end{bmatrix} = roth\ \Gamma_{12} \cdot \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = roth\ \Gamma_{12} \cdot \mathbf{i}_1, \qquad (146A)$$

where $\gamma > 0$ if $\Delta ct > 0$, $roth\ \Gamma_{12} = F(\gamma_{12}, \mathbf{e}_\alpha)$ due to (363). Its time-like invariant is

$$\mathbf{i}'_{1k} \cdot I^{\pm} \cdot \mathbf{i}_{1k} = \mathbf{sinh}'\gamma_{1k} \cdot \mathbf{sinh}\ \gamma_{1k} - \cosh^2 \gamma_{1k} = \sinh^2 \gamma_{1k} \cdot \mathbf{e}'_\alpha \mathbf{e}_\alpha - \cosh^2 \gamma_{1k} = -1. \ (147A - I)$$

$$\mathbf{i}'_{1k} \cdot \mathbf{i}_{1k} = -\mathbf{sin}' i\gamma_{1k} \cdot \mathbf{sin}\ i\gamma_{1k} - \cos^2 i\gamma_{1k} = \sin^2 i\gamma_{1k} \cdot \mathbf{e}'_\alpha \mathbf{e}_\alpha - \cos^2 i\gamma_{1k} = -1. \ (147A - II)$$

Here for unity hyperboloid-II as the Lambert's sphere of the imajinary radius $\pm i$, with both these pseudo-Euclidean and anti-Euclidean sine-cosine identical invariants, we denote:
sinh γ_{1k} is the 3×1-vector projection of $\mathbf{i}_{1k}$ into $\langle \mathcal{E}^3 \rangle^{(1)}$ parallel to $\vec{ct}^{(1)}$,
$\cosh \gamma_{1k}$ is the scalar projection of $\mathbf{i}_{1k}$ into $\vec{ct}^{(1)}$ parallel to $\langle \mathcal{E}^3 \rangle^{(1)}$. In addition,
tanh γ_{1k} is the cross 3×1-vector projection of $\mathbf{i}_{1k}$ into $\langle \mathcal{E}^3 \rangle^{(1)}$ parallel to $\vec{ct}^{(k)}$,
sech γ_{1k} is the cross scalar projection of $\mathbf{i}_{1k}$ into $\vec{ct}^{(1)}$ parallel to $\langle \mathcal{E}^3 \rangle^{(k)}$.
Consider two-steps geodesic motions $\mathbf{i}_{12}, \mathbf{i}_{23} \Rightarrow \mathbf{i}_{13}$ of an element on hyperboloid II along two hyperbolae in bases $\tilde{E}_1$ and $\tilde{E}_2$, with its polar clear description (see before in (111A)):

$$\{roth \Gamma_{23} \cdot\}_{\tilde{E}_2} \cdot \overset{\mathbf{i}_{12}}{\begin{bmatrix} \sinh\ \gamma_{12} \cdot \mathbf{e}_\alpha \\ \cosh \gamma_{12} \end{bmatrix}} = \{roth\ \Gamma_{23}\}_{\tilde{E}_2} \cdot \{roth\ \Gamma_{12}\}_{\tilde{E}_1} \cdot \overset{\mathbf{i}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = \qquad (148A)$$

$$= \{roth\ \Gamma_{12} \cdot (roth\ \Gamma_{23})_{\tilde{E}_1} \cdot roth^{-1}\ \Gamma_{12}\}_{\tilde{E}_2} \cdot roth\ \Gamma_{12} \cdot \overset{\mathbf{i}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdot \overset{\mathbf{i}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} =$$

$$= \{roth \Gamma_{12} \cdot\}_{\tilde{E}_1} \cdot \overset{\mathbf{i}_{23}}{\begin{bmatrix} \sinh\ \gamma_{23} \cdot \mathbf{e}_\beta \\ \cosh \gamma_{23} \end{bmatrix}} = roth\ \Gamma_{13} \cdot rot\ \Theta_{13} \cdot \overset{\mathbf{i}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} \equiv roth\ \Gamma_{13} \cdot \overset{\mathbf{i}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = \overset{\mathbf{i}_{13}}{\begin{bmatrix} \sinh\ \gamma_{13} \cdot \mathbf{e}_\sigma \\ \cosh \gamma_{13} \end{bmatrix}}.$$

Four final matrices are in canonical form in $\tilde{E}_1$. **This means clear solution of the task:** *To find geodesic passed through points 2 and 3.* We'll consider such two-steps summation on hyperboloid II in general tensor-vector-scalar (tvs) presentation to the end of big Ch. 10A.

The trajectory of hyperbolic geodesic motion $\mathbf{i}_{12} \rightarrow \mathbf{i}_{13}$ is in the cut of the hyperboloid II by the eigen pseudoplane of matrix $\{roth\ \Gamma_{12} \cdot (roth\ \Gamma_{23})_{\tilde{E}_1} \cdot roth^{-1}\Gamma_{12} \cdot\}_{\tilde{E}_2}$ including these two points with the hyperbola. Intersection of this pseudoplane with the projective hyperplane is a straight line segment in $\langle\langle \mathcal{E}^n \rangle\rangle$, it corresponds to this geodesic trajectory. A hyperbolic triangle on a hyperboloid II with iR is realized as a cycle of 3 geodesic motions:

$$\{roth\ \Gamma_{12}\}_{\tilde{E}_1} \mathbf{u}_1 = \mathbf{u}_{12}, \quad \{roth\ \Gamma_{23}\}_{\tilde{E}_2} \mathbf{u}_{12} = \mathbf{u}_{13}, \quad \{roth\ \Gamma_{31}\}_{\tilde{E}_3} \mathbf{u}_{13} = \mathbf{u}_1.$$

By (148A), for a *point element* $\mathbf{u}_1$, rotation Θ_{13} annihilates. The triangle cycle returns a *nonpoint object* in the start, but the object is turned in the base $\tilde{E}_3$ at angle Θ_{13}. The point of application of the nonpoint object moves as $\mathbf{u}_1 \rightarrow \mathbf{u}_{12} \rightarrow \mathbf{u}_{13} \rightarrow \mathbf{u}_1$ along three hyperbolic geodesic lines. Summation of two-step non-collinear hyperbolic motions, according to polar decomposition (111A), is represented as the motion along geodesic line γ_{13} in direction $\mathbf{e}_\sigma$ with the induced orthospherical rotation $rot\ \Theta_{13}$, but only for a nonpoint element.

The hyperboloid I (see Figure 4) has $R = \pm 1$.

Represent the 4×1-radius-vector of the unity hyperboloid I and its principal pseudonormal $\mathbf{p} = \mathbf{r}_{(I)} = \mathbf{p}_{(I)}$ also tangent $\mathbf{i}_{(II)}$ to hyperboloid II and pseudonormal to a world line in $\tilde{E}_1$

$$\mathbf{p} = \mathbf{r}_{(I)} = \mathbf{p}_{(I)} = \mathbf{p}_{12} = \begin{bmatrix} \cosh\ \gamma_{12} \cdot \mathbf{e}_\alpha \\ \sinh \gamma_{12} \end{bmatrix} = roth\ \Gamma_{12} \cdot \begin{bmatrix} \mathbf{e}_\alpha \\ 0 \end{bmatrix} = roth\ \Gamma_{12} \cdot \mathbf{p}_{1(\alpha)}, \quad (149A)$$

where $\gamma > 0$ if $\Delta ct > 0$, $roth\ \Gamma_{12} = F(\gamma_{12}, \mathbf{e}_\alpha)$. (Here $\mathbf{i}_{12}$ and $\mathbf{p}_{12}$ on II and I are conjugate – see at Figure 4 for $\mathbf{u}$ and $\mathbf{v}$ under radius-parameter R.) Its space-like invariant is

$$\mathbf{p}'_{1k} \cdot I^{\pm} \cdot \mathbf{p}_{1k} = \mathbf{cosh}'\gamma_{1k} \cdot \mathbf{cosh}\ \gamma_{1k} - \sinh^2 \gamma_{1k} = \cosh^2 \gamma_{1k} \cdot \mathbf{e}'_\alpha \mathbf{e}_\alpha - \sinh^2 \gamma_{1k} = +1. \ (150A - I)$$

or

$$\mathbf{p}'_{1k} \cdot \mathbf{p}_{1k} = \mathbf{cos}' i\gamma_{1k} \cdot \mathbf{cos}\ i\gamma_{1k} + \sin^2 i\gamma_{1k} = \cos^2 i\gamma_{1k} \cdot \mathbf{e}'_\alpha \mathbf{e}_\alpha + \sin^2 i\gamma_{1k} = +1. \ (150A - II)$$

Here for the hyperboloid-I as the sphere of the real-valued radius ± 1, we denote:
$\mathbf{cosh}\ \gamma_{1k}$ is the 3×1-vector projection of $\mathbf{p}_{1k}$ into $\langle \mathcal{E}^3 \rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(1)}$,
$\sinh \gamma_{1k}$ is the scalar projection of $\mathbf{p}_{1k}$ into $\overrightarrow{ct}^{(1)}$ parallel to $\langle \mathcal{E}^3 \rangle^{(1)}$. In addition,
$\mathbf{coth}\ \gamma_{1k}$ is the cross 3×1-vector projection of $\mathbf{p}_{1k}$ into $\langle \mathcal{E}^3 \rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(k)}$,
cosech γ_{1k} is the cross scalar projection of $\mathbf{p}_{1k}$ into $\overrightarrow{ct}^{(1)}$ parallel to $\langle \mathcal{E}^3 \rangle^{(k)}$.

With regard to the hyperboloid I, there is a dilemma with two possible variants of the tensor hyperbolic angle for points on it with the constant module of radius-vector $\rho(\mathbf{v})$:

1) or, as its argument leave the hyperbolic angle Γ so that for both hyperboloids their principal angles γ are symmetric with respect to the isotropic cone (see as at Figure 4).

2) or, as its argument one choose the complementary angle Υ (see as at Figure 4). But then the cosine-sine matrix of hyperbolic rotation must be replaced by the corresponding cotangent-cosecant rotation matrix with the complementary principal angle υ.

Both these variants are valid, but we choose below the first variant with the principal angle γ for two-steps motions on the hyperboloid I. Their matrices are bonded as follows:

$$\begin{matrix} roth\ \Gamma & = & \overline{roth}\ \Upsilon \end{matrix}$$

$$\left| \begin{array}{c|c} \frac{\cosh \gamma \cdot \overleftarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha}'}{\sinh \gamma \cdot \mathbf{e}'_\alpha} & \frac{\sinh \gamma \cdot \mathbf{e}_\alpha}{\cosh \gamma} \end{array} \right| \begin{matrix} \cdots \\ \cdots \end{matrix} \left| \begin{array}{c|c} \frac{\coth \upsilon \cdot \overleftarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha \mathbf{e}_\alpha}'}{\mathrm{csch}\ \upsilon \cdot \mathbf{e}'_\alpha} & \frac{\mathrm{csch}\ \upsilon \cdot \mathbf{e}_\alpha}{\coth \upsilon} \end{array} \right|. \quad (151A)$$

For the hyperboloid I, we begin two-steps transformations starting immediately from the 2-nd stage, when matrices are already expressed in the basis $\tilde{E}_1$, as it was shown in (148A):

$$\{roth\ \Gamma_{12}\}_{\tilde{E}_1} \cdot \overset{\mathbf{p}_{23}}{\begin{bmatrix} \cosh\ \gamma_{23} \cdot \mathbf{e}_\kappa \\ \sinh \gamma_{23} \end{bmatrix}} = \{roth\ \Gamma_{12}\}_{\tilde{E}_1} \cdot \{roth\ \Gamma_{23}\}_{\tilde{E}_1} \cdot \overset{\mathbf{p}_{1(\kappa)}}{\begin{bmatrix} \mathbf{e}_\kappa \\ 0 \end{bmatrix}} =$$

$$= roth\ \Gamma_{13} \cdot rot\ \Theta_{13} \cdot \overset{\mathbf{p}_{1(\kappa)}}{\begin{bmatrix} \mathbf{e}_\kappa \\ 0 \end{bmatrix}} = \{roth\ \Gamma_{13}\}_{\tilde{E}_1} \cdot \overset{\mathbf{p}^*_{1(\kappa)}}{\begin{bmatrix} \mathbf{e}^*_\kappa \\ 0 \end{bmatrix}} = \overset{\mathbf{p}_{13}}{\begin{bmatrix} \cosh\ \gamma_{13}{}^* \cdot \mathbf{e}^*_\sigma \\ \sinh \gamma_{13}{}^* \end{bmatrix}}. \quad (152A)$$

Here the directional cosine vector $\mathbf{e}^*_\beta$ of the second motion is orthospherically shifted, with respect to the original vector $\mathbf{e}_\beta$. The two-steps hyperbolic motions on the unity hyperboloid I are realized with topological constraints corresponding to the cotangent hyperplane model or more visually to the tangent cylindrical model outside the Cayley oval (sect. 12.1). They are possible iff hyperplane cotangent or cylindrical tangent projections of these motions may be connected by straight cotangents ($\mathbf{coth}\ \gamma_{1j}$) or tangent ($\mathbf{tanh}\ \gamma_{1j}$) segments without topological obstacles. *We'll continue considerations of such relations in Ch. 10A.*

As a result, the points of the unity hyperboloids II and I and in corresponding to them two concomitant hyperbolic and hyperbolic-elliptical geometries (see above and in Ch. 12) have the additional cotangent–cosecant negative and positive pseudo-Euclidean invariants:

$$\mathbf{i}' \cdot I^{\pm} \cdot \mathbf{i} = \mathbf{r}'_{(II)} \cdot I^{\pm} \cdot \mathbf{r}_{(II)} = \mathbf{csch}'\gamma \cdot \mathbf{csch}\ \gamma - \coth^2\gamma = -1 = i^2. \qquad (II)$$

$$\mathbf{p}' \cdot I^{\pm} \cdot \mathbf{p} = \mathbf{r}'_{(I)} \cdot I^{\pm} \cdot \mathbf{r}_{(I)} = \mathbf{coth}'\gamma \cdot \mathbf{coth}\ \gamma - \mathrm{csch}^2\gamma = +1 = 1^2. \qquad (I)$$

Recall, that due to the formulae of pseudo-Euclidean trigonometry and hyperbolic non-Euclidean geometry, we have the correspondences for the complementary hyperbolic angles:

$$\sinh(\Gamma, \Upsilon) = \mathrm{csch}\ (\Upsilon, \Gamma) \ \Leftrightarrow\ \sinh(\Gamma, \Upsilon) \cdot \sinh(\Upsilon, \Gamma) = I,$$

$$\cosh(\Gamma, \Upsilon) = \coth\ (\pm\Upsilon, \Gamma) \ \Leftrightarrow\ \tanh(\pm\Gamma, \Upsilon) = \mathrm{sech}(\Upsilon, \Gamma).$$

This determines strictly the geometric interdependence of these complementary angles shown at Figure 4 (Ch. 12), i. e., cotangent and cosecant cross projections of the angle Γ or Υ may be interpret as the usual orthoprojections of the angles Υ or Γ!

In both these cases, for the hyperboloids II and I in $\langle \mathcal{P}^{n+1} \rangle$, one may interpret clear these hyperbolic angles through their trigonometric projections by tangent and cotangent projective models either on the projective hyperplane or on the projective hypercylinder with respect to the trigonometric n-ball equivalent geometrically to the Cayley n-oval absolute.

With any own reflector metric tensor of $\langle \mathcal{P}^{n+1} \rangle$ *hyperboloids II and I are conjugated:*

$$\boxed{\mathbf{i}' \cdot I^{\pm} \cdot \mathbf{p} = \mathbf{p}' \cdot I^{\pm} \cdot \mathbf{i} = 0 \ \Leftrightarrow\ \mathbf{r}'_{(II)} \cdot I^{\pm} \cdot \mathbf{r}_{(I)} = \mathbf{r}'_{(I)} \cdot I^{\pm} \cdot \mathbf{r}_{(II)} = 0}\ .$$

* * *

Further, we describe in general form an algorithm for evaluating main characteristics of summary polysteps rotation (motion) in $\langle \mathcal{P}^{n+1} \rangle$ and $\langle \mathcal{P}^{3+1} \rangle \equiv \langle \mathcal{E}^3 \boxtimes \overrightarrow{ct} \rangle$ (see before in sect. 11.3, 11.4 and (111A)) in the tensor, vector and scalar forms. The algorithm starts with application of formula (485) for correct transformation of the initial unity base $\tilde{E}_1$. On the final step of the algorithm, the polar representation, according to (474)–(476) and (111A)–(118A), is used. On these stages, the homogeneous modal transformations are

$$\tilde{E}_t = roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdots roth\ \Gamma_{(t-1),t} \cdot \tilde{E}_1 = T_{1t} \cdot \tilde{E}_1,$$

$$T_{1t} = roth\ \Gamma_{1t} \cdot rot\ \Theta_{1t} = rot\ \Theta_{1t} \cdot roth\ \overset{\angle}{\Gamma}_{1t}\ .$$

$$T_{1t} \cdot T'_{1t} = roth^2\ \Gamma_{1t} = roth\ 2\Gamma_{1t}, \qquad T'_{1t} \cdot T_{1t} = roth^2\ \overset{\angle}{\Gamma}_{1t} = roth\ 2\ \overset{\angle}{\Gamma}_{1t},$$

$$rot\ \Theta_{1t} = roth^{-1}\Gamma_{1t} \cdot T_{1t} = roth\ (-\Gamma_{1t}) \cdot T_{1t}.$$

The latter gives $rot\ \Theta_{1t}$ as defect Θ_{1t} of the *Closed cycle of principal rotations!* We use $\mathbf{e}_\sigma$ and $\mathbf{e}_{\underset{\sigma}{\angle}}$, they are the directional vectors in structures (362), (363) for Γ_{tj} and $\overset{\angle}{\Gamma}_{tj}$;
$\cos\theta_{1t} = \mathbf{e}'_\sigma \cdot \mathbf{e}_{\underset{\sigma}{\angle}} = \dfrac{tr\ rot\ \Theta_{1t} - 2}{n-1}$ is the cosine form of orthospherical scalar shift θ in canonical structure (497). This formula is valid in $\langle \mathcal{P}^{n+1} \rangle$, see (497) and (120A).

The matrix $roth\ \Gamma_{1t}$ is evaluated at $n = 3$ in canonical forms (362) or generally – in form (363) or in cell form (324). The matrix $rot\ \Theta_{1t}$ is evaluated at $n = 3$ in canonical form (497) or generally – in cell form (259). Lorentzian contraction is evaluated with the use of the summary rotation angle Γ_{1t} and the hyperbolic deformational matrix with canonical structures (364), (365), in particular, for objects of Ch. 4A. However, $\mathbf{tanh}\,\Gamma_{1t}$ (the velocity) and $\mathrm{sech}\ \Gamma_{1t}$ (as the relativistic factor) may be computed directly from $\mathbf{sinh}\,\Gamma_{1t}$ and $\mathbf{cosh}\,\Gamma_{1t}$.

The canonical and polar forms of Lorentzian homogeneous transformation, in that number, for arbitrary and summarized polysteps principal motions:

$$T_{1t} = roth\ \Gamma_{12} \cdots roth\ \Gamma_{(t-1),t} = roth\Gamma \cdot rot\ \Theta = rot\ \Theta \cdot roth\ \overset{\angle}{\Gamma} = \qquad (153A).$$

$$= \left[\begin{array}{c|c} I_{3\times3} + (\cosh\gamma - 1)\cdot \mathbf{e}_\sigma \mathbf{e}'_\sigma & \sinh\gamma \cdot \mathbf{e}_\sigma \\ \hline \sinh\gamma \cdot \mathbf{e}'_\sigma & \cosh\gamma \end{array}\right] \cdot \left[\begin{array}{c|c} [rot\ \Theta]_{3\times3} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \end{array}\right] =$$

$$= \left[\begin{array}{c|c} [rot\ \Theta]_{3\times3} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} I_{3\times3} + (\cosh\gamma - 1)\cdot \mathbf{e}_{\overset{\angle}{\sigma}} \mathbf{e}'_{\overset{\angle}{\sigma}} & \sinh\gamma \cdot \mathbf{e}_{\overset{\angle}{\sigma}} \\ \hline \sinh\gamma \cdot \mathbf{e}'_{\overset{\angle}{\sigma}} & \cosh\gamma \end{array}\right] =$$

$$= \left[\begin{array}{c|c} [rot\ \Theta]_{3\times3} + (\cosh\gamma - 1)\cdot \mathbf{e}_\sigma \mathbf{e}'_{\overset{\angle}{\sigma}} & \sinh\gamma \cdot \mathbf{e}_\sigma \\ \hline \sinh\gamma \cdot \mathbf{e}'_{\overset{\angle}{\sigma}} & \cosh\gamma \end{array}\right] =$$

$$= \left[\begin{array}{c|c} [rot\ \Theta]_{3\times3} + (\cosh\gamma - 1)\cdot \cos\theta \cdot \overleftarrow{\mathbf{e}_\sigma \mathbf{e}'_{\overset{\angle}{\sigma}}} & \sinh\gamma \cdot \mathbf{e}_\sigma \\ \hline \sinh\gamma \cdot \mathbf{e}'_{\overset{\angle}{\sigma}} & \cosh\gamma \end{array}\right] \text{(Compare with symmetric tensor (100A)).}$$

$$\mathbf{e}_\sigma \mathbf{e}'_\sigma = \overleftarrow{\mathbf{e}_\sigma \mathbf{e}'_\sigma},\ \mathbf{e}_{\overset{\angle}{\sigma}} \mathbf{e}'_{\overset{\angle}{\sigma}} = \overleftarrow{\mathbf{e}_{\overset{\angle}{\sigma}} \mathbf{e}'_{\overset{\angle}{\sigma}}} = [rot'\Theta]_{3\times3} \cdot \overleftarrow{\mathbf{e}_\sigma \mathbf{e}'_\sigma} \cdot [rot\ \Theta]_{3\times3},\ \mathbf{e}_\sigma \mathbf{e}'_{\overset{\angle}{\sigma}} = \cos\theta \cdot \overleftarrow{\mathbf{e}_\sigma \mathbf{e}'_{\overset{\angle}{\sigma}}}. \qquad (154A)$$

If some $roth\ \Gamma_{ij}$ are collinear or if $n = q = 1$, then they are grouped. Formula (153A) gives also **General Law of summing principal rotations (motions)** in $\langle\mathcal{P}^{n+1}\rangle$, expressed in hyperbolic form (363) or at $n = 3$ in (362) in the original base $\tilde{E}_1 = \{I\}$. Now, with (153A), our readers may one time again be convinced in truety of all formulae for summing *two-steps rotations (motions)* inferred by explicit multiplications in beginning of this Ch. 7A.

The matrix $S = roth\ \Gamma$ is emanated, for example, from the last and lowest elements t_{44} and t_{k4} for general matrix T in (153A). They permit to express the matrix S in the base $\tilde{E}_1$ in canonical forms (362), (363) in $\langle\mathcal{P}^{n+1}\rangle$ and evaluate scalar and vector trigonometric functions in the angle γ with its directional vector $\mathbf{e}_\sigma$ and the angle θ. The matrix $rot\ \Theta$ in $\langle\mathcal{P}^{3+1}\rangle$ is computed in canonical form (497) with the use of (499) for $\sin\theta_{13}$ with the sign of θ, and $\mathbf{e}_N$. For $n = 3$ and k=1, 2, 3 we obtain, with final velocities $\mathbf{v}^*$ and $\mathbf{v}$, the following

$$\left.\begin{array}{l} \cosh\gamma = t_{44}, \sinh\gamma = \sqrt{\cosh^2\gamma - 1} = v^*/c,\ \tanh\gamma = v/c; \tanh\gamma_k = t_{k4}/t_{44}; \\ \cos\sigma_k = t_{k4}/\sinh\gamma,\ \cos\overset{\angle}{\sigma}_k = t_{4k}/\sinh\gamma,\ \mathbf{e}_\sigma = \{\cos\sigma_k\}, \mathbf{e}_{\overset{\angle}{\sigma}} = \{\cos\overset{\angle}{\sigma}_k\}. \\ \cos\theta_{13} = \mathbf{e}'_\sigma \cdot \mathbf{e}_{\overset{\angle}{\sigma}};\ \ \overrightarrow{\mathbf{r}_N}(\theta_{13}) = \mathbf{e}_{\overset{\angle}{\sigma}} \otimes \mathbf{e}_\sigma = \mp\sin\theta_{13} \cdot \overrightarrow{\mathbf{e}_N}\ \text{(last for } n = 3). \end{array}\right\} \qquad (155A)$$

Scalar final results do not change under the mirror permutation of particular motions. It leads merely to substitution in (153A): $T \to T'$ with $\Theta \to -\Theta$, $\mathbf{e}_\sigma \to \mathbf{e}_{\overset{\angle}{\sigma}}$.

Theorem. *In general, any polysteps noncollinear hyperbolic rotations* $roth\ \Gamma_{1t}$ *in* $\langle\mathcal{P}^{n+1}\rangle$ *or motions on hyperboloids are represented as hyperbolic one and single orthospherical shift.* Such interpretation of Law (153A) of summing hyperbolic motions in $\tilde{E}_1 = \{I\}$ is confirmed with polar decomposition (111A) in the pseudo-Euclidean space, where $rot\ \Theta$ is revealed, for example, in the hyperbolically shifted $\tilde{E}_{1h} = roth\ \Gamma_{1t} \cdot \tilde{E}_1$. In physical space-time $\langle\mathcal{E}^3 \boxtimes \overrightarrow{ct}\rangle$, it is confirmed experimentally by the Thomas precession of the electron spin – see further.

In the sequel, in accordance with our trigonometric approach, we shall use Cartesian sub-base $\tilde{E}_1^{(3)}$ of the universal base $\tilde{E}_1 = \{I\}$ analogous to projective *homogeneous coordinates* in the Euclidean projective hyperspace $\langle\langle\mathcal{E}^3\rangle\rangle$ (see in Ch. 12). Consider again the tangents (velocities) summation in scalar and vectorial trigonometric forms (138A) and (125A) inside the trigonometric ball as analog of the Cayley oval absolute with radii $R = 1$ for tangents and $R = c$ for velocities. Hyperbolic tangent models of principal motions are preferred, because they are limited by finite parameter 1 or R! This scale factor belongs to the finite tangent flat model of a hyperboloid II and to the finite tangent cylindrical model of a hyperboloid I. Indeed, there holds: $\tanh\gamma << \gamma < \sinh\gamma$. The hyperbolic cotangent models are infinite as well as sine one. Besides, in the tangent-cotangent models, the hyperbolic geodesics are straight lines, which are coaxial each other – see at Figure 4.

Consider in details the tangent flat model of principal topologically unlimited motions on a hyperboloid II (Figure 4A). It is identical to the projective Klein's model of the real-valued hyperbolic space, see in sect. 12.1. Though the analogous tangent model of a hyperboloid I is realized on the cylindrical model with taking into account topological constraints! We choose the origin O of this tangent subbase $\tilde{E}_1^{(3)}$ as the start point (1) of first tangent projection [12], the origin O' in the subbase is the following point (2) of second tangent projection [23], where both the projections are summarized, and so on up to the last summand. There is one to one correspondence between all these origins O in this limited tangent subbase $\tilde{E}_1^{(3)}$ and all these points k inside the Cayley oval. (*The coordinate velocity is* $\mathbf{v}_{ij} = \mathbf{c} \cdot \mathbf{tanh}\, \gamma_{ij}$.)

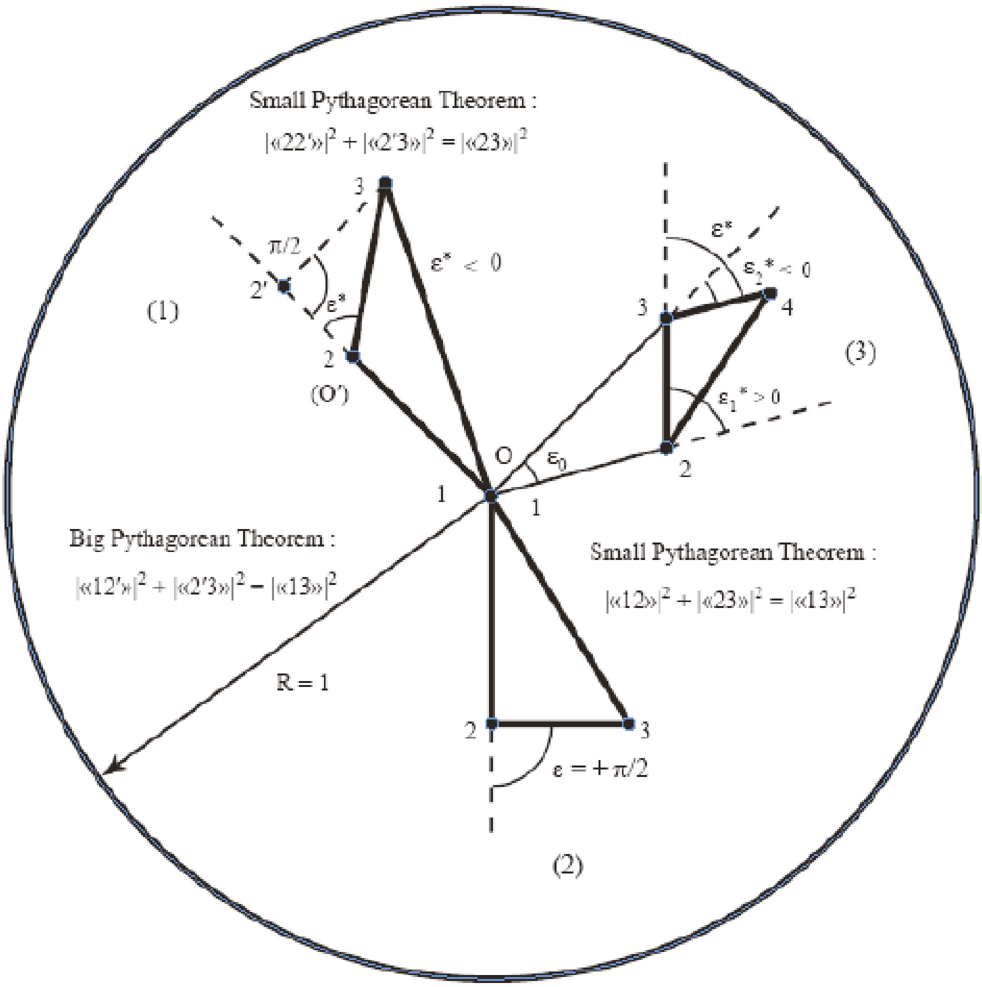

Figure 4A. Summing tangent projections of hyperbolic motions in the tangent (velocity) model due to the theorem on presentation of their sum in biorthogonal Pythagorean form.

<u>Variant 1</u>. *Centered* triangle in $\tilde{E}_1^{(3)}$:

$$[12] = \mathbf{tanh}\ \gamma_{12}, \quad [23] = \mathbf{tanh}\ \gamma_{23} \cdot k_1^* \cdot k_2 \cdot k_3^*, \quad [13] = \mathbf{tanh}\ \gamma_{13},$$

$$[22'] = \mathbf{tanh}\ \overline{\overline{\gamma}}_{23}, \quad [2'3] = \mathbf{tanh}\ \overset{\perp}{\gamma}_{23}, \quad \varepsilon^* = \pi - A_{123}^*, \quad A_{132}^* = \varepsilon^* - \varepsilon_0.$$

Variant 2. Centered right triangle in $\tilde{E}_1^{(3)}$:

$[12] = \mathbf{tanh}\ \gamma_{12}$, $[23] = \mathbf{tanh}\ \gamma_{23} \cdot \mathrm{sech}\ \gamma_{12}$, $[13] = \mathbf{tanh}\ \gamma_{13}$, $\varepsilon = A_{123} = \pi/2$.

Variant 3. Decentered triangle coplanar with center O in $\tilde{E}_1^{(3)}$: $\varepsilon_0 = A_{213}$,

$[23] = \mathbf{tanh}\ \gamma_{23}$, $\varepsilon_1^* = \pi - A_{123}^*$, $[34] = \mathbf{tanh}\ \gamma_{34}$, $\varepsilon_2^* = \pi - A_{134}^*$, $[24] = \mathbf{tanh}\ \gamma_{24}$,

$\varepsilon^* = \pi - A_{234}^* = \varepsilon_1^* + \varepsilon_2^* - \varepsilon_0 = \pi - \{\pi - \varepsilon_2^* - [\pi - \varepsilon_0 - (\pi - \varepsilon_1^*)]\}$.

* * *

The matrix of pure hyperbolic rotation in the base of its own determination $\tilde{E}_1$ can be considered as matrix-function $roth\ \Gamma_{12} = F(\gamma, \mathbf{e}_\alpha)$ due to its canonical form (363). Each such matrix with these two parameters γ and the vector of directional cosine $\mathbf{e}_\alpha$ implements motion of point (1) and determines any other point (k) inside the oval.

All centered tangent projections $\mathbf{tanh}\ \gamma_{12}$ are radiated from the point (1), i. e., center O of the tangent subbase $\tilde{E}_1^{(3)}$ (for example, along $\mathbf{e}_\alpha$). They are not distorted in Euclidean metric of the Euclidean projective space $\langle\langle\mathcal{E}^3\rangle\rangle$, i. e., its Euclidean length in $\tilde{E}_1^{(3)}$ corresponds exactly to $\mathbf{tanh}\,\gamma_{12}$. Moreover, the central spherical angles ε_0 between $\mathbf{tanh}\,\gamma_{1i}$ and $\mathbf{tanh}\,\gamma_{1j}$ in the tangent model are not distorted too. We shall take advantage of these facts!

Following motion γ_{23} starts at point (2). If it is directed along $\mathbf{e}_\alpha$, then in $\langle\langle\mathcal{E}^3\rangle\rangle$ the second motion in its tangent projection $\{\mathbf{tanh}\,\gamma_{23}\}_{\tilde{E}_1}$ is expressed in the same tangent subbase $\tilde{E}_1^{(3)}$ with these three coefficients of distortions in Euclidean subspace $\langle\langle\mathcal{E}^3\rangle\rangle$:

$$k_1 = \frac{\{\tanh\gamma_{13}\}_{\tilde{E}_1}}{\{\tanh\gamma_{12}\}_{\tilde{E}_1} + \{\tanh\gamma_{23}\}_{\tilde{E}_2}} = 1/(1 + \tanh\gamma_{23}\cdot\tanh\gamma_{12}) < 1.$$

$$k_2 \cdot k_3 = \frac{\{\tanh\gamma_{13}\}_{\tilde{E}_1} - \{\tanh\gamma_{12}\}_{\tilde{E}_1}}{\{\tanh\gamma_{23}\}_{\tilde{E}_2}} = \frac{\{\tanh\gamma_{23}\}_{\tilde{E}_1}}{\{\tanh\gamma_{23}\}_{\tilde{E}_2}} = \mathrm{sech}^2\gamma_{12} << 1,$$

where $k_2 = k_3 = \mathrm{sech}\ \gamma_{12}$. The first distortion is caused by *hyperbolic* summation of segments γ_{12} and γ_{23} as one for two *collinear segments*. The sequential distortion is combined from two factors. The first one $k_2 = \mathrm{sech}\ \gamma_{12}$ is Einsteinian dilation of time in the base $\tilde{E}_2$, the second one $k_3 = \mathrm{sech}\ \gamma_{12}$ is contraction of distance as result of *cross projecting at tangent mapping of distance* between two cross-bases (it is formally analogous in result to Lorentzian contraction of extent, when a distance in $\langle\mathcal{E}^3\rangle^{(2)}$ is reduced in $\tilde{E}_1$ due to its cross projecting into $\langle\mathcal{E}^3\rangle^{(1)}$ parallel to $\overrightarrow{ct}^{(2)}$, see in Ch. 4A).

In the triangle 123 (Figure 4A(1)), only the term [23] is distorted by k_2, k_3. Due to Pythagorean theorem (138A) in the big right triangle $\mathbf{12'3}$, its parallel projection [22'] is the difference of distorted parallel projection [12'] and undistorted term [12], i. e., [22'] is distorted by k_1^*, k_2, k_3; its normal projection [2'3] is distorted only by k_1^*, k_2:

$$\tanh\overline{\overline{\gamma}}_{23} = \frac{\cos\varepsilon\cdot\tanh\gamma_{23}\cdot\mathrm{sech}^2\gamma_{12}}{1 + \cos\varepsilon\cdot\tanh\gamma_{23}\cdot\tanh\gamma_{12}} = \cos\varepsilon\cdot\tanh\gamma_{23}\cdot k_1^*\cdot k_2\cdot k_3. \qquad (156A)$$

$$\tanh\overset{\perp}{\gamma}_{23} = \frac{\sin\varepsilon\cdot\tanh\gamma_{23}\cdot\mathrm{sech}\ \gamma_{12}}{1 + \cos\varepsilon\cdot\tanh\gamma_{23}\cdot\tanh\gamma_{12}} = \sin\varepsilon\cdot\tanh\gamma_{23}\cdot k_1^*\cdot k_2. \qquad (157A)$$

Note, the distorting coefficients of type $k*$ depend on the angle ε, and the coefficients of type k_3 act only on the parallel projection of $\mathbf{tanh}\ \gamma_{23}$ according to the Herglotz Principle (the last see initially in Ch. 2A).

Due to *Big Pythagorean theorem* (125A), (138A) in the right triangle 12'3 in $\tilde{E}_1$, there hold

$$\tanh^2 \gamma_{13} = \tanh^2[\gamma_{12} + \overline{\overline{\gamma}}_{23}] + \tanh^2 \overset{\perp}{\gamma}_{23},$$

$$\cos\varepsilon_0 = \mathbf{e}'_\sigma \cdot \mathbf{e}_\alpha = \tanh\,[\gamma_{12} + \overline{\overline{\gamma}}_{23}]/\tanh\gamma_{13}, \quad \sin\varepsilon_0 = \mathbf{e}'_\sigma \cdot \mathbf{e}_\beta = \tanh \overset{\perp}{\gamma}_{23}\,/\tanh\gamma_{13}.$$

With squared (156A) and (157A), we obtain in $\tilde{E}_1^{(3)}$ the *Small Pythagorean theorem* for the *right triangles* **22'3** and **123** as (130A), due to variants (1) and (2) at Figure 4A:

$$\mathbf{tanh}\gamma_{23} = \mathbf{tanh}\gamma_{13} - \mathbf{tanh}\gamma_{12} \to \{\tanh\ \gamma_{23}\}_{\tilde{E}_1} = \{\tanh\ \gamma_{23}\}_{\tilde{E}_2} \cdot k_1^* \cdot k_2 \cdot k_3^* =$$

$$= \sqrt{\tanh^2 \overline{\overline{\gamma}}_{23} + \tanh^2 \overset{\perp}{\gamma}_{23}} = \tanh\gamma_{23} \cdot k_1^* \cdot \mathrm{sech} \cdot \sqrt{\cos^2\varepsilon \cdot \mathrm{sech}^2\ \gamma_{12} + \sin^2\varepsilon}.$$

(Compare k_2 and k_3^* with coefficients of Lorentzian contraction – collinear (53A) and non-collinear (54A).) The Small Pythagorean theorem gives the general variant at Figure 4A(1) and the simplest variant at Figure 4A(2). For sine and tangent orthogonal summation, both Small Pythagorean theorems were inferred in (129A), (130A). Note, that we may apply geometrically the sine vectorial summation (without k_3) according to Pythagorean theorem (124A), (135A). But sine projections are non-limited by R. But in the spherical geometry (Ch. 8A) the sine projections are limited by R !

The decentered angles subject to distortions too. Consider distortion of the angle ε^* between **tanh** γ_{12} and **tanh** γ_{23} (Figure 4A(1)). Cross projecting transfers the origin of distorted vector **23** into point O'. The distorted angle ε^* is expressed in terms of the distorted projection **tanh** γ_{23} due to formulae of Euclidean scalar trigonometry:

$$\cos\varepsilon^* = \frac{\tanh\overline{\overline{\gamma}}_{23}}{\{\tanh\ \gamma_{23}\}_{\tilde{E}_2}} = \frac{\cos\varepsilon \cdot \mathrm{sech}\ \gamma_{12}}{\sqrt{\cos^2\varepsilon \cdot \mathrm{sech}^2\ \gamma_{12} + \sin^2\varepsilon}} = \cos\varepsilon \cdot k_3/k_3^* < \cos\varepsilon. \qquad (158A)$$

In STR ε^* is a distorted spherical angle between velocities $\mathbf{v}_{12}$ and $\mathbf{v}_{23}$ in the space $\langle\langle\mathcal{E}^3\rangle\rangle$. If $\varepsilon = \pi/2$, there is no distortion: $\cos\varepsilon^* = \cos\varepsilon = 0$, see this variant at Figure 4A(2).

For coplanar decentered motions in the plane $\langle\mathcal{E}^2\rangle \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle\rangle$ at Figure 4A(3), such angle ε^* is expressed in terms of distorted partial angles ε_1^*, ε_2^* and undistorted central angle ε_0 between **tanh** γ_{12} and **tanh** γ_{23}. These *open angles* π are not distorted too, that follows from (158A). By theorems of Euclidean scalar trigonometry, there holds:

$$\varepsilon^* = \varepsilon_1^* + \varepsilon_2^* - \varepsilon_0 = \pi - A_{234}^* = \pi - \{\pi - \varepsilon_2^* - [\pi - \varepsilon_0 - (\pi - \varepsilon_1^*)]\}, \qquad (159A)$$

$$\cos\varepsilon_1^* = \frac{\cos\varepsilon_1 \cdot \mathrm{sech}\ \gamma_{12}}{\sqrt{\cos^2\varepsilon_1 \cdot \mathrm{sech}^2\ \gamma_{12} + \sin^2\varepsilon_1}}, \quad \sin\varepsilon_1^* = \frac{\sin\varepsilon_1}{\sqrt{\cos^2\varepsilon_1 \cdot \mathrm{sech}^2\ \gamma_{12} + \sin^2\varepsilon_1}};$$

$$\cos\varepsilon_2^* = \frac{\cos\varepsilon_2 \cdot \mathrm{sech}\ \gamma_{13}}{\sqrt{\cos^2\varepsilon_2 \cdot \mathrm{sech}^2\ \gamma_{13} + \sin^2\varepsilon_2}}, \quad \sin\varepsilon_2^* = \frac{\sin\varepsilon_2}{\sqrt{\cos^2\varepsilon_2 \cdot \mathrm{sech}^2\ \gamma_{13} + \sin^2\varepsilon_2}};$$

$$\left.\begin{array}{c} \cos\varepsilon^* = \cos[\varepsilon_1^* + \varepsilon_2^* - \varepsilon_0] = \\ = \dfrac{[\cos\varepsilon_0 \cdot (\cos\varepsilon_1 \cdot \cos\varepsilon_2 \cdot \mathrm{sech}\ \gamma_{12} \cdot \mathrm{sech}\ \gamma_{13} - \sin\varepsilon_1 \cdot \sin\varepsilon_2) + \sin\varepsilon_0 \cdot (\sin\varepsilon_1 \cdot \cos\varepsilon_2 \cdot \mathrm{sech}\ \gamma_{13} + \cos\varepsilon_1 \cdot \sin\varepsilon_2 \cdot \mathrm{sech}\ \gamma_{12})]}{\sqrt{(\cos^2\varepsilon_1 \cdot \mathrm{sech}^2\ \gamma_{12} + \sin^2\varepsilon_1) \cdot (\cos^2\varepsilon_2 \cdot \mathrm{sech}^2\ \gamma_{13} + \sin^2\varepsilon_2)}}. \end{array}\right\}$$

Such summation of **tanh** γ_{23} and **tanh** γ_{34} is realized as $[12]+[23]^* = [13]$ under ε_1^* and $[13]+[34]^* = [14]$ under ε_2^*, see at Figure 4A(3). Further we have again variant 4A(1).

But, generally, with *non-coplanar summands*, for example, $\mathbf{tanh}\gamma_{34} \notin \langle\mathcal{E}^2\rangle \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle$, for the summation in $\langle\mathcal{E}^3\rangle$, (159A) do not hold. We choose $\mathbf{tanh}\gamma_{13} \cdot \mathbf{e}_{\sigma(13)} = \mathbf{tanh}\gamma_{13} \cdot \mathbf{e}_\sigma$ due to (138A) as the first segment and $\mathbf{tanh}\gamma_{34} \cdot \mathbf{e}_{\beta(34)}$ as the *third* segment. Further, we use (156A)-(159A) for this two-steps motions in $\langle\mathcal{E}^2\rangle \equiv \langle\mathbf{e}_{\sigma(13)}, \mathbf{e}_{\beta(34)}\rangle$, etc.!

* * *

Kinematics of a material body *progressive* movement is determined by kinematics of the material point M, which is the barycenter of homogeneous body. For the point M, distinction between non-relativistic and relativistic kinematics can be seen in projective representations of the point movement in the universal base $\tilde{E}_1 = \{I\}$ as original one. (For the current coordinate of the proper distance along the movement, we use in $\tilde{E}_1$ the greek notation $\chi = x^{(1)}$, introduced in (73A), by analogy with the proper time!)

In Lagrange space-time $\langle \mathcal{L}^{3+1} \rangle \equiv \langle \mathcal{E}^3 \oplus \overrightarrow{t} \rangle$:
the increment and differentials of *progressive* movement, with decomposition (137A) in $\langle \mathcal{E}^3 \rangle$, along a world line of point M change as follows:

$$\Delta \mathbf{x}^{(1)} = d\mathbf{x}^{(1)} + d^2\mathbf{x}^{(1)}/2! + \cdots = dx^{(1)} \cdot \mathbf{e}_\alpha + d^2x^{(1)} \cdot \mathbf{e}_\beta/2! + \cdots, \; d\mathbf{x}^{(1)} = d\chi \cdot \mathbf{e}_\alpha,$$

$$d^2\mathbf{x}^{(1)} = d^2\chi \cdot \mathbf{e}_\beta = d^2\chi \cdot (\cos\varepsilon \cdot \mathbf{e}_\alpha + \sin\varepsilon \cdot \mathbf{e}_\nu) = \overline{\overline{d^2\chi}} \cdot \mathbf{e}_\alpha + \overset{\perp}{d^2\chi} \cdot \mathbf{e}_\nu \equiv$$

$$\equiv d(d\chi \cdot \mathbf{e}_\alpha) = [\partial d\chi]_\alpha \cdot \mathbf{e}_\alpha + d\chi[\partial \mathbf{e}_\alpha]_{dx} = [\partial d\chi]_\alpha \cdot \mathbf{e}_\alpha + d\chi \left\{ ||\partial \mathbf{e}_\alpha|| \cdot \frac{\partial \mathbf{e}_\alpha}{||\partial \mathbf{e}_\alpha||} \right\}_{dx} =$$

$= [\partial d\chi]_\alpha \cdot \mathbf{e}_\alpha + d\chi \cdot [\partial\alpha]_{dx} \cdot \mathbf{e}_\nu$. Here we used for $\mathbf{e}_\beta$ decomposition (137A). That is why

$$[\partial d\chi]_\alpha = \cos\varepsilon \cdot d^2\chi = \overline{\overline{d^2\chi}}, \qquad d\chi \cdot [\partial\alpha]_{dx} = \sin\varepsilon \cdot d^2\chi = \overset{\perp}{d^2\chi};$$

$$\mathbf{v}(t) = \frac{d\mathbf{x}^{(1)}}{dt} = v_0 \cdot \mathbf{e}_\alpha(t_0) + \int_{t_0}^{t} \mathbf{g}(t)dt;$$

$$\mathbf{g}(t) = \frac{d^2\mathbf{x}^{(1)}}{dt^2} = g(t) \cdot \mathbf{e}_\beta(t) = \frac{\overline{\overline{d^2\chi}}}{dt^2} \cdot \mathbf{e}_\alpha(t) + \frac{\overset{\perp}{d^2\chi}}{dt^2} \cdot \mathbf{e}_\nu(t) = \overline{\overline{g}}(t) \cdot \mathbf{e}_\alpha(t) + \overset{\perp}{g}(t) \cdot \mathbf{e}_\nu(t),$$

$$\overline{\overline{g}}(t) = \cos\varepsilon(t) \cdot g(t) = \left[\frac{\partial d\chi}{dt^2}\right]_\alpha, \quad \overset{\perp}{g}(t) = \sin\varepsilon(t) \cdot g(t) = \frac{d\chi}{dt} \cdot \left[\frac{\partial\alpha}{dt}\right]_{dx} = v(t) \cdot w_\alpha(t), \text{etc.}$$

Orthospherical rotation $d\alpha$ or w_α does not change here a *progressive* nature of the movement.
The Law of Mechanical Energy Conservation holds as $[\cos\varepsilon \cdot F](t)d\chi(t) = d[mv^2/2]$.
The Law of Angular Momentum Conservation holds as $[\sin\varepsilon \cdot F \cdot \mathbf{e}_\nu](t)dt = d[mv \cdot \mathbf{e}_\alpha]$.
We see, that in Classic Mechanics similar Laws act separately and independently.

* * *

In Minkowski space-time $\langle \mathcal{P}^{3+1} \rangle \equiv \langle \mathcal{E}^3 \boxtimes \overrightarrow{ct} \rangle$:
with (80A), (137A), (145A), there hold:

$$\text{In } \tilde{E}_1: \quad \Delta\mathbf{x}^{(1)} \neq d\mathbf{x}^{(1)} + d^2\mathbf{x}^{(m)}/2! + \cdots, \quad d\mathbf{x}^{(1)} = dx \cdot \mathbf{e}_\alpha = d\chi \cdot \mathbf{e}_\alpha;$$

$$\left.\begin{aligned} &\text{In } \tilde{E}_m: \quad d^2\mathbf{x}^{(m)} = d^2x^{(m)} \cdot \mathbf{e}_\beta = d\gamma^{(m)} \cdot d(c\tau) \cdot \mathbf{e}_\beta = \\ &= d^2x^{(m)} \cdot (\cos\varepsilon \cdot \mathbf{e}_\alpha + \sin\varepsilon \cdot \mathbf{e}_\nu) = \overline{\overline{d^2x^{(m)}}} \cdot \mathbf{e}_\alpha + \overset{\perp}{d^2x^{(m)}} \cdot \mathbf{e}_\nu \Leftrightarrow \\ &\Leftrightarrow d\gamma = d\gamma \cdot \mathbf{e}_\beta = \cos\varepsilon\, d\gamma\, \mathbf{e}_\alpha + \sin\varepsilon\, d\gamma \cdot \mathbf{e}_\nu) = \overline{\overline{d\gamma}} \cdot \mathbf{e}_\alpha + \overset{\perp}{d\gamma} \cdot \mathbf{e}_\nu. \end{aligned}\right\}.$$

In $\langle \mathcal{P}^{3+1} \rangle$, the differentials $d\mathbf{x}^{(1)}$ and $d^2\mathbf{x}^{(m)}$ are not summed immediately unlike the case in $\langle \mathcal{L}^{3+1} \rangle$, as they are situated in different subspaces $\langle \mathcal{E}^3 \rangle$ and thus should be summed hyperbolically with the use of motion angle γ and its differentials $d\gamma^{(m)} = d\gamma$.

Then from the differential $\mathbf{d^2x}^{(m)}$ in the instantaneous base $\tilde{E}_m$ we obtain the *current inner 3-acceleration* (as pure Euclidean characteristic in $\langle\mathcal{E}^3\rangle^{(m)}$, gotten before for collinear motions in (79A), (82A), and $3D$ Absolute Pythagorean Theorem in (145A) in $\langle\mathcal{P}^{3+1}\rangle$:

$$\mathbf{g}(\tau) = c\frac{\mathbf{d}\gamma}{d\tau} = \mathbf{F}/m_0 = g(\tau)\cdot\mathbf{e}_\beta = \frac{dv^{(m)}\cdot\mathbf{e}_\beta}{d\tau} = \frac{d^2x\cdot\mathbf{e}_\beta}{d\tau^2} = \frac{\overline{\overline{d^2x}}\cdot\mathbf{e}_\alpha + \overset{\perp}{d^2x}\cdot\mathbf{e}_\nu}{d\tau^2} = \overline{\overline{g}}(\tau)\cdot\mathbf{e}_\alpha + \overset{\perp}{g}(\tau)\cdot\mathbf{e}_\nu.$$

In the base $\tilde{E}_m$, projective differentials $\overline{\overline{d^2x}}$ and $\overset{\perp}{d^2x}$ are situated together in $\langle\mathcal{E}^3\rangle^{(m)}$. Then in the instantaneous base $\tilde{E}_m$, these projections of the inner acceleration are folowing:

$c\frac{\overline{\overline{d\gamma}}}{d\tau} = \frac{\overline{\overline{dv^{(m)}}}}{d\tau} = \overline{\overline{g}}(\tau) = \cos\varepsilon\cdot g(\tau) = \overline{\overline{F}}/m_0$ is the *parallel proper 3-acceleration* with $\mathbf{e}_\alpha$.

$c\frac{\overset{\perp}{d\gamma}}{d\tau} = \frac{\overset{\perp}{dv^{(m)}}}{d\tau} = \sin\varepsilon\cdot g(\tau) = \overset{\perp}{g}(\tau) = v^*(\tau)\cdot w^*_\alpha(\tau) = \overset{\perp}{F}/m_0$ is the *normal proper 3-acceleration* with binormal unity vector $\mathbf{e}_\nu$.

By (119A) we get $\cos\varepsilon = \mathbf{e}'_\beta\mathbf{e}_\alpha$, $0\le\varepsilon\le\pi$ (acceleration in $[0;\pi/2)$, deceleration in $(\pi/2;\pi]$).

Evaluate differential variations of the basic scalar and vectorial trigonometric functions projected from the hyperboloids II and I [see preliminary in (132A), (146A) for II and in (133A), (149A) for I], including Euclidean projections inside and outside the trigonometric ball with $R=1$ and scalar projections on the time arrow, with produced further from them the space-like and time-like physical characteristics as velocities, accelerations, momentums and energy. We'll use formulae for two-steps motions (122A), (124A), (135A), (137A), (138A) for II with 3D Absolute Pythagorean theorems of type (145A). Analogous formulae for I will be gotten in last Ch. 10A. In result, at diffirentiation in the base $\tilde{E}_1$, we'll must obtain all trigonometric angular differentials with proportional to them physical vector and scalar characteristics. Here $d\alpha$ is the angle of the orthospherical rotation of the velocity $\mathbf{v}_\alpha$ or of $\mathbf{e}_\alpha$ (as above in the 4D Lagrange space-time), and it is the scalar value of $d\mathbf{e}_\alpha$.

For the correct results in such scalar and $3D$ evaluations of physical characteristics, we must use such a metric reflector tensor of the space-time $\langle\mathcal{P}^{3+1}\rangle$, in order to take into account usual adopted mathematical forms of them, connected with gotten trigonometric prototypes. For this correspondence, we use below the tensors $\{I^\mp\}$ and $\{I^\pm\}$, in accordance with the imaginary time-arrow of Poincaré and the classical real-valued Euclidean subspace.

On the hyperboloid I, constrained by its cylindrical topology (Ch. 12A), along the time arrow and in the Euclidean directions $\mathbf{e}_\beta\neq\mathbf{e}_\alpha$, there hold:

$$d\sinh\gamma = \cosh\gamma_q\ d\gamma_q = \cosh\gamma\ d\gamma. \tag{160A}$$

$$\mathbf{cosh}\ \gamma = \sinh\gamma\cdot\mathbf{coth}\ \gamma = \cosh\gamma\cdot\mathbf{e}_\alpha\ \rightarrow\ \mathbf{coth}\ \gamma = \frac{\mathbf{dx}}{dy} = \frac{dx}{dy}\cdot\mathbf{e}_\alpha = \coth\gamma\cdot\mathbf{e}_\alpha,$$

$$\left.\begin{aligned} &d\ \mathbf{cosh}\ \gamma = d(\cosh\gamma\cdot\mathbf{e}_\alpha) = \sinh\gamma\ d\gamma\cdot\mathbf{e}_\alpha + \cosh\gamma\ d\alpha_2\cdot\mathbf{e}_\mu = \\ &= \sinh\gamma_q\ d\gamma_q\cdot\mathbf{e}_\kappa = \sinh\gamma_q\cdot[\cos\epsilon\ d\gamma_q\cdot\mathbf{e}_\alpha + \sin\epsilon\ d\gamma_q\cdot\mathbf{e}_\mu], \\ &|d\ \mathbf{cosh}\ \gamma|^2 = \sinh^2\gamma\ d\gamma^2 + \cosh^2\gamma\ d\alpha_2^2 = \sinh^2\gamma_q\ (d\gamma_q)^2 = \\ &= \sinh^2\gamma_q\cdot[(\cos\epsilon\ d\gamma_q)^2 + (\sin\epsilon\ d\gamma_q)^2] = \sinh^2\gamma_q\ [(\overline{\overline{d\gamma_q}})^2 + (\overset{\perp}{d\gamma_q})^2], \\ &\mathbf{cosh}\ \gamma = \cosh\gamma\cdot\mathbf{e}_\alpha(\gamma) = \\ &= \cosh\gamma_0\cdot\mathbf{e}_{\alpha(0)} + \int_{\gamma_0}^{\gamma}[\sinh\gamma\ d\gamma\cdot\mathbf{e}_\alpha + \cosh\gamma\ d\alpha_2\cdot\mathbf{e}_\mu]. \end{aligned}\right\} \tag{161A}$$

See in detail in (238A).

On the hyperboloid II (at its top sheet), along the time arrow and in the Euclidean directions $\mathbf{e}_\beta \neq \mathbf{e}_\alpha$, there hold:

$$\cosh\gamma = \frac{d(ct)}{d(c\tau)} \;\rightarrow\; d\cosh\gamma = d\frac{d(ct)}{d(c\tau)} = \sinh\gamma_p\, d\gamma_p = \sinh\gamma\, d\gamma. \qquad (162A)$$

$$\mathbf{sinh}\,\gamma = \frac{d\mathbf{x}}{d(c\tau)} = \frac{d\chi}{d(c\tau)}\cdot\mathbf{e}_\alpha = \sinh\gamma\cdot\mathbf{e}_\alpha = \frac{\mathbf{v}^*}{c},\;\; \mathbf{tanh}\,\gamma = \frac{d\mathbf{x}}{d(ct)} = \frac{d\chi}{d(ct)}\cdot\mathbf{e}_\alpha = \tanh\gamma\cdot\mathbf{e}_\alpha = \frac{\mathbf{v}}{c};$$

$$\left.\begin{array}{l} d\,\mathbf{sinh}\,\gamma = d(\sinh\gamma\cdot\mathbf{e}_\alpha) = \cosh\gamma\; d\gamma\cdot\mathbf{e}_\alpha + \sinh\gamma\; d\alpha_1\cdot\mathbf{e}_\nu = \\ = \cosh\gamma_p\; d\gamma_p\cdot\mathbf{e}_\beta = \cosh\gamma_p\cdot[\cos\varepsilon\; d\gamma_p\cdot\mathbf{e}_\alpha + \sin\varepsilon\; d\gamma_p\cdot\mathbf{e}_\nu], \\ \\ |d\,\mathbf{sinh}\,\gamma|^2 = \cosh^2\gamma\; d\gamma^2 + \sinh^2\gamma\; d\alpha_1^2 = (\cosh\gamma_p\; d\gamma_p)^2 = (\cosh\gamma\; d\gamma)^2 = \\ = \cosh^2\gamma_p\cdot[(\cos\varepsilon\; d\gamma_p)^2 + (\sin\varepsilon\; d\gamma_p)^2] = \cosh^2\gamma_p\;[(\overline{\overline{d\gamma_p}})^2 + (\overset{\perp}{d\gamma_p})^2]. \\ \\ \mathbf{sinh}\,\gamma = \sinh\gamma\cdot\mathbf{e}_\alpha(\gamma) = \\ = \sinh\gamma_0\cdot\mathbf{e}_{\alpha(0)} + \int_{\gamma_0}^{\gamma}[\cosh\gamma\; d\gamma\cdot\mathbf{e}_\alpha + \sinh\gamma\; d\alpha_1\cdot\mathbf{e}_\nu]. \end{array}\right\} \quad (163A)$$

If $\gamma_p = 0$, we get the Local Absolute Pythagorean theorem (145A). See in detail in (228A).

$$\left.\begin{array}{l} d\,\mathbf{tanh}\,\gamma = d(\tanh\gamma\cdot\mathbf{e}_\alpha) = \mathrm{sech}^2\gamma\; d\gamma\cdot\mathbf{e}_\alpha + \tanh\gamma\; d\alpha_1\cdot\mathbf{e}_\nu = \\ = \mathrm{sech}^2\gamma_p\; d\gamma_p\cdot\mathbf{e}_\beta = \mathrm{sech}^2\gamma_p\cdot[\cos\varepsilon\; d\gamma_p\cdot\mathbf{e}_\alpha + \sin\varepsilon\; d\gamma_p\cdot\mathbf{e}_\nu], \\ \\ |d\,\mathbf{tanh}\,\gamma|^2 = \mathrm{sech}^4\; d\gamma^2 + \tanh^2\gamma\; d\alpha_1^2 = (\mathrm{sech}^2\gamma_p\; d\gamma_p)^2 = (\mathrm{sech}^2\gamma\; d\gamma)^2 = \\ = \mathrm{sech}^4\gamma_p\cdot[(\cos\varepsilon\; d\gamma_p)^2 + (\sin\varepsilon\; d\gamma_p)^2] = \mathrm{sech}^4\gamma_p\cdot[(\overline{\overline{d\gamma_p}})^2 + (\overset{\perp}{d\gamma_p})^2]; \\ \\ \mathbf{tanh}\,\gamma = \tanh\gamma\cdot\mathbf{e}_\alpha(\gamma) = \\ = \tanh\gamma_0\cdot\mathbf{e}_{\alpha(0)} + \int_{\gamma_0}^{\gamma}[\mathrm{sech}^2\gamma\; d\gamma\cdot\mathbf{e}_\alpha + \tanh\gamma\; d\alpha_1\cdot\mathbf{e}_\nu]. \end{array}\right\} \quad (164A)$$

Relations (161A)–163A) give us differential and integral summation on the Minkowski hyperboloids I and II of these three vector trigonometric functions with the change of their angular arguments and directions, in addition, to two-steps summations. General tensor-vector-scalar (tvs) summation with metric forms along a world-line see in last Ch. 10A.

We see in (164A), that $|d\mathbf{tanh}\;\gamma| \ll |d\gamma'|$, which causes the limitation of the tangent motion model by $R = 1$ in the trigonometric ball (the Cayley oval) at Figure 4A. In this limited flat tangent model, one may begin the motion either from the origin O (at $\gamma' = 0$ and with $\mathbf{e}_\alpha$) or from the non-centered point O' (at $\gamma' > 0$). In any case, summation or integration is realized in the projective hyperspace $\langle\langle\mathcal{E}^3\rangle\rangle$ inside this trigonometric ball with $R = 1$ (for *coordinate velocity* v with $R = c$). On the other hand, we see in non-limited flat sine model (163A), that analogous motion summation or integration is realized with the same direction vector $\mathbf{e}_\alpha$ (!), in all the Euclidean projective hyperspace, in that number for *proper velocity* v^*. The angle γ (with its vector of the directional cosines) is main angular argument of these motion models. For transferring to accelerations we use bond (79A).

In last Ch. 10A, we will give complete $4D$ representations for these hyperbolic sine–cosine differentials variations with parallel strict inference of the general $4D$ metric forms along a world line and on both its concomitant hyperboloids with accompanying calculations of adjacent geometric and physical characteristics, in particular, as movable tetrahedron – all in tensor trigonometry form with vector and scalar orthoprojections (i. e., in "tvs" forms).

* * *

The *summary 3-vector of proper velocity* $\mathbf{v}^*(\tau)$ of a particle M or the barycenter of a body M may be strictly inferred trigonometrically with (163A) and from the parallel and normal inner accelerations with the use of the proper time, but formally in $\langle\langle\mathcal{E}^3\rangle\rangle^{(1)}$):

$$\mathbf{v}^*(\tau)-\mathbf{v}^*(\tau_0)=c\cdot(\mathbf{sinh}\ \gamma-\mathbf{sinh}\ \gamma_0)=v^*(\tau)\cdot\mathbf{e}_\alpha(\tau)-v^*(\tau_0)\cdot\mathbf{e}_\alpha(\tau_0)= \qquad (165A)$$

$$=c\int_{\tau_0}^{\tau}\cos\varepsilon(\tau)\cdot\cosh\gamma_p(\tau)\cdot\frac{d\gamma_p}{d\tau}\,d\tau\cdot\mathbf{e}_\alpha(\tau)+c\int_{\tau_0}^{\tau}\sin\varepsilon(\tau)\cdot\cosh\gamma_p(\tau)\cdot\frac{d\gamma_p}{d\tau}\,d\tau\cdot\mathbf{e}_\nu(\tau)=$$

$$=\int_{\tau_0}^{\tau}\cosh\gamma(\tau)\cdot\left[c\cdot\frac{d\gamma}{d\tau}\right]\,d\tau\cdot\mathbf{e}_\alpha(\tau)+\int_{\tau_0}^{\tau}\left[c\cdot\sinh\gamma(\tau)\cdot\frac{d\alpha}{d\tau}\right]\,d\tau\cdot\mathbf{e}_\nu(\tau)=$$

$$=\int_{\tau_0}^{\tau}\frac{\overline{\overline{dv^*}}}{d\tau}\,d\tau\cdot\mathbf{e}_\alpha(\tau)+\int_{\tau_0}^{\tau}v^*(\tau)\cdot w^*_\alpha(\tau)\,d\tau\cdot\mathbf{e}_\nu(\tau)=$$

$$=\int_{\tau_0}^{\tau}\cosh\gamma(\tau)\cdot\overline{\overline{g}}(\tau)\,d\tau\cdot\mathbf{e}_\alpha(\tau)+\int_{\tau_0}^{\tau}\overset{\perp}{g}(\tau)\,d\tau\cdot\mathbf{e}_\nu(\tau),$$

where: $d\alpha$ - is the differential of the non-relativistic spherical rotation of the vector $\mathbf{e}_\alpha(\tau)$; $\cosh\gamma\cdot\overline{\overline{g}}(\tau)=\frac{\overline{\overline{dv^*}}}{d\tau}=\overline{\overline{g}}\,^*(\tau)$ is the *tangential inner acceleration*, $v^*=c\cdot\sinh\gamma$ – see (82A); $c\frac{\overset{\perp}{d\gamma}}{d\tau}=\frac{\overset{\perp}{dv}{}^{(m)}}{d\tau}=\overset{\perp}{g}[t(\tau)]=v^*(\tau)\cdot w^*_\alpha(\tau)$ is the *normal inner acceleration* in time τ. Parallel and normal inner accelerations in $\tilde{E}_m$ satisfy the **3***D Relative Pythagorean theorem*: $\overline{\overline{g}}\,^{*2}+\overset{\perp}{g}{}^2=g^2$ (see about it in details in last Ch. 10A). $w^*_\alpha(\tau)=d\alpha/d\tau$ is the *proper angular velocity* of the Euclidean part of rotations of a world line (or of $\mathbf{e}_\alpha$) at a point M in $\langle\mathcal{E}^3\rangle^{(m)}$.

The *summary 3-vector of coordinate velocity* $\mathbf{v}(t)$ at the point M may be strictly inferred also trigonometrically with (164A) and with the use of the coordinate time in $\langle\langle\mathcal{E}^3\rangle\rangle^{(1)}$):

$$\mathbf{v}(t)-\mathbf{v}(t_0)=c\cdot(\mathbf{tanh}\ \gamma-\mathbf{tanh}\ \gamma_0)=v(t)\cdot\mathbf{e}_\alpha(t)-v(t_0)\cdot\mathbf{e}_\alpha(t_0)= \qquad (166A)$$

$$=c\int_{t_0}^{t}\cos\varepsilon\cdot\mathrm{sech}^2\gamma_p(t)\cdot\frac{d\gamma_p}{dt}\,dt\cdot\mathbf{e}_\alpha(t)+c\int_{t_0}^{t}\sin\varepsilon\cdot\mathrm{sech}^2\gamma_p(t)\cdot\frac{d\gamma_p}{dt}\,dt\cdot\mathbf{e}_\nu(t)=$$

$$=\int_{\tau_0}^{\tau}\mathrm{sech}^2\gamma(\tau)\cdot\left[c\cdot\frac{d\gamma}{d\tau}\right]\,d\tau\cdot\mathbf{e}_\alpha(\tau)+\int_{\tau_0}^{\tau}\mathrm{sech}^2\gamma(\tau)\cdot\left[c\cdot\sinh\gamma(\tau)\cdot\frac{d\alpha}{d\tau}\right]d\tau\cdot\mathbf{e}_\nu(\tau)=$$

$$=\int_{t_0}^{t}\frac{\overline{\overline{dv}}}{dt}dt\cdot\mathbf{e}_\alpha(t)+\int_{t_0}^{t}v(t)\cdot w^*_\alpha[\tau(t)]\,dt\cdot\mathbf{e}_\nu(t)=$$

$$=\int_{t_0}^{t}\mathrm{sech}^3\gamma(t)\cdot\overline{\overline{g}}[\tau(t)]\,dt\cdot\mathbf{e}_\alpha(t)+\int_{t_0}^{t}\mathrm{sech}\ \gamma(t)\cdot\overset{\perp}{g}[\tau(t)]\,dt\cdot\mathbf{e}_\nu[\tau(t)],$$

where $t_0=\tau_0,\ t=t(\tau)$ along motion (85A). The parallel *coordinate acceleration* as (83A) is

$$\overline{\overline{g}}^{(1)}(t)=\mathrm{sech}^3\ \gamma\cdot\overline{\overline{g}}[\tau(t)]=\frac{\overline{\overline{dv}}}{dt}. \qquad (167A)$$

The *normal coordinate acceleration* with normal τ (84A) and parallel (85A) t to v times is

$$\overset{\perp}{g}{}^{(1)}(t)=\mathrm{sech}\ \gamma\cdot\overset{\perp}{g}[\tau(t)]=\frac{\overset{\perp}{dv}}{dt}=v(t)\cdot w^*_\alpha[\tau(t)]=w(t)\cdot v^*_\alpha[\tau(t)]. \qquad (168A)$$

(But, in fact, the time in the normal direction of motion streams as proper time τ.) From here we get the STR formulae for parallel and normal force parts acting on M in $\tilde{E}_1$:

$$\overline{\overline{F}}=\cos\varepsilon\cdot m_0\,g=m_0\cdot\cosh^3\gamma\cdot\overline{\overline{g}}^{(1)}(t),\quad \overset{\perp}{F}=\sin\varepsilon\cdot m_0\,g=m_0\cdot\cosh\gamma\cdot\overset{\perp}{g}{}^{(1)}(t).$$

The current *proper distance* is evaluated by analogous two ways with the separation in two time parameters $t_0 = \tau_0$, and $t = t(\tau)$ under condition (84A), (85A) of simultaneity. In the base $\tilde{E}_1$, from (165A) and (166A) we obtain two identical integrals for $\mathbf{x}$ at $\tau < t$:

$$\mathbf{x}_\tau(\tau) - \mathbf{x}_0 \equiv \mathbf{x}_t(t) - \mathbf{x}_0 = \int_{\tau_0}^{\tau} v^*(\tau) \cdot \mathbf{e}_\alpha(\tau)\ d\tau \equiv \int_{t_0}^{t} v(t) \cdot \mathbf{e}_\alpha(t)\ dt \equiv$$

$$\equiv \int_{\tau_0}^{\tau} \left[v_0^* \cdot \mathbf{e}_\alpha(\tau_0) + \int_{\tau_0}^{\tau} \cosh\gamma(\tau) \cdot \overline{\overline{g}}(\tau)\ d\tau \cdot \mathbf{e}_\alpha(\tau) + \int_{\tau_0}^{\tau} \overset{\perp}{g}(\tau)\ d\tau \cdot \mathbf{e}_\nu(\tau) \right] d\tau =$$

$$= \int_{\tau_0}^{\tau} \left[v_0^* \cdot \mathbf{e}_\alpha(\tau_0) + \int_{\tau_0}^{\tau} \overline{\overline{g}}^{\,*}(\tau)\ d\tau \cdot \mathbf{e}_\alpha(\tau) + \int_{\tau_0}^{\tau} \overset{\perp}{g}(\tau)\ d\tau \cdot \mathbf{e}_\nu(\tau) \right] d\tau \equiv$$

$$\equiv \int_{t_0}^{t} \left[v_0 \cdot \mathbf{e}_\alpha(t_0) + \int_{t_0}^{t} \mathrm{sech}^3\gamma(t) \cdot \overline{\overline{g}}(t)\ dt \cdot \mathbf{e}_\alpha(t) + \int_{t_0}^{t} \mathrm{sech}\gamma(t) \cdot \overset{\perp}{g}(t)\ dt \cdot \mathbf{e}_\nu(t) \right] dt =$$

$$= \int_{t_0}^{t} \left[v_0 \cdot \mathbf{e}_\alpha(t_0) + \int_{t_0}^{t} \overline{\overline{g}}^{(1)}(t)\ dt \cdot \mathbf{e}_\alpha(t) + \int_{t_0}^{t} \overset{\perp}{g}{}^{(1)}(t)\ dt \cdot \mathbf{e}_\nu(t) \right] dt. \qquad (169A)$$

Variation of the time-like hyperbolic cosine differential (not according to its expression by scalar product (162A) on the hyperboloid II), is proportional to the work of the tangential inner force (81A) causing a rectilinear part of *free progressive movement of material point M*:

$$\left.\frac{d(ct)}{d(c\tau)}\right|_{\tau_0}^{\tau} = \int_{\gamma_0}^{\gamma} d\cosh\gamma = \int_{\gamma_0}^{\gamma} \mathbf{sinh}\gamma\ \mathbf{d}\gamma = \int_{\gamma_0}^{\gamma} (\sinh\gamma \cdot \mathbf{e}_\alpha)\ (d\gamma \cdot \mathbf{e}_\beta) = \int_{\gamma_0}^{\gamma} \cos\varepsilon(\tau) \cdot \sinh\gamma\ d\gamma =$$

$$= \frac{1}{c^2} \cdot \int_{\tau_0}^{\tau} \cos\varepsilon(\tau) \cdot v^*(\tau) \cdot g(\tau) d\tau = \frac{1}{c^2} \cdot \int_{t_0}^{t} \cos\varepsilon[\tau(t)] \cdot v[\tau(t)] \cdot g[\tau(t)]\ dt = \frac{1}{c^2} \cdot \int_{\chi_0}^{\chi} \cos\varepsilon(\chi) \cdot g(\chi)\ d\chi =$$

$$= \frac{1}{m_0 c^2} \cdot \int_{\chi_0}^{\chi} \cos\varepsilon(\chi) \cdot F(\chi)\ d\chi = \frac{1}{m_0 c^2} \cdot \int_{\chi_0}^{\chi} \overline{\overline{F}}(\chi)\ d\chi = \frac{A}{m_0 c^2} = \frac{A}{E_0} = \frac{\Delta E}{E_0} = k_E. \qquad (170A)$$

If $\gamma_0 = 0$, $(v_0 = 0)$, then $\boxed{k_E = \cosh\gamma - 1 = A/E_0} \ \Rightarrow\ \boxed{E = \cosh\gamma \cdot E_0 = E_0 + A = mc^2}$.

k_E is a *factor of energy increment*: $k_E \cdot E_0 = A$. We infer, that during progressive motion of a body its total energy $E = mc^2$ is the *hyperbolic cosine orthoprojection* of the *tensor of energy–momentum* $\mathcal{T}_E = c \cdot \mathcal{T}_P$ (Ch. 5A) onto the axis $\overrightarrow{ct}^{(1)}$; it is conservative under $\mathbf{F} = \mathbf{0}$.

In 1900, genius Henri Poincaré in his well-known now article [62] inferred first (!!!) the fundamental physical relation between energy and mass as $m = E/c^2$ identical to $E = mc^2$, for the light's energy, as a kind of electromagnetic radiation. Later in 1905, Albert Einstein inferred relation $m = E/c^2$ (but as often for him, without reference to article above – see in the end of Ch.12) for the thermal radiation energy of a hot body, on the basis of the Planck quantum theory of radiation by massive body [88]. In 1908, Gilbert Lewis confirmed the analogous relation $E = mc^2$ (of course, with reference to Einstein's article) between increments of relativistic kinetic energy of a moving body and of its relativistic inertial mass [68]. However, only after the historical event when the very respected scientist Lise Meitner accurately considered the fact of uranium fission in the experiments of her colleagues – chemists Otto Hahn and Friedrich Strassmann (bombarding thorium with neutrons) and explained the mass defect in such a process by this fundamental relation $m = E/c^2$, physicists and not only they will paid superextra great attention onto this formula, with well-known further consequences for all peoples! However, according to the Rules of Scientific Ethics, priority in the discovery of this fundamental formula belongs to Henri Poincaré, if the present scientific community continues and will continue to comply with these Rules.

In Ch. 5A, we marked that as a true progenitor of concepts momentum and energy, in the relativistic sense, should be considered the *own 4-momentum* $\mathbf{P}_0 = m_0\mathbf{c}$ (98A-II). It is 4-th column of tensor of momentum $\mathcal{T}_P$ (101A), proportional with coefficient m_0c to our trigonometric measureless tensor of motion (100A) in the space-time $\langle\mathcal{P}^{3+1}\rangle$, i. e., we have:

$$\mathbf{P}_0 = P_0\cdot\mathbf{i}_\alpha = m_0\cdot\mathbf{c}_\alpha = P_0\cdot\begin{bmatrix}\mathbf{sinh}\ \gamma\\ \cosh\gamma\end{bmatrix} = P_0\cdot\begin{bmatrix}\sinh\ \gamma\cdot\mathbf{e}_\alpha\\ \cosh\gamma\end{bmatrix} = \begin{bmatrix}\mathbf{p}\\ P\end{bmatrix}.$$

It is preserved under $\mathbf{F}=\mathbf{0}\leftrightarrow\mathbf{P}_0=\mathbf{Const}$. The scalar value $P_0 = m_0c = E_0/c$ is pseudo-Euclidean invariant for the particle or body M. The own 4-momentum $\mathbf{P}_0$ is a hypotenuse of the *pseudo-Euclidean right triangle of three momenta*. Its sides are in the *pseudoplane of motion* $\langle\mathbf{e}_\alpha,\mathbf{i}_1\rangle$, which is similar to ones of the interior right triangle at Figure 1A(1), because $\mathcal{T}_P = P_0\cdot roth\,\Gamma$. We get again the *Absolute pseudo-Euclidean Pythagorean Theorem of three momenta* (98A-I):

$$\mathbf{P}_0 = P_0\cdot\mathbf{i} = P\cdot\mathbf{i}_1 + p\cdot\mathbf{j}\Rightarrow(iP_0)^2 = (iP)^2 + p^2 = -P_0^2 = -P^2+p^2.\ (\text{for tensor } I^\pm).$$

We may adopt that $m = P/c$, $E = P\cdot c$, $\mathbf{p} = m\mathbf{v}$. The own momentum $\mathbf{P}_0 = P_0\cdot\mathbf{i} = m_0\mathbf{c}$, as absolute 4-vector in $\langle\mathcal{P}^{3+1}\rangle$, is the *geometric invariant* along a world line to the Lorentzian transformations, where $\mathbf{c} = c\cdot\mathbf{i}$ is 4-velocity of Poincaré. $\mathbf{P}_0$ and $\mathbf{c}$ are always tangential to a world line. *Its variable cosine projection* onto the time arrow $\overrightarrow{ct}^{(1)}$ is the *total momentum* $\mathbf{P} = P\cdot\mathbf{i}_1 = P_0\cdot\cosh\gamma\cdot\mathbf{i}_1$. *Its variable 3-vector sine projection* into the Euclidean space $\langle\mathcal{E}^3\rangle^{(1)}$ is the *real momentum* $\mathbf{p} = p\cdot\mathbf{j} = P_0\cdot\mathbf{sinh}\ \gamma = P_0\cdot\sinh\gamma\cdot\mathbf{e}_\alpha = m_0\mathbf{v}^* = m\mathbf{v}$. Both these relative momenta are expressed in the base $\tilde{E}_1$ of the Minkowskian space-time $\langle\mathcal{P}^{3+1}\rangle$.

This illustrates, that during progressive movement the real momentum $\mathbf{p} = m\mathbf{v}$ of body or particle M is the *hyperbolic sine orthoprojection* of the *tensor of momentum* $\mathcal{T}_P$ into $\langle\mathcal{E}^3\rangle^{(1)}$. The tensors $\mathcal{T}_E$ and $\mathcal{T}_P$ with 4-momentum $\mathbf{P}_0$ are conservative under $\mathbf{F}=\mathbf{0}$.

The real momentum $\mathbf{p}(t)$ as sine projection of $\mathbf{P}_0$ into $\langle\mathcal{E}^3\rangle^{(1)}$ due to (165A) changes as

$$\equiv\mathbf{p}(t) = p[\tau_0(t_0)]\cdot\mathbf{e}_\alpha[\tau_0(t_0)] + m_0\int_{t_0}^{t}\{\ \overline{\overline{g^*}}[\tau(t)]\cdot\mathbf{e}_\alpha[\tau(t)] + \overset{\perp}{g}\ [\tau(t)]\cdot\mathbf{e}_\nu[\tau(t)\}\ d\tau(t) =$$

$$= p[\tau_0(t_0)]\cdot\mathbf{e}_\alpha[\tau_0(t_0)] + \int_{t_0}^{t}\{\ \overline{\overline{F^*}}[\tau(t)]\cdot\mathbf{e}_\alpha[\tau(t)] + \overset{\perp}{F}\ [\tau(t)]\cdot\mathbf{e}_\nu[\tau(t)\}\ d\tau(t).$$

* * *

In STR and external non-Euclidean geometry on the hyperboloid II in $\langle\mathcal{P}^{3+1}\rangle\equiv\langle\mathcal{E}^3\rangle\boxtimes\overrightarrow{y}$, according to (141A) and (144A-I) above, any progressive non-collinear motion of a particle or a body M is accompanied by the induced orthospherical shift $d\theta$ or precession in time w_β^* of the 3-rd normal axis $\mathbf{e}_\mu^{(m)}$ of the current normal plane $\langle\mathcal{E}^2\rangle_{Ns}^{(m)}\equiv\langle\mathbf{e}_\alpha^{(m)},\mathbf{e}_\beta^{(m)}\rangle\equiv\langle\mathbf{e}_\alpha^{(m)},\mathbf{e}_\nu^{(1)}\rangle$, – both rotated with the angular velocity w_α^* under hyperbolic inclination γ to the immobile axis $\mathbf{e}_\mu^{(1)}$ of the base $\tilde{E}_1$ and hence with the cosine slope to $\mathbf{e}_\mu^{(1)}$. The latter and rotated $\mathbf{e}_\mu^{(m)}$ have the common point of application O as the center of $\tilde{E}_1$. The rotated (normally to $\mathbf{e}_\alpha^{(m)}$) vector $\mathbf{e}_\nu^{(1)}$ has the point of application in the body M barycenter. Accordingly, the slower this rotation, the smaller these induced effects up to zero. Initially we have the elements from (144A-I): $\mathbf{tanh}\ (\gamma_\iota/2) = \tanh(\gamma_\iota/2)\cdot\mathbf{e}_\alpha^{(m)}$, $\mathbf{d}\gamma_p = d\gamma_p\cdot\mathbf{e}_\beta^{(m)}$, $\mathbf{e}_\alpha^{(m)}\times\mathbf{e}_\nu^{(1)} = \mathbf{e}_\mu^{(m)}\equiv\overrightarrow{\mathbf{e}_N}^{(m)}$.

To develop expression (144A-I) in $\langle\mathcal{E}^3\rangle\subset\langle\mathcal{P}^{3+1}\rangle\equiv\langle\mathcal{E}^3\rangle\boxtimes\overrightarrow{ct}$, taking in account (162A), we add the so-called *normal relations* at $\gamma_p = 0$ [see more at $\gamma_p\neq\mathbf{0}$ in (230A), Ch. 10A]:

$$\sin\varepsilon\ d\gamma_p = \overset{\perp}{d\gamma_p} = \sinh\gamma_\iota\ d\alpha_1\Leftrightarrow\sin\varepsilon\cdot g_\beta = \overset{\perp}{g_\beta} = v_\iota^*\cdot w_\alpha^*. \qquad (171A)$$

It follows if compare normal increments in (132A) and (135A) at $\mathbf{e}_\nu$, or both in (163A). We'll obtain normal relations with generalization and its rigorous justification in the Absolute Pythagorean theorems in last Ch. 10A, produced by the *differential tensor trigonometry*.

With our tensor trigonometric approach continuing (144A) with (171A), we get generating chain of clarity understood tvs formulae for the induced orthospherical shift and precession.

$$\left.\begin{aligned}
&-\mathbf{d}\theta = -d\theta\cdot\overrightarrow{\mathbf{e}_N} = \mathbf{tanh}\ (\gamma/2)\otimes\mathbf{d}\gamma = \frac{\mathbf{tanh}\ \gamma}{1+\mathrm{sech}\ \gamma}\otimes\mathbf{d}\gamma = \frac{\mathbf{sinh}\gamma}{\cosh\gamma+1}\otimes\mathbf{d}\gamma = \\
&= \frac{\sinh\gamma}{\cosh\gamma+1}\cdot\mathbf{e}_\alpha\otimes(d\gamma\cdot\mathbf{e}_\beta) = \frac{\cosh\gamma-1}{\sinh\gamma}\cdot\mathbf{e}_\alpha\otimes(d\gamma\cdot\mathbf{e}_\beta) = \\
&= \tanh\frac{\gamma}{2}\cdot\sin\varepsilon\ d\gamma\cdot\overrightarrow{\mathbf{e}_N} = \frac{\cosh\gamma-1}{\sinh\gamma}\cdot\sin\varepsilon\ d\gamma\cdot\overrightarrow{\mathbf{e}_N} = \frac{\cosh\gamma-1}{\sinh\gamma}\ \overset{\perp}{d\gamma}\cdot\overrightarrow{\mathbf{e}_N} = \\
&= \frac{\cosh\gamma-1}{\sinh\gamma}\cdot\sinh\gamma\ d\alpha\cdot\overrightarrow{\mathbf{e}_N} = (\cosh\gamma-1)\ d\alpha\cdot\overrightarrow{\mathbf{e}_N} = k_E\ d\alpha\cdot\overrightarrow{\mathbf{e}_N} = \\
&= [(d\alpha)^* - d\alpha]\cdot\overrightarrow{\mathbf{e}_N} \approx 1/2\ \gamma^2\ d\alpha\cdot\overrightarrow{\mathbf{e}_N}; \\
&-\frac{\mathbf{d}\theta}{d\tau} = w_\theta^*\cdot\overrightarrow{\mathbf{e}_N} = \tanh\frac{\gamma}{2}\cdot\sin\varepsilon\cdot\frac{d\gamma}{d\tau}\cdot\overrightarrow{\mathbf{e}_N} = \tanh\frac{\gamma}{2}\cdot\sinh\gamma\ \cdot\frac{d\alpha}{d\tau}\cdot\overrightarrow{\mathbf{e}_N}, \\
&-\frac{\mathbf{d}\theta}{dt} = w_\theta\cdot\overrightarrow{\mathbf{e}_N} = \tanh\frac{\gamma}{2}\cdot\sinh\gamma\cdot w_\alpha\cdot\overrightarrow{\mathbf{e}_N} = \tanh\frac{\gamma}{2}\cdot\frac{v^*\cdot w_\alpha}{c}\cdot\overrightarrow{\mathbf{e}_N} = \\
&= (\cosh\gamma-1)\cdot w_\alpha\cdot\overrightarrow{\mathbf{e}_N} = k_E\ w_\alpha\cdot\overrightarrow{\mathbf{e}_N} = (w_\alpha^* - w_\alpha)\cdot\overrightarrow{\mathbf{e}_N} \approx 1/2\ (\tfrac{v}{c})^2\cdot w_\alpha\cdot\overrightarrow{\mathbf{e}_N}.
\end{aligned}\right\}\quad(172A)$$

Thus, in the instantaneous plane $\langle\mathcal{E}^2\rangle^{(m)} \equiv \langle\mathbf{e}_\alpha^{(m)},\mathbf{e}_\nu^{(1)}\rangle$, the orthospherical as if orbital rotation w_α^* of an electron in a hydrogen atom $\mathcal{H}$, due to its planetary model of Bohr, as a microscopic gyroscope with its *orbital momentum* L (in addition to its proper momentum named by *spin*), induces mathematically (!) and in a result physically, the orthospherical precession of the electron orbit axis $\overrightarrow{\mathbf{e}_N}^{(m)} \equiv \mathbf{e}_\alpha^{(m)}\times\mathbf{e}_\nu^{(1)} = \mathbf{e}_\mu^{(m)}$ contrary to direction of w_α^* and with a lower angular velocity w_θ^*, fixed as w_θ in the base $\bar{E}_1$ (called sometimes as the laboratory system) with its 3-rd immobile axis $\overrightarrow{\mathbf{e}_N} = \mathbf{e}_\mu^{(1)}$. In accordance with the Lorentzian group, this precession causes the negative difference $-w_\theta$ in $\bar{E}_1$ between relativistic and non-relativistic maps of the electron rotation as $(w_\alpha^* - w_\alpha)$, perceived by Observer in $\langle\mathcal{E}^3\rangle^{(1)}$.

At the value $\varepsilon = \pi/2$ for the electron rotation, this precession causes the additional correction to spin–orbital interaction in normal direction with the coefficient "1/2", which came to be known as the *Thomas half*. Such interpretation by Llewellyn Thomas in 1926) [93] was the first independent confirmation of STR with its foundation as a theory of the new relativistic space-time with the Lorentzian transformations of coordinates, having a *group nature*, developed by the great Henri Poincaré in 1905 [63] in result of his very successful collaboration with the contemporary to him eminent physicist Hendrik Lorentz!

Expression (172A) gives *immediate and simplest tensor trigonometric explanations of the induced orthospherical shift with the Thomas precession and angular deviations in both non-Euclidean geometries* associated in Chs. 7A, 8A with this shift under the angle $\pm d\theta$ as:

$$\boxed{d\theta = (1-\cosh\gamma)\ d\alpha = [d\alpha - (d\alpha)^*] < 0 \leftrightarrow d\theta = (1-\cos\varphi)\ d\alpha = [d\alpha - (d\alpha)^*] > 0.}\quad(173A)$$

This angular shift is caused by "angular dissonance" between the true local orthospherical increment $d\alpha$ *in* $\langle\mathcal{E}^3\rangle^{(m)}$ *on the trajectory of non-collinear motion, fixed from an electron moving on its orbit, and its cosine projection* $(d\alpha)^*$*, perceived in* $\langle\mathcal{E}^3\rangle^{(1)}$*, according to STR; and in hyperbolic geometry on the hyperboloid II, in spherical geometry on the hyperspheroid.*

Translating this angular shift in time, as in (172A), with the use of these two physical "relativistic factors γ and β ", we come in the base $\bar{E}_1$ to the well-known in STR *physical* relativistic formula by L. Föppl and P. Daniell [91], who in 1913 (!) in Göttingen predicted theoretically such a kind of precession as the kinematic effect of STR (quite possible, with the use for such rotations the time dilation, introduced by Herman Minkowski before in 1908 (see in Ch. 3A) and published also in "Göttingen Nachrichten" [66] without trigonometry:

$$w_\theta = d\theta/dt = -(\cosh\gamma - 1)\cdot w_\alpha \equiv -w_\alpha[1/\sqrt{1-\beta^2} - 1] = -w_\alpha\cdot(\gamma-1).\qquad(A)$$

The Thomas precession is caused by the fact, that the Euclidean normal plane $\langle\mathcal{E}^2\rangle_N^{(m)}$ of the sine binormal rotation $d\alpha_1$ has its current local slope $\cosh\gamma_t$ exactly in the place of a particle M (here the electron). It is interpreted either as the difference of the same angular velocity in two bases $\tilde{E}_m$ and $\tilde{E}_1$, or as if the rotation of the difference $[d\alpha-(d\alpha)^*]<0$ with velocity w_θ, which is fixed separately by Observer in the immobile laboratory system $\tilde{E}_1$.

Similar reverse angular dissonance can be observed even at a home. To see it, you need to swirl the water in a round sink. As a result of braking only of the lower layer of water due to friction, we'll see an imaginary counter-rotation of water at a much lower angular speed!

This precession, due to (172A), is approximated by area of triangle with sides v/c, g/c and angle ε between them. Besides, expression (172A) is represented exactly in the physical relativistic form, but without "c" , through *angular velocities* g/v and g/v^*:

$$\frac{\mathbf{d}\theta}{d\tau}=\mathbf{e}_\alpha\times\left[\frac{\mathbf{g}}{v}-\frac{\mathbf{g}}{v^*}\right]=\mathbf{e}_\alpha\times\mathbf{e}_\beta\left[\frac{g}{v}-\frac{g}{v^*}\right]=-\sin\varepsilon\cdot\left[\frac{g}{v}-\frac{g}{v^*}\right]\cdot\overrightarrow{\mathbf{e}_N}=-\left[\frac{\overset{\perp}{g}}{v}-\frac{\overset{\perp}{g}}{v^*}\right]\cdot\overrightarrow{\mathbf{e}_N}. \qquad (B)$$

This induced orthospherical precession is explained by the matrix formulae of type (111A) for differential summing non-collinear two-steps hyperbolic motions Γ and $d\Gamma$ in $\langle\mathcal{P}^{3+1}\rangle$, with appearance in result of the same induced orthospherical precession $d\theta/dt$. In vector formulae for two-steps hyperbolic motion, due to (141A), the sign "$-$" for rotation of precessing axis $\overrightarrow{\mathbf{e}_N}$ illustrates the following **Rule** $sgn\ \theta_{13}=-sgn\ \varepsilon$ in the pseudo-Euclidean space of the theory of relativity and in the hyperbolic space of velocities. These are also mathematical and physical clear confirmation of the imaginary nature of hyperbolic motion angles as $i\gamma$.

From the point of view of the hypothetic Observer in the uninertial base $\tilde{E}_m$, this induced orthospherical precession with internal rotation of a body M in $\langle\mathcal{E}^3\rangle^{(m)}$ is caused by the manifestation of the *Coriolis acceleration* g_C from the force $F_C=mg_C$ in the base $\tilde{E}_m$. Then from (172A) and using connecting relation (79A), we obtain this *Coriolis acceleration* of the body or particle in the space-time $\langle\mathcal{P}^{3+1}\rangle$ acting under angular hyperbolic velocity $\eta_\gamma=d\gamma/d\tau=g/c$ of the base $\tilde{E}_m$ with exact formula and with approximation:

$$\mathbf{g}_C=\left[c\cdot\frac{\mathbf{d}\theta}{d\tau}+c\cdot\frac{\mathbf{d}\theta}{dt}\right]\approx 2c\cdot\frac{d\theta}{d\tau}\cdot\overrightarrow{\mathbf{e}_N}=2[c\cdot w_\theta^*]\cdot\overrightarrow{\mathbf{e}_N}\approx-\sin\varepsilon\cdot v\cdot\frac{d\gamma}{d\tau}\cdot\overrightarrow{\mathbf{e}_N}=-\sin\varepsilon\cdot v\cdot\eta^*\cdot\overrightarrow{\mathbf{e}_N}. \quad (C)$$

The Thomas precession may have an *ephemeral* character, so, for rectilinear motions. This has a place if $\mathbf{e}_\beta=\text{const}$, and it is not obligatory that $\mathbf{e}_\beta=\mathbf{e}_\alpha$. This is accelerated (decelerated) physical movement in the plane $\langle\mathcal{E}^2\rangle\equiv\langle\mathbf{e}_{\alpha(0)},\mathbf{e}_\beta\rangle\equiv\langle\mathbf{e}_{\alpha(0)},\mathbf{e}_\mu\rangle\equiv\text{Const}$ with $\mathbf{v}_0$ under the angle ε_0 to $\mathbf{e}_\beta=\text{const}$. In the origin of the base $\tilde{E}_1$, such a world line slope corresponds to $\tanh\gamma_0=v_0/c$ with $\mathbf{e}_{\alpha(0)}$. Execute the hyperbolic modal transformation of the base as $roth\ \Gamma\cdot\tilde{E}_1=\tilde{E}_{1h}$ with $\gamma=\gamma_0$ and $\mathbf{e}_\alpha=\mathbf{e}_{\alpha(0)}$. Then, in this new base, we annihilate the rotation $d\theta$, because in it $\tanh\gamma$ ($\mathbf{v}$) and $\mathbf{d}\gamma$ ($\mathbf{g}$) are collinear vectors ($\sin\varepsilon=0$). Such modal transformation is equivalent to translation in the base with $\mathbf{v}=\mathbf{v}_0$.

Thomas precession was the first in 1926 real confirmation of the Theory of Relativity with its basis Poincaré – Minkowski $4D$ space-time, thanks to the remarkable work of Llewellyn Thomas [93], which was deservedly awarded the Nobelean Prize. He has explained with the STR group approach the anomalous normal effect of Pieter Zeeman with the spin-orbital interaction of an electron in the hydrogen atom. In Ch. 9A, we'll show that the Thomas precession has a relation to executing of the Law of Energy conservation by its own part.

In our time, the nature of mass inertia with confirmation of the Mach Principle and the Principle of Relativity by Galileo–Poincaré was inferred in works of the very eminent now scientist Peter Higgs [82] by the Higgs field with its quantum particle "bozon", discovered experimentally in 2012. The Higgs theory has confirmed in fact the Poincaré – Minkowski space-time of our Universe! We have maintained the same opinion since 1-st edition of this our book in 2004 [15], despite fierce resistance from some aggressive apologists of the GTR.

Chapter 8A

Trigonometric models of two-steps and polysteps motions in quasi-Euclidean and spherical geometries

Definition of the *quasi-Euclidean* oriented space $\langle \mathcal{Q}^{n+1} \rangle$ (sect. 5.7) is similar, but only in a certain extent, to that for the pseudo-Euclidean Minkowski space $\langle \mathcal{P}^{n+1} \rangle$ (sect. 12.1) – see together in sect. 6.3. The *reflector-tensor* $I^{\pm}$ or $I^{\mp}$ (17A) is also important in $\langle \mathcal{Q}^{n+1} \rangle$. It determines orientation and admitted own transformations in this space. But the metric of the quasi-Euclidean space is Euclidean! In geometry of $\langle \mathcal{Q}^{n+1} \rangle$, the quasi-Euclidean tensor trigonometry act with their spherical functions and reflectors. They are defined in canonical forms, with respect to the universal base $\tilde{E}_1 = \{I\}$, mainly, by the principal rotations $rot\ \Phi$ (313), (314) with the frame axis and by secondary ones $rot\ \Theta$ (how in hyperbolic case too).

The main geometric (with the radius R) and trigonometric (with the unity radius) object of this binary space $\langle \mathcal{Q}^{n+1} \rangle$ is an *oriented hyperspheroid*, centralized in the universal base $\tilde{E}_1 = \{I\}$ with the origin O for all admitted $\tilde{E}_k$. It is oriented along its frame axis $\overrightarrow{y}^{(1)}$ – see at Figure 4 in Ch. 12 (similar to orientation of both Minkowski hyperboloids in $\langle \mathcal{P}^{n+1} \rangle$). The origin O is also a center of all orthospherically connected universal quasi-Cartesian bases $\tilde{E}_{1u} = rot\ \theta \cdot \tilde{E}_1$. The rotations $rot\ \Theta$, admitted usually by the coaxially oriented reflector tensor $I^{\pm}$ (17A-I), express, in the external *quasi-Euclidean geometry in* $\langle \mathcal{Q}^{n+1} \rangle$, the induced or free orthospherical rotations under summing non-collinear principal spherical rotations $rot\ \Phi$; but, in the internal *spherical geometry on the hyperspheroid*, these rotations give angular excess in closed geometric figures, composed from geodesic large circles on the hyperspheroid. The *absolute space* $\langle \mathcal{Q}^{n+1} \rangle$ is represented in any quasi-Cartesian base $\tilde{E}_k$ as the spherically orthogonal direct sum of relative axis $\overrightarrow{y}^{(k)}$ and Euclidean subspace $\langle \mathcal{E}^n \rangle^{(k)}$:

$$\langle \mathcal{Q}^{n+1} \rangle \equiv \langle \mathcal{E}^n \rangle^{(k)} \boxplus \overrightarrow{y}^{(k)} \equiv \text{CONST}, \quad \Delta y > 0, \tag{174A}$$

where $\langle \mathcal{E}^n \rangle$ is a Euclidean hyperplane, $\overrightarrow{y}$ is an oriented down or up frame axis for angle φ.

From a point of view of the *quasi-Euclidean tensor trigonometry*, also the subspace $\langle \mathcal{E}^n \rangle^{(k)}$ is k-th Euclidean hyperplane and $\overrightarrow{y}^{(k)}$ is a k-th cosine axis. The imaginarization of the axis $\overrightarrow{y}$ transforms our real-valued quasi-Euclidean binary space $\langle \mathcal{Q}^{n+1} \rangle$ into the complex-valued quasi-Euclidean binary space of index $q = 1$ by Poincaré (see in sect. 6.1), isometric to the real-valued pseudo-Euclidean binary space by Minkowski with the same $I^{\pm}$.

The following operations are admitted in $\langle \mathcal{Q}^{n+1} \rangle$ with right bases:
1) rotations of the two types: as principal spherical $rot\ \Phi$ and orthospherical $rot\ \Theta$;
2) parallel translations preserving the space structure (174A) with reflector tensor $I^{\pm}$.

The *principal tensors of rotations (motions)* $\langle rot\ \Phi \rangle$ execute principal spherical rotations (motions) with the frame axis $\overrightarrow{y}$ at spherical angles Φ in $\langle \mathcal{Q}^{n+1} \rangle$; the *free or induced tensors of rotations* $\langle rot\ \Theta \rangle$ execute orthospherical rotations and shifts at orthospherical angles Θ in the Euclidean part of $\langle \mathcal{Q}^{n+1} \rangle$ in (174A), – according to general conditions (257) from Ch. 5:

$$\left. \begin{aligned} & rot\ \Phi \cdot I^{\pm} \cdot rot\ \Phi = I^{\pm}, \\ & rot'\ \Theta \cdot I^{\pm} \cdot rot\ \Theta = I^{\pm} = rot\ \Theta \cdot I^{\pm} \cdot rot'\ \Theta. \end{aligned} \right\} \text{(with reflector tensor } I^{\pm}) \tag{175A}$$

That is why, for analysis of homogeneous composite rotation (motion) T, we shall use the polar decomposition (the right-oriented universal base should be chosen as original one):

$$\tilde{E} = T \cdot \tilde{E}_1 = rot\ \Phi \cdot rot\ \Theta \cdot \tilde{E}_1 = rot\ \Theta \cdot rot\ \overset{\angle}{\Phi} \cdot \tilde{E}_1. \tag{176A}$$

$$T = rot\ \Phi \cdot rot\ \Theta = rot\ \Theta \cdot rot\ \overset{\angle}{\Phi}, \qquad det\ T = +1. \tag{177A}$$

The hyperspheroid of *radius* R embedded into $\langle \mathcal{Q}^{n+1}\rangle$, *as a perfect hypersurface*, is an object, where its *internal* spherical geometry is in one-to-one correspondence with the quasi-Euclidean tensor trigonometry of $\langle \mathcal{Q}^{n+1}\rangle$ up to the coefficint of similarity R, both having the same orientation. Abstract spherical-hyperbolic analogy (322) in $\tilde{E}_{(01)}$, see (443), and (323) in $\tilde{E}_{(02)}$, see (444), takes place. Specific analogy, for example, as sine-tangent (331), can be used locally in any universal bases, see in sect. 6.1, 6.2. Thus, the principal spherical rotations are expressed in $\tilde{E}_1$, according to abstract analogy (323) as follows:

$$\Gamma \leftrightarrow i\Gamma \leftrightarrow \Phi, \quad roth\ \Gamma \leftrightarrow rot\ i\Gamma \leftrightarrow rot\ \Phi, \quad (\tilde{E}_{(1h)} \leftrightarrow \tilde{E}_{(02)} \leftrightarrow \tilde{E}_{(1s)}), \tag{178A}$$

On the base, we expose the materials of this Chapter in parallel with ones of Ch. 7A !
The *spherical tensor of motion rot* Φ with the frame axis $\overrightarrow{y}$ in $\langle \mathcal{Q}^{2+1}\rangle$ has, due to (313), (314), the following canonical structure in $\tilde{E}_1$ corresponding to the reflector tensor $I^{\pm}$:

$$\{rot\ \Phi\}_{3\times 3} = \cos\Phi + i\cdot\sin\Phi \qquad\qquad \{rot\ \Theta\}_{3\times 3} \qquad \text{reflector tensor } I^{\pm}$$

$$\begin{array}{|c|c|}\hline \cos\varphi_i\cdot\overleftarrow{\mathbf{e}_\alpha\cdot\mathbf{e}_\alpha{}'}+\overrightarrow{\mathbf{e}_\alpha\cdot\mathbf{e}_\alpha{}'} & \mp\sin\varphi_i\cdot\mathbf{e}_\alpha \\ \hline \pm\sin\varphi_i\cdot\mathbf{e}'_\alpha & \cos\varphi_i \\ \hline \end{array} \cdots\cdots \begin{array}{|c|c|}\hline \{rot\ \Theta\}_{2\times 2} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \\ \hline \end{array} \cdots\cdots \begin{array}{|c|c|}\hline I_{2\times 2} & \mathbf{0} \\ \hline \mathbf{0}' & -1 \\ \hline \end{array}\ . \tag{179A}$$

The orthospherical rotation in the angle Θ as a rule is also secondary for principal angle.

According to abstract spherical-hyperbolic analogy (323), all formulae of the hyperbolic geometry from Ch. 7A with relation (119A) are transformed into their analogues in the spherical geometry. With right correspondence between principal motions in both geometries measured either by natural pseudo-Euclidean and Euclidean measures of length or by angular Lambert's hyperbolic and spherical measures of angle, there hold:

$$a_{(H)} = \lambda = \gamma\cdot R, \quad \rightarrow \quad a_{(S)} = l = \varphi\cdot R, \tag{180A}$$

Further, we infer formulae of the spherical tensor trigonometry ($q = 1$) often with the use of this spherical–hyperbolic analogy (with corresponding to it commentaries). For two-step noncollinear motions, by (176A, 177A), we obtain the modal transformations with a new base expressed in $\tilde{E}_1 = \{I\}$, as spherical analogs of (111A):

$$\tilde{E}_3 = rot\ \Phi_{12}\cdot rot\ \Phi_{23}\cdot\tilde{E}_1 = \{rot\ \Phi_{12}\cdot rot\ \Phi_{23}\cdot rot'\ \Phi_{12}\}_{\tilde{E}_2}\cdot rot\ \Phi_{12}\cdot\tilde{E}_1 =$$
$$= rot\ \Phi_{13}\cdot rot\ \Theta_{13}\cdot\tilde{E}_1 = \{rot\ \Phi_{13}\cdot rot\ \Theta_{13}\cdot rot'\ \Phi_{13}\}_{\tilde{E}_{1s}}\cdot rot\ \Phi_{13}\cdot\tilde{E}_1 = \tag{181A}$$

$$= rot\ \Theta_{13}\cdot rot\ \overset{\angle}{\Phi}_{13}\cdot\tilde{E}_1 = \{rot\ \Theta_{13}\cdot rot\ \overset{\angle}{\Phi}_{13}\cdot rot'\ \Theta_{13}\}_{\tilde{E}_{1u}}\cdot rot\ \Theta_{13}\cdot\tilde{E}_1 = T_{13}\cdot\tilde{E}_1 = \{T_{13}\}.$$

These formulae are given for the direct order of the two principal motions

Corollary. *Two-step noncollinear spherical rotations (motions) rot* Φ_{1j} *in* $\langle \mathcal{Q}^{n+1}\rangle$ *or on the hyperspheroid may be represented as sequential spherical and orthospherical ones.*

Some characteristics of such motions in direct and inverse orders are expressed as

$$rot\ \overset{\angle}{\Phi}_{13} = rot'\ \Theta_{13}\cdot rot\ \Phi_{13}\cdot rot\Theta_{13} = rot\ (-\Theta_{13})\cdot rot\ \Phi_{13}\cdot rot\ (+\Theta_{13}), \tag{182A}$$

due to (113A) : $\mathbf{e}_{\overset{\angle}{\sigma}} = \{rot\ (+\Theta_{13})\}_{3\times 3}\cdot\mathbf{e}_\sigma$ (under rule $\varepsilon > 0 \rightarrow \theta_{13} > 0$) $\Rightarrow \cos\theta_{13} = \mathbf{e}'_{\overset{\angle}{\sigma}}\cdot\mathbf{e}_\sigma$.

Rotation $\pm\theta$ is expressed in $\tilde{E}_{1s} = rot\ \overset{\angle}{\Phi}\cdot\tilde{E}_1$. (If $n = 2$, it acts in the plane $\langle \mathcal{E}^2\rangle^{(1s)}$). If $n = 3$, we have $\overrightarrow{\mathbf{r}_N}(\theta) = \mathbf{e}_{\overset{\angle}{\sigma}}\otimes\mathbf{e}_\sigma = \pm\sin\theta\cdot\overrightarrow{\mathbf{e}_N}$, $\quad\overrightarrow{\mathbf{r}_N}(\varepsilon) = \mathbf{e}_\alpha\otimes\mathbf{e}_\beta = \pm\sin\varepsilon\cdot\overrightarrow{\mathbf{e}_N}$.

There is the essential difference between the angles Γ and Φ: in $\tilde{E}_1$, Γ is symmetric, Φ is antisymmetric. In their diagonal forms, Γ is real-valued, Φ is imaginary-valued. As consequence, all these trigonometric formulae are identical, when angles are represented in symmetric forms: Γ in the base $\tilde{E}_1$, $-i\Phi$ in the base $\tilde{E}_{(01)}$ – see (271), (277).

The next formula holds due to this peculiarity in the real-valued original base $\tilde{E}_1$:

$$rot\ \Phi_{13} = \sqrt{rot\ \Phi_{12} \cdot rot\ (2\Phi_{23}) \cdot rot\ \Phi_{12}} = \sqrt{rot\ (2\Phi_{13})} =$$

$$= \sqrt{[rot\ \Phi_{12} \cdot rot\ \Phi_{23}] \cdot [rot\ \Phi_{23} \cdot rot\ \Phi_{12}]} = \sqrt{T\ T^\star}, \qquad (183A)$$

The formula is analogous to (114A), but square roots are *trigonometric* (see in sect. 5.6). We have a peculiarity, which relates to spherical case for permutation of motions with change of order into contrary. From the original $\tilde{E}_1 = \{I\}$, as in (181A), this leads to the base $\tilde{E}_3^\star = rot\ \Phi_{23} \cdot rot\ \Phi_{12} = T_{13}^\star \cdot \tilde{E}_1$, where $T^\star$ is quasi-analog of $T\ '$ in (116A), but $T_{13}^\star \neq T_{13}'$!

From the direct formulae (181A), we obtain the orthospherical analog of (115A):

$$rot\ (+\Theta_{13}) = rot\ \Phi_{12} \cdot rot\ \Phi_{23} \cdot rot\ \overset{\angle}{\Phi}_{31} = rot\ \Phi_{31} \cdot rot\ \Phi_{12} \cdot rot\ \Phi_{23} = rot\ \Phi_{31} \cdot T_{13}. \quad (184A)$$

It represents this orthospherical rotation as result of the closed cycle of rotations (motions) $rot\ \Phi_{ij}$ in the spherical triangle 123 and adds (183A). It is executed as in (115A) from points 1 and 3 in bases of particular rotations (motions) actions along of the triangle sides!

In order that a result of (183A) was $rot\ \Phi_{13}$, we adopted for two-step rotations (motions) inverse to (181A) the expression analogous to (116A) (without transition in $\tilde{E}_{(01)}$!):

$$\tilde{E}_3^\star = rot\ \Phi_{23} \cdot rot\ \Phi_{12} \cdot \tilde{E}_1 = T_{13}^\star \cdot \tilde{E}_1 = rot\ (-\Theta_{13}) \cdot rot\ \Phi_{13} \cdot \tilde{E}_1 =$$

$$= rot\ \overset{\angle}{\Phi}_{13} \cdot rot\ (-\Theta_{13}) \cdot \tilde{E}_1 = \{rot\ \overset{\angle}{\Phi}_{13} \cdot rot\ (-\Theta_{13}) \cdot rot'\ \overset{\angle}{\Phi}_{13}\}_{\tilde{E}_1} \cdot rot\ \overset{\angle}{\Phi}_{13} \cdot \tilde{E}_1. \qquad (185A)$$

This expression is completely compatible with (182A), gotten from (181A)! For inverse order of rotations (motions), we obtain the analogs of (117A), (118A) with inverse cycle (184A):

$$rot\ \overset{\angle}{\Phi}_{13} = \sqrt{rot\ \Phi_{23} \cdot rot\ (2\Phi_{12}) \cdot rot\ \Phi_{23}} = \sqrt{rot\ (2\ \overset{\angle}{\Phi}_{13})} = \sqrt{T^\star\ T}, \qquad (186A)$$

$$rot\ (-\Theta_{13}) = rot\ \overset{\angle}{\Phi}_{13} \cdot rot\ \Phi_{23} \cdot rot\ \Phi_{12} = rot\ (-\Phi_{32}) \cdot rot\ (-\Phi_{21}) \cdot rot\ (-\Phi_{13}) = T_{13}^\star \cdot rot\ \Phi_{31}. \quad (187A)$$

The angles Φ_{13} and $\overset{\angle}{\Phi}_{13}$ differ by vectors of directional cosines. Due to (182A), its scalar summarized angle φ_{13} (including for polysteps motions) does not depend on ordering of summands (direct or inverse). The case when the directional cosines of motions are either equal or additively opposite each other corresponds to collinear motions. Choice of direct or inverse order of summands in two-steps spherical rotations (motions) T or $T^\star$ is redused to these partial angles substitution analogous to (121A):

$$\varphi_{12} \leftrightarrow \varphi_{23}, \qquad \alpha_k \leftrightarrow \beta_k, \ k = 1, 2. \qquad (188A)$$

Formulae of two-steps motions summation in $\langle Q^{n+1}\rangle$ in their direct and inverse follow are obtained either with multiplying two modal matrices in (183A) and (186A), or using (as in the end of Ch. 10A) immediate summation of these two motions, or alternatively applying *abstract spherical-hyperbolic analogy* (178A). The scalar cosine of summarized angle φ_{13} is expressed as abstract analog of hyperbolic (122A), and *with the external* angle $\varepsilon = \pi - A_{123}$:

$$\left.\begin{aligned} \cos\varphi_{13} &= \cos\varphi_{12} \cdot \cos\varphi_{23} - \cos\varepsilon \cdot \sin\varphi_{12} \cdot \sin\varphi_{23} = \\ &= \cos\varphi_{13} = \cos\varphi_{12} \cdot \cos\varphi_{23} + \cos A_{123} \cdot \sin\varphi_{12} \cdot \sin\varphi_{23}. \end{aligned}\right\} \qquad (189A)$$

It is similar to the cosine formula with $+\cos A_{123}$ in the spherical geometry for solution of a triangle **123** on a sphere, what no has a relation to our first formula of tensor trigonometric two-steps cosine summation of principal segment-arcs (of big circles) on the hyperspheroid. This formula shows that cosine scalar summation of motions on the hyperspheroid does not depend on ordering φ_{12}, φ_{23} (similar to hyperbolic analog in Ch. 7A).

Motion on the surface of the hyperspheroid with increasing y-coordinate preserves the angles φ_{ij} positivity. That is why, for positive angles of motions and distances in the *spherical Lambert measure*, the "parallelogram rule" takes place (as in Euclidean geometry and non-Euclidean hyperbolic geometry):

$$|\varphi_{12} - \varphi_{23}| \leq \varphi_{13} \leq \varphi_{12} + \varphi_{23}.$$

It is analogous to (123A) and follows from (189A). Due to the inequalities and $\varphi_{ij} > 0$, distance in spherical geometry is a norm. The whole quasi-Euclidean space has Euclidean metric, that is why the length of a geodesic spherical arc $d\varphi$ and an orthospherical arc $d\theta$ are Euclidean. The nD hyperspheroid in $\langle Q^{n+1}\rangle$, in its *sine model*, is mapped entirely into the two-side closed projective n-dimensional hypersurface $[\langle\langle \mathcal{E}^n\rangle\rangle]$, also with topology of n-sphere (see in Ch. 12 and Figure 4), but only inside the Cayley oval of radius R (trigonometric circle at $R = 1$) with its whole internal border. In internal geometry of the hyperspheroid, the *scalar and vector* formulae for the sine and tangent of the arcs sum hold in direct and contrary orders of motions. Thus, the scalar sine formula is evaluated from (189A), including two commutative variants as the *mirror Pythagorean sums*, provided that $\varphi_{12} \leftrightarrow \varphi_{23}$; and, of course, it is a spherical abstract analog of (124A):

$$\sin^2\varphi_{13} = 1 - \cos^2\varphi_{13} =$$
$$= \sin^2\varphi_{12} + \sin^2\varphi_{23} - (1 + \cos^2\varepsilon)\cdot\sin^2\varphi_{12}\cdot\sin^2\varphi_{23} + 2\cos\varepsilon\cdot\cos\varphi_{12}\cdot\cos\varphi_{23}\cdot\sin\varphi_{12}\cdot\sin\varphi_{23} =$$
$$= (\sin\varphi_{12}\cdot\cos\varphi_{23} + \cos\varepsilon\cdot\sin\varphi_{23}\cdot\cos\varphi_{12})^2 + (\sin\varepsilon\cdot\sin\varphi_{23})^2 =$$
$$= (\sin\varphi_{23}\cdot\cos\varphi_{12} + \cos\varepsilon\cdot\sin\varphi_{12}\cdot\cos\varphi_{23})^2 + (\sin\varepsilon\cdot\sin\varphi_{12})^2. \qquad (190A)$$

Tangent direct formula follows from (189A), (190A) as spherical abstract analog of (125A):

$$\tan^2\varphi_{23} = \left[\frac{\tan\varphi_{12} + \cos\varepsilon\cdot\tan\varphi_{23}}{1 - \cos\varepsilon\cdot\tan\varphi_{23}\cdot\tan\varphi_{12}}\right]^2 + \left[\frac{\sin\varepsilon\cdot\tan\varphi_{23}\cdot\sec\ \varphi_{12}}{1 - \cos\varepsilon\cdot\tan\varphi_{23}\cdot\tan\varphi_{12}}\right]^2. \qquad (191A)$$

They express the spherical *Big and Small Pythagorean Theorems* in $\langle Q^{n+1}\rangle$, which act in quasi-Euclidean and spherical geometries also for sine and tangent segments as projections into $[\langle\langle \mathcal{E}^n\rangle\rangle]$. They act in two variants: for direct and inverse orders of these segments.

Further, with Tensor Trigonometry as before in Ch. 7A, we infer all *vector* trigonometric formulae for summation of two-steps motions on the hyperspheroid and in the spherical type of the non-Euclidean geometry, or identically of two-steps principal spherical rotations with the frame axis (from sect. 5.12) in $\langle Q^{n+1}\rangle$. These spherical vector formulae with directional cosines have also the same abstract analogy with summing hyperbolic motions on the hyperboloid II and rotations in $\langle \mathcal{P}^{n+1}\rangle$. And the metric form on the hyperspheroid, given in the end of Ch. 6A, has abstract analogy with one on the hyperboloid II in (132A), Ch. 7A, etc.. The result of such vector summation depends on ordering of summands φ_{12} and φ_{23}. So, the summary *vector sines* in two contrary variants of ordering two motions, expressed in the initial Cartesian sub-base, are the following:

$$\left.\begin{array}{l}
(1)\ \ \mathbf{sin}\ \varphi_{13} = \sin\varphi_{13}\cdot\mathbf{e}_\sigma = \\
= (\cos\varphi_{23}\cdot\sin\varphi_{12} + \cos\varepsilon\cdot\cos\varphi_{12}\cdot\sin\varphi_{23})\cdot\mathbf{e}_\alpha + \sin\varepsilon\cdot\sin\varphi_{23}\cdot\mathbf{e}_\nu = \\
= [\cos\varphi_{23}\cdot\sin\varphi_{12} - \cos\varepsilon\cdot(1 - \cos\varphi_{12})\cdot\sin\varphi_{23}]\cdot\mathbf{e}_\alpha + \sin\varphi_{23}\cdot\mathbf{e}_\beta; \\
(2)\ \ \mathbf{sin}\ \overset{\angle}{\varphi}_{13} = \sin\varphi_{13}\cdot\mathbf{e}_{\overset{\angle}{\sigma}} = \\
= (\cos\varphi_{12}\cdot\sin\varphi_{23} + \cos\varepsilon\cdot\cos\varphi_{23}\cdot\sin\varphi_{12})\cdot\mathbf{e}_\beta + \sin\varepsilon\cdot\sin\varphi_{12}\cdot\mathbf{e}_{\overset{\angle}{\nu}} = \\
= [\cos\varphi_{12}\cdot\sin\varphi_{23} - \cos\varepsilon\cdot(1 - \cos\varphi_{23})\cdot\sin\varphi_{12}]\cdot\mathbf{e}_\beta + \sin\varphi_{12}\cdot\mathbf{e}_\alpha.
\end{array}\right\} \qquad (192A)$$

From here, under conditions $\varphi_{12} = \varphi$ and $\varphi_{23} = d\varphi$, we obtain the same metric form (109A) of the hyperspheroid from its Pole, but in the vector form – see more in Ch. 10A.

$$\mathbf{e}_\nu = \left\{\frac{\cos\beta_k - \cos\varepsilon\cdot\cos\alpha_k}{\sin\varepsilon}\right\}_{k=1,2,3} = \frac{\mathbf{e}_\beta - \cos\varepsilon\cdot\mathbf{e}_\alpha}{\sin\varepsilon} = \frac{\overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}\cdot\mathbf{e}_\beta}{||\overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}\cdot\mathbf{e}_\beta||} \ \text{– see before in (136}A\text{)}.$$

The vector $\mathbf{e}_\nu$ (and $\mathbf{e}_{\overleftarrow{\nu}}$ for inversely ordered summary motions at $\mathbf{e}_\alpha \leftrightarrow \mathbf{e}_\beta$ is used in biorthogonal decompositions of principal motion increment into tangential and normal parts. They are executed through biorthogonal representation of the 2-nd vector in the sum: $\mathbf{e}_\beta = \cos\varepsilon \cdot \mathbf{e}_\alpha + \sin\varepsilon \cdot \mathbf{e}_\nu, \quad \mathbf{e}'_\nu \cdot \mathbf{e}_\alpha = 0, \quad \mathbf{e}'_\nu \cdot \mathbf{e}_\beta = \sin\varepsilon \quad (\varepsilon \in [0;\pi])$.

In the spherical geometry, this finite vector sine summation is seen descriptively on the projective hyperplane at Figure 4, Ch. 12, similar to also finite tangent summation in the hyperbolic geometry, for example, as at Figure 4A, Ch. 7A. Sine formulae, in squared and vector variants as (124A), (135A) and as (125A), (138A), have in $\tilde{E}_1^{(3)}$ similar interpretations in $\langle \mathcal{E}^3 \rangle^{(1)}$:

$$\mathbf{sin}\ \varphi_{23} = \overset{=}{\mathbf{sin}}\ \varphi_{23} + \overset{\perp}{\mathbf{sin}}\ \varphi_{23} \ \rightarrow\ \mathbf{sin}\varphi_{13} = (\mathbf{sin}\ \varphi_{12} + \overset{=}{\mathbf{sin}}\ \varphi_{23}) + \overset{\perp}{\mathbf{sin}}\ \varphi_{23}.$$

Both these relations are compatible. So, as results, we obtain the **Big Pythagorean Theorem** in its squared variant corresponding to (124A), and, as a consequence, the **Small Pythagorean Theorem** for the second segment in $\tilde{E}_1^{(3)}$, with the trivial case corresponding to (129A):

$$\sin^2\varphi_{13} = \sin^2(\varphi_{12} + \overset{=}{\varphi}_{23}) + \sin^2\overset{\perp}{\varphi}_{23}\ , \quad \sin^2\varphi_{23} = \sin^2\overset{=}{\varphi}_{23} + \sin^2\overset{\perp}{\varphi}_{23}\ .$$

In these formulae, $\sin\overset{=}{\varphi}_{13} = \cos\varepsilon \cdot \sin\varphi_{13}\ ,\ \ \sin\overset{\perp}{\varphi}_{23} = \sin\overset{\perp}{\varphi}_{13} = \sin\varepsilon \cdot \sin\varphi_{13}$. Their *cosines*, are, as due to (122A), the scalar projections into $\overrightarrow{y}$ parallel to $\langle \mathcal{E}^3 \rangle$.

Formula for the vector tangent is analogous to (138A), and given only for completeness:

$$\mathbf{tan}\ \varphi_{13} = \tan\varphi_{13} \cdot \mathbf{e}_\sigma = \left(\frac{\tan\varphi_{12} + \cos\varepsilon \cdot \tan\varphi_{23}}{1 - \cos\varepsilon \cdot \tan\varphi_{23} \cdot \tan\varphi_{12}}\right) \cdot \mathbf{e}_\alpha + \left(\frac{\sin\varepsilon \cdot \tan\varphi_{23} \cdot sec\ \varphi_{12}}{1 - \cos\varepsilon \cdot \tan\varphi_{23} \cdot \tan\varphi_{12}}\right) \cdot \mathbf{e}_\nu. \quad (193A)$$

* * *

As the abstract spherical analogs on the 2D hyperspheroid of the cosine-sine differentials (160A, 161A) on the hyperboloid II, we obtain:

$$\left.\begin{aligned} &d\cos\varphi_p = \sin\gamma_p\ d\gamma_p = \sin\gamma_i\ d\gamma_i;\\ &|d\mathbf{sin}\ \varphi(\gamma)|^2 = \cos^2\varphi\ d\varphi^2 + \sin^2\varphi\ d\alpha^2 = \cos^2\varphi_p\ (d\varphi_p)^2 =\\ &= \cos^2\varphi_p \cdot [(\cos\varepsilon\ d\varphi_p)^2 + (\sin\varepsilon\ d\varphi_p)^2] = \cos^2\varphi_p \cdot [\overline{\overline{d\varphi_p}}^2 + \overset{\perp}{d\varphi_p}^2] < 1;\\ &\sin\varphi \cdot \mathbf{e}_\alpha(\gamma) = \sin\varphi_0 \cdot \mathbf{e}_{\alpha(0)} + \textstyle\int_{\varphi_0}^{\varphi} [\cos\varphi\ d\varphi \cdot \mathbf{e}_\alpha + \sin\varphi\ d\alpha \cdot \mathbf{e}_\nu]. \end{aligned}\right\} \quad (194A)$$

Here $d\alpha$ is the angle of the secondary orthospherical rotation of Euclidean basis vector.

* * *

Besides, principal angles φ and γ are the *covariant parallel angles* in the spherical and hyperbolic geometries – see in Ch. 6 and Ch. 1A. They are accompanied by the complementary *countervariant parallel angles* υ (by Lobachevsky) and ξ. All relations between them were inferred entirely in the end of Ch. 6. Simplest additive bond of spherical scalar and tensor angles $\varphi \leftrightarrow \xi$ is a peculiarity of the spherical geometry. With (317) or by analogy with (496), we give the rotation at complementary tensor spherical angle as follows:

$$\overline{rot}\ \Phi \qquad = \qquad rot\ \Xi$$

$$\left\| \begin{array}{c|c|c|c|c} \dfrac{\sin\varphi \cdot \overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}'}{\pm\cos\varphi \cdot \mathbf{e}'_\alpha} & \dfrac{\mp\cos\varphi \cdot \mathbf{e}_\alpha}{\sin\varphi} & \begin{matrix}\cdots\\ \cdots\end{matrix} & \dfrac{\cos\xi \cdot \overleftarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha\mathbf{e}_\alpha}'}{\pm\sin\ \xi \cdot \mathbf{e}'_\alpha} & \dfrac{\mp\sin\ \xi \cdot \mathbf{e}_\alpha}{\cos\xi} \end{array} \right\|.$$

* * *

From (189A), for summing conventionally orthogonal particular spherical segments or motions, the scalar cosine multiplicative formula hold, with its generalization:

$$\cos\varphi_{13} = \cos\varphi_{12} \cdot \cos\varphi_{23}, \qquad (\varepsilon = \pm\pi/2),$$

$$\cos\varphi = \prod_{k=1}^{t} \cos\varphi_{(k)}, \quad \varepsilon_{ij} = \pm\pi/2, \quad 1 \le i, j \le t \le n, \quad i \ne j, \ (\text{on axes } \overrightarrow{y} \text{ and } \overrightarrow{y}^{(k)}). \qquad (195A)$$

It is the spherical abstract analog of hyperbolic formula (131A) in Ch. 7A. The final *scalar* angle φ and the distance $a = R \cdot \varphi$ do not depend on ordering of conventionally orthogonal particular angles.

If all t orthogonal segments are infinitesimal, then the *Infinitesimal Pythagorean Theorem* holds for now non-conventionally orthogonal infinitesimal spherical segments with the angular measure of Lambert φ.

For the sine of conventionally orthogonal motions sum, we obtain:
$\sin^2\varphi_{13} = \sin^2\varphi_{12} + (\sin\varphi_{23}\cdot\cos\varphi_{12})^2 = \sin^2\varphi_{23} + (\sin\varphi_{12}\cdot\cos\varphi_{23})^2$, $(\varphi = l/R)$.

Suppose that, instead of the possible k orthogonal spherical motion' angles, we deal with only their orthogonal differentials at zero values of these angles at the point M. Then we have the Rule of their squared Pythagorean summation on the hyperspheroid (till $k = n$):
$(d\varphi_{13})^2 = (d\varphi_{12})^2 + (d\varphi_{23})^2$.

The Rule is analogous to the squared Pythagorean summation of the inner hyperbolic differentials and inner accelerations in the instantaneous local base of STR – see in Ch. 9A. See analogous quadrics, as decomposition of the inner differential, below in (197A).

The projective sine measure $R\sinh\,(l/R)$ may be used also in the flat sine model of the hyperspheroid of radius R, which follows the Big and Small Pythagorean Theorems. Decomposition of $d\mathbf{sin}\,\varphi$ in (192A) and summation in (193A) are executed in this model in the trigonometric ring (ball), limited in the projective hyperplane $[\langle\langle\mathcal{E}^n\rangle\rangle]$ by the radius R. (At $R\to\infty$, it is Euclidean as for the hyperboloid II in Ch. 12.)

We can use the same formulae (136A) and (139A) for the vectors of directional cosines:

$$\mathbf{e}_\nu = \frac{\mathbf{e}_\beta - \cos\varepsilon\cdot\mathbf{e}_\alpha}{\sin\varepsilon} = \frac{\overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}\cdot\mathbf{e}_\beta}{||\overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha}\cdot\mathbf{e}_\beta||},\quad \mathbf{e}_{\measuredangle\nu} = \frac{\mathbf{e}_\alpha - \cos\varepsilon\cdot\mathbf{e}_\beta}{\sin\varepsilon} = \frac{\overrightarrow{\mathbf{e}_\beta\mathbf{e}'_\beta}\cdot\mathbf{e}_\alpha}{||\overrightarrow{\mathbf{e}_\beta\mathbf{e}'_\beta}\cdot\mathbf{e}_\alpha||}.$$

And we obtain: $\cos\theta_{13} = \mathbf{e}'_{\measuredangle\sigma}\cdot\mathbf{e}_\sigma$; $\mathbf{e}_\beta = \cos\varepsilon\cdot\mathbf{e}_\alpha + \sin\varepsilon\cdot\mathbf{e}_\nu \leftrightarrow \mathbf{e}_\alpha = \cos\varepsilon\cdot\mathbf{e}_\beta + \sin\varepsilon\cdot\mathbf{e}_{\measuredangle\nu}$,

$$\mathbf{e}'_\nu\cdot\mathbf{e}_{\measuredangle\nu} = -\cos\varepsilon = +\cos A_{123},\quad \mathbf{e}_\alpha\cdot\mathbf{e}_{\measuredangle\nu} = \mathbf{e}_\beta\cdot\mathbf{e}_\nu = +\sin\varepsilon = +\sin A_{123}.$$

Vectors $\mathbf{e}^{(1)}_\alpha$, $\mathbf{e}^{(2)}_\beta$, $\mathbf{e}_\nu$, $\mathbf{e}_\sigma$, $\mathbf{e}_{\measuredangle\nu}$ x $\mathbf{e}_{\measuredangle\sigma}$ are *formally* inside an angle π in the plane $\langle\mathcal{E}^2\rangle\equiv\langle\mathbf{e}_\alpha,\mathbf{e}_\beta\rangle$.

Due to **General Signs Rule**, see in (182A) and in sect. 12.2, for spherical geometry we have: $\boxed{sgn\ \theta_{13} = +sgn\ \varepsilon\,!}$. If $\varepsilon > 0$, then $\theta_{13} > 0$, and if $\varepsilon < 0$, then $\theta_{13} < 0$, i. e., the leg 13 is shifted orthospherically in direction always with increasing the sum of angles in the spherical triangle 123. Plane of this orthospherical rotation is $\langle\mathcal{E}^2\rangle\equiv\langle\mathbf{e}_\alpha,\mathbf{e}_\beta\rangle$. If $n = 3$, then vectors $\mathbf{e}_\alpha,\mathbf{e}_\beta,\overrightarrow{\mathbf{e}_N}$ and $\mathbf{e}_{\measuredangle\sigma},\mathbf{e}_\sigma,\overrightarrow{\mathbf{e}_N}$ form the right ($\varepsilon > 0$) or left $\varepsilon < 0$ triples. They correspond to counter-clockwise scalar angles in *right-handed* bases. (Oriented vector $\overrightarrow{\mathbf{r}_N}(\theta) = \mathbf{e}_{\measuredangle\sigma}\otimes\mathbf{e}_\sigma = \pm\sin\theta\cdot\overrightarrow{\mathbf{e}_N}$ determines right screw of rotations if $n = 3$.)

Formula (143A) from Ch. 7A for $\cos\theta_{13}$ is transformed by similar way as it was on the hyperboloid II. For two-steps principal spherical motions, formula gives the angular *excess* of geodesic spherical triangle 123 on the hyperspheroid. For two conventionally orthogonal (at maximum $|\theta|$) and general motions, we obtain these expression for orthospherical shifting θ_{13}:

$$\cos\theta_{13} = \frac{\cos\varphi_{12} + \cos\varphi_{23}}{\cos\varphi_{12}\cdot\cos\varphi_{23} + 1} > 0,$$

$$\sin\theta_{13} = \frac{+\sin\varphi_{12}\cdot\sin\varphi_{23}}{\cos\varphi_{12}\cdot\cos\varphi_{23} + 1};\quad \sin d\theta = d\theta = +\sin\varepsilon\cdot\frac{\sin\varphi\ d\varphi}{1+\cos\varphi} = +\sin\varepsilon\cdot\tan\,(\varphi/2)\ d\varphi.$$

As before, in infinitesimal considerations we shall apply the useful formulae for the cosine of the first angular differential *(with exactness up to 2-nd power of differentials)*:
$\cos d\varphi = 1 - (d\varphi)^2/2$ and $\cos d\theta = 1 - (d\theta)^2/2$.

In both sine formulae (194A), put these values of angles: $\varphi_{12} = \varphi$, $\varphi_{23} = d\varphi$. The latter is the differential of an arc φ under angle ε to the segment φ. Further, by abstract spherical-hyperbolic analogy (323) to (172A) at $n\geq 2$, and similar to inferring hyperbolic formulae (144A) in Ch. 7A (using direct and inverse ordering variants of (194A) with the angles φ and $d\varphi$ and relations (141A)), we'll obtain the differential of the unduced orthospherical shift in $\langle\mathcal{E}^n\rangle^{(1)}$ of $\langle\mathcal{Q}^{n+1}\rangle$ and, in particular, in the plane $\langle\mathcal{E}^2\rangle^{(1)}\equiv\langle\mathbf{e}_\alpha,\mathbf{e}_\beta\rangle$. In addition, we use the angle $d\alpha$ of the current rotational shift of the unity directive angle $\mathbf{e}_\alpha$ in the plane $\langle\mathcal{E}^2\rangle^{(m)}\equiv\langle\mathbf{e}_\alpha,\mathbf{e}^{(m)}_\beta\rangle$.

With the spherical sign Rule and *normal relation* $\sin\varepsilon \, d\varphi = \overset{\perp}{d\varphi} = \sin\varphi \, d\alpha$, in result we get:

$$\mathbf{e}_{\underset{\sigma}{\angle}} \times \mathbf{e}_\sigma = d\theta \cdot \overrightarrow{\mathbf{e}_N}^{(m)} = \frac{\sin\varphi \cdot \mathbf{e}_\alpha}{1+\cos\varphi} \otimes d\varphi \cdot \mathbf{e}_\beta^{(m)} = \tan\frac{\varphi}{2} \, d\varphi \cdot \overrightarrow{\mathbf{r}_N} = +\tan\frac{\varphi}{2} \cdot \sin\varepsilon \, d\varphi \cdot \overrightarrow{\mathbf{e}_N} = \quad (196A)$$

$$= \tan\frac{\varphi}{2} \, \overset{\perp}{d\varphi} \cdot \overrightarrow{\mathbf{e}_N} = \frac{1-\cos\varphi}{\sin\varphi} \cdot \sin\varphi \, d\alpha \cdot \overrightarrow{\mathbf{e}_N} = (1-\cos\varphi) \, d\alpha \cdot \overrightarrow{\mathbf{e}_N} \to \boxed{d\theta = (1-\cos\varphi) \, d\alpha = 2\sin^2(\varphi/2) \, d\alpha}.$$

Note that the *normal relation* (of type above) will obtain with a rigorous justification in the differential 3D Relative Pythagorean theorems in the last Ch. 10A of the Appendix. *It is abstract analog of* (171A). *In the Euclidean sub-space* $\langle\mathcal{E}^n\rangle^{(1)}$, *this shift is caused by difference between the real orthospherical rotation differential* $\mathbf{de}_\alpha = \mathbf{d}\alpha$ *in* $\langle\mathcal{E}^n\rangle^{(m)}$ *on a curved trajectory (maybe closed) and its spherical (here) cosine projection onto the projective hyperplane* $\langle\langle\mathcal{E}^n\rangle\rangle^{(1)}$!

It has positive values due to same directions of θ and ε. The angles φ and $d\varphi$ are expressed in the bases $\tilde{E}_1$ and $\tilde{E}_m$ of $\langle\mathcal{Q}^{n+1}\rangle$. This *differential variant* of the induced orthospherical shift and rotation θ is useful in spherical geometry. For example, on the Globe, it gives the change of latitude – see further. (But recall that for two arcs, the *single* normal $\overrightarrow{\mathbf{e}_N}$ exists only in $\langle\mathcal{Q}^{3+1}\rangle$!) Thus, for a triangle 123 in $\langle\mathcal{Q}^{3+1}\rangle$, formed by $d\varphi_{12}$ and $d\varphi_{23}$, with their also orthospherical external angle ε (using the expression for vector element of the area [21, p. 526])), we infer bonded formula $dS(d\theta)$:

$$d\theta_{13} \cdot \overrightarrow{\mathbf{e}_N} = \sin\varepsilon \cdot \frac{(d\varphi_{12}) \cdot (d\varphi_{23})}{2} \cdot \overrightarrow{\mathbf{e}_N} = \sin\varepsilon \cdot \frac{(dl_{12}) \cdot (dl_{23})}{2R^2} \cdot \overrightarrow{\mathbf{e}_N} = \frac{dS_{123}}{R^2} \cdot \overrightarrow{\mathbf{e}_N}.$$

Due to the Harriot's result in spherical geometry or generally to the Gauss–Bonnet Theorem [21, p. 533], the area of the geodesic triangle 123 (on a surface of positive constant Gaussian curvature and the angular excess of this spherical triangle (here with external ε) $d\delta_{123} = 2\pi - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ are connected as $d\delta_{123} = dS_{123}/R^2 = K_G \, dS_{123} > 0$. As results, we obtain the differential and integral formulae for connection of these two specific angles

$$d\theta_{13} = d\delta_{123} = \frac{dS_{123}}{R^2} = K_G \, dS_{123} \;\Rightarrow\; \theta_{13} = \delta_{123} = \frac{S_{123}}{R^2} = K_G \cdot S_{123}$$

in the geodesic triangles on the hyperspheroid and, hence, in the other curvilinear spherical non-Euclidean spaces too. These formulae mean: the angle θ_{13} of orthospherical shifting and Harriot's angular excess $+\delta_{123}$ in a spherical triangle 123 are equal, as well as it was for Lambert's angular defect $-\delta_{123}$ in a hyperbolic triangle 123 (Ch. 7A)!!!

An inference of both these expressions consists in contour and surface integrating with further applying their infinitesimal identity. This is internal point of view on the hyperspheroid geometry. It (as well as any sphere) cannot be bent without loss of its metrical properties, and, hence, it is a surface of constant positive radius. (The same is valid for the hyperboloid II as a sphere of the imaginary constant radius iR, see in Ch. 12.)

The orthospherical tensor of rotation Θ_{13}, in accordance with tensor formulae (184A), (187A), is identical to *tensor angular excess* of a geodesic triangle on the hyperspheroid. Angular deviations (scalar and tensor) take place due to dependence of parallel displacement on a surface with curvature on its way. The scalar or tensor angular excesses are expressed through the orthospherical shift θ or Θ as the result of a closed cycle of geodesic motions along the triangle sides! Taking into account the analogous results in Ch. 7A, we formulate the following our result for the spherical geometry on the hyperspheroid with frame axis, which adds the previous our results for the hyperboloid II and for the hyperbolic non-Euclidean geometry.

General Corollary (Theorem). *The induced orthospherical rotation* Θ *is a true cause of the Harriot, Lambert and, in general, Gauss–Bonnet angular deviations in convex geodesic figures in non-Euclidean geometries, including their spherical and hyperbolic types!*

The special case is summation of two-steps or polysteps motions when both particular angles are *infinitesimally small.* Suppose that, for example, in formulae (193A), (196A) with $n = 2$ both these principal spherical angles are infinitesimal. So, for right triangle 123 with $\cos\varepsilon = 0$, we obtain as $\varphi_{12} \to 0, \varphi_{23} \to 0$:

$$\varphi_{13} = \sqrt{\varphi_{12}^2 + \varphi_{23}^2}, \quad \theta_{13} = \frac{\varphi_{12} \cdot \varphi_{23}}{2} = \frac{a_{12} \cdot a_{23}}{2R^2} = S_{123} \cdot K_G.$$

For k-steps principal spherical motions on the hypersperoid, according to formula (193), the following generalization holds:

$$\lim_{\varphi_{(\jmath)} \to 0} l = R \cdot \sqrt{\sum_{\jmath=1}^{k} \varphi^2_{(\jmath)}}.$$

$$V = \varphi_{(1)} \cdots \varphi_{(k)} \cdot R^k, \qquad k \le n, \quad \varepsilon = \pm\pi/2.$$

They are the simplest infinitesimal formulae for the geometry on the hyperspheroid as of the Euclidean geometry. This confirms the *infinitesimal character of Euclidean metric* on the hyperspheroid of radius R.

Corollary. *Geometry of the hyperspheroid is infinitesimally Euclidean.*

Conclusion. *Orthospherical induced shifting* Θ *gives the clear mathematical explanation, with the use of Tensor Trigonometry, to the Harriot angular excess in closed figures in the spherical geometry, in that number, on the surface of the hyperspheroid!*

Commutativity of the partial angles of motion (arcs) takes place in the scalar variant of conventionally orthogonal summation formulae. In particular, the first differential of the total angle arc is represented on the tangent n-dimensional Euclidean subspace $\langle \mathcal{E}^n \rangle$ to the n-dimensional hyperspheroid embedded in the quasi-Euclidean space $\langle \mathcal{Q}^{n+1} \rangle \equiv \langle \mathcal{E}^n \boxplus \overrightarrow{y} \rangle$ with reflector tensor $I^{\pm}$ (as on the hyperboloid II in Ch. 7A):

$$(d\varphi)^2 = \sum_{k=1}^{n} [d\varphi_{(k)}]^2, \qquad (dl)^2 = (Rd\varphi)^2 = \sum_{k=1}^{n} [dl_{(k)}]^2, \qquad \varepsilon_{(1\jmath)} = \pm\pi/2, \tag{197A}$$

According to the Big Pythagorean theorem (see it in sine versions: scalar (190A) and vectorial (192A)), in spherical geometry of the hyperspheroid, it is possible to use Cartesian sub-base $\tilde{E}_1^{(n)}$ of the original base $\tilde{E}_1 = \{I\}$, as sine projective *homogeneous coordinates* into the Euclidean projective hyperplane $\langle\langle \mathcal{E}^n \rangle\rangle$, but only inside the ball with radius R or for the quasi-Euclidean tensor trigonometry at $R = 1$ (similar to tangent model of the hyperbolic geometry on the hyperboloid II in Ch. 12). The sine model of principal motions with its Pythagorean theorem are preferred here, because they are bounded by finite parameter either $\mathbf{1}$ as trigonometric one or R as geometric one for considerations of geometric problems.

* * *

In $\langle \mathcal{Q}^{2+1} \rangle$, for analysis and interpretation of two-steps motions on the hyperspheroid by differential method it is useful to apply decomposition of the inner total differential $d\varphi_\beta$ along the instantaneous axis $x^{(m)}$ into its spherical orthoprojections, parallel (along $\mathbf{e}_\alpha$) and orthogonal (along $\mathbf{e}_\nu$) ones with respect to the current vector of principal motions $\mathbf{e}_\alpha$ at the local point M in the current base $\tilde{E}_m$. We decompose this current inner differential of the increment of motion with the spherical differential causing it into the parallel and normal parts by the Pythagorean Theorem in the current Euclidean sub-space $\langle \mathcal{E}^3 \rangle^{(m)}$, with respect to the direction of $\mathbf{e}_\alpha$, as follows:

$$\left.\begin{aligned} d\varphi_\beta \cdot \mathbf{e}_\beta &= \cos\varepsilon \; d\varphi_\beta \cdot \mathbf{e}_\alpha + \sin\varepsilon \; d\varphi_\beta \cdot \mathbf{e}_\nu = \overline{\overline{d\varphi_\beta}} \cdot \mathbf{e}_\alpha + \overset{\perp}{d\varphi_\beta} \cdot \mathbf{e}_\nu \to (d\varphi_\beta)^2 = \left(\overline{\overline{d\varphi_\beta}}\right)^2 + \left(\overset{\perp}{d\varphi_\beta}\right)^2, \\ dl_\beta \cdot \mathbf{e}_\beta &= \cos\varepsilon \cdot dl_\beta \cdot \mathbf{e}_\alpha + \sin\varepsilon \cdot dl_\beta \cdot \mathbf{e}_\nu = \overline{\overline{dl_\beta}} \cdot \mathbf{e}_\alpha + \overset{\perp}{dl_\beta} \cdot \mathbf{e}_\nu \to dl_\beta^2 = \left(\overline{\overline{dl_\beta}}\right)^2 + \left(\overset{\perp}{dl_\beta}\right)^2. \end{aligned}\right\} \tag{198A}$$

It is the spherical Local *Absolute Euclidean Pythagorean theorem* for spherically orthogonal decomposition in the Cartesian sub-base $\tilde{E}_m^{(3)}$ of the brutto differential $d\varphi \cdot \mathbf{e}_\beta$, with respect to the directional vector $\mathbf{e}_\alpha$ of the hyperbolic angle of motion φ. The parallel part accelerates motion along the curve, the normal part rotates the direction of motion with its curve.

* * *

Consider the hyperspheroid of radius R including trigonometric one if $R = 1$.

Hyperspheroid (see at Figure 4) has $R = +1$. (Radius may be R), $\varphi > 0$ if $\Delta y > 0$. Represent it by $\mathbf{t}(\varphi) = \mathbf{r}(\varphi)$ as its radius-vector and the principal tangent to a regular curve and by $\mathbf{n}(\varphi)$ as the principal quasinormal to the same regular curve in $\langle \mathcal{Q}^{2+1} \rangle$ under absract analogy with hyperboloid II an I in $\langle \mathcal{P}^{2+1} \rangle$ (see in the end of Ch. 6A). They are expressed in $\tilde{E}_1$ with the clockwise φ counted off $\overrightarrow{y}^{(1)}$ and with counterclockwise φ counted off $\langle \mathcal{E}^n \rangle^{(1)}$. With presentations from its North Pole and Equator, we have the following two variants:

$$\mathbf{t}(\varphi) = \begin{bmatrix} \mathbf{sin}\ \varphi \\ \cos\varphi \end{bmatrix} = \begin{bmatrix} \sin\varphi \cdot \mathbf{e}_\alpha \\ \cos\varphi \end{bmatrix}, \quad \mathbf{n}(\varphi) = \begin{bmatrix} \mathbf{cos}\ \varphi \\ -\sin\varphi \end{bmatrix} = \begin{bmatrix} \cos\varphi \cdot \mathbf{e}_\alpha \\ -\sin\varphi \end{bmatrix}. \quad (199A - II, I)$$

$$\mathbf{t}(\varphi)'_{1k} \cdot \mathbf{t}(\varphi)_{1k} = \mathbf{sin}'\ \varphi_{1k} \cdot \mathbf{sin}\ \varphi_{1k} + \cos^2\varphi_{1k} = \sin^2\varphi_{1k} \cdot \mathbf{e}'_\alpha \mathbf{e}_\alpha + \cos^2\varphi_{1k} = 1. \quad (200A - II)$$

$$\mathbf{n}(\xi)'_{1k} \cdot \mathbf{n}(\xi)_{1k} = \mathbf{cos}'\ \varphi_{1k} \cdot \mathbf{cos}\ \varphi_{1k} + \sin^2\varphi_{1k} = \cos^2\varphi_{1k} \cdot \mathbf{e}'_\alpha \mathbf{e}_\alpha + \sin^2\varphi_{1k} = 1. \quad (200A - I)$$

$\mathbf{sin}\ \varphi_{1k}$ is the $n \times 1$-vector orthoprojection of $\mathbf{t}(\varphi)_{1k}$ into $\langle \mathcal{E}^n \rangle^{(1)}$ parallel to $\overrightarrow{y}^{(1)}$,
$\cos\varphi_{1k}$ is the scalar orthoprojection of $\mathbf{t}(\varphi)_{1k}$ into $\overrightarrow{y}^{(1)}$ parallel to $\langle \mathcal{E}^n \rangle^{(1)}$.
$\mathbf{cos}\ \varphi_{1k}$ is the $n \times 1$-vector orthoprojection of $\mathbf{n}(\varphi)_{1k}$ into $\langle \mathcal{E}^n \rangle^{(1)}$ parallel to $\overrightarrow{y}^{(1)}$,
$\sin\varphi_{1k}$ is the scalar orthoprojection of $\mathbf{n}(\varphi)_{1k}$ into $\overrightarrow{y}^{(1)}$ parallel to $\langle \mathcal{E}^n \rangle^{(1)}$ – (see Figure 4).

Consider for the 1-st case the geodesic motions $\mathbf{t}_{12}, \mathbf{t}_{23} \to \mathbf{t}_{13}$ on the hyperspheroid along large circles in $\tilde{E}_1$ and $\tilde{E}_2$ with tensor of motion (179A), polar decomposition as in (181A) and by analogy with (148A):

$$= \{rot\ \Phi_{23}\}_{\tilde{E}_2} \cdot \overset{\mathbf{t}_{12}}{\begin{bmatrix} \sin\ \varphi_{12} \cdot \mathbf{e}_\alpha \\ \cos\varphi_{12} \end{bmatrix}} = \{rot\ \Phi_{23}\}_{\tilde{E}_2} \cdot \{rot\ \Phi_{12}\}_{\tilde{E}_1} \cdot \overset{\mathbf{t}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = \quad (201A)$$

$$= \{rot\ \Phi_{12} \cdot (rot\ \Phi_{23})_{\tilde{E}_1} \cdot rot'\ \Phi_{12}\} \cdot rot\ \Phi_{12} \cdot \overset{\mathbf{t}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = \{rot\ \Phi_{12}\}_{\tilde{E}_1} \cdot \{rot\ \Phi_{23}\}_{\tilde{E}_1} \cdot \overset{\mathbf{t}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} =$$

$$= rot\ \Phi_{13} \cdot rot\ \Theta_{13} \cdot \overset{\mathbf{t}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = rot\ \Phi_{13} \cdot \overset{\mathbf{t}_1}{\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}} = \overset{\mathbf{t}_{13}}{\begin{bmatrix} \sin\varphi_{13} \cdot \mathbf{e}_\sigma \\ \cos\varphi_{13} \end{bmatrix}}.$$

We'll continue this in Ch. 10A. A spherical triangle on a hyperspheroid with radius R can be easy implemented as a cycle of 3 geodesics. If the start apex is a central element $\mathbf{u}_1$, then $rot\ \Phi_{12} \cdot \mathbf{u}_1 = \mathbf{u}_{12}$, $\{rot\ \Phi_{12} \cdot rot\ \Phi_{23} \cdot rot'\ \Phi_{12}\}_{\tilde{E}_2} \cdot \mathbf{u}_{12} = \mathbf{u}_{13}$, $\{rot\ \Phi_{31}\}_{\tilde{E}_3} \cdot \mathbf{u}_{13} = \mathbf{u}_1$. The triple can be converted into a non-centered triangle with the admissible transformation. A trajectory of spherical motion $\mathbf{u}_{12} \to \mathbf{u}_{13}$ is in the cut of unity hyperspheroid by the eigen quasiplane of rotation $\{rot\ \Phi_{23}\}_{\tilde{E}_2}$. Intersection of this quasiplane with the projective hyperplane is a straight line segment $\mathbf{u}_{23}$ in $\langle\langle \mathcal{E}^n \rangle\rangle$, it corresponds to this geodesic trajectory.

Thus, *for any two points* $\mathbf{u}_{12}$ *and* $\mathbf{u}_{13}$ *on the hyperspheroid of radius R, there exists a unique geodesic line passing through them.* However, there is *a special case*, when two points of the hyperspheroid are polar (as North Pole C_{II} and South one at Figure 4). Such two points produce only spherical digons. (It is a polygon with two sides and two vertices.) This illustrates the following well-known Theorem of spherical geometry: any two points of a semisphere (beside nonpolar ones) of a sphere can be connected by a unique arc of a large circle (as geodesic line), this arc is shortest in the natural Euclidean and Lambert angular length measures. Therefore, this gives the *matrix tensor trigonometric way* for solving such a problem. In the base $\tilde{E}_2 = rot\ \Phi_{12} \cdot \tilde{E}_1$, the geodesic motion $\mathbf{u}_{12} \to \mathbf{u}_{13}$ is going along the shortest arc with length $a_{23} = R \cdot \varphi_{23}$. By (201A), for only a point element $\mathbf{u}_1$, orthospherical rotation Θ_{13}, in fact, annihilates. A triangle cycle of motions returns a nonpoint object into the start point, but the geometric object in it is turned in the base $\tilde{E}_3$ at induced angle Θ_{13}. Hence, the application point of this non-point object is transformed here as $\mathbf{u}_1 \to \mathbf{u}_{12} \to \mathbf{u}_{13}$ along the spherical geodesic lines $R\varphi_{12}$ and $R\varphi_{23}$ as arcs from the large circles of radius R.

Let us apply the $2D$ hyperspheroid with the frame Earth axis from sect. 5.12., with North and South Poles (at $\varphi_0 = 0, \xi_0 = \pm\pi/2$ and $\theta_0 = 0$) and the greenwich reper meridian **for a tensor trigonometric model of any angular motions on the Earth globe** with two its fixed angular coordinates: $\xi = \pi/2 - \varphi$ (a latitude) and θ (a longitude). For this, we'll use both motion tensors from (179A) and general as $T_{(m)} = rot\ \Phi \cdot rot\ \Theta$ – see it below in (202A). (So, $rot\ \Theta$ may be free and induced orthospherical rotation along the globe parallels.) We can begin the angular motion from some choosing point with its two coordinates: $\xi_0 = \pi/2 - \varphi_0$ (as initial latitude with respect to the Equator) and θ_0 (as initial longitude with respect to the greenwich meridian). Accordingly, in it, we have the initial values of the two tensors as $rot\ \Phi_0$ and $rot\ \Theta_0$ with $T_{(0)} = rot\ \Phi_0 \cdot rot\ \Theta_0$. At the given motion in Northern hemisphere for measuring we chose the counterclockwise $\varphi = \pi/2 - \xi$ with its zero point C_{II} as the North Pole, but after the Equator transition for the same motion in the Southern hemisphere for measuring we chose the clockwise $\varphi = \pi/2 - \xi$ by change of its sign how on the upper and lower parts of the globe. For the Southern hemisphere, we change only the sign of a latitude. For the Western hemisphere, we change only the sign of a longitude. See also in Ch. 10A.

* * *

Now, we describe in general form an algorithm for evaluating main characteristics of summary polysteps rotation (motion) in $\langle Q^{n+1}\rangle$ and $\langle Q^{2+1}\rangle \equiv \langle \mathcal{E}^2 \boxplus \overrightarrow{y}\rangle$ in the scalar, vectorial, and tensor forms. The algorithm starts with application of formula (485) for right transformation of the original base $\tilde{E}_1 = \{I\}$. On the final step of the algorithm, the polar representation (176A), (177A) according to (181A)-(184A) is used. On these stages, with T and T^* from (183A), the homogeneous modal transformations are

$$\tilde{E}_t = rot\ \Phi_{12} \cdot rot\ \Phi_{23} \cdots rot\ \Phi_{(t-1),t} \cdot \tilde{E}_1 = T_{1t} \cdot \tilde{E}_1 = \{T_{1t}\},$$

$$\tilde{E}_t^* = rot\ \Phi_{(t-1),t} \cdots rot\ \Phi_{23} \cdot rot\ \Phi_{12} \cdot \tilde{E}_1 = T_{1t}^* \cdot \tilde{E}_1 = \{T_{1t}^*\}.$$

$$T_{1t} = rot\ \Phi_{1t} \cdot rot\ \Theta_{1t} = rot\ \Theta_{1t} \cdot rot\ \overset{\angle}{\Phi}_{1t}, \quad T_{1t}^* = rot\ \overset{\angle}{\Phi}_{1t} \cdot rot\ (-\Theta_{1t}) = rot\ (-\Theta_{1t}) \cdot rot\ \Phi_{1t}.$$

$$T_{1t} \cdot T_{1t}^* = rot^2\ \Phi_{1t} = rot\ 2\Phi_{1t}, \qquad T_{1t}^* \cdot T_{1t} = rot^2\ \overset{\angle}{\Phi}_{1t} = rot\ 2\ \overset{\angle}{\Phi}_{1t}.$$

$$rot\ \overset{\angle}{\Phi}_{1t} = rot'\ \Theta_{1t} \cdot rot\ \Phi_{1t} \cdot rot\ \Theta_{1t}; \quad rot\ \Theta_{1t} = rot'\ \Phi_{1t} \cdot T_{1t}.$$

The matrix $rot\ \Phi_{1t}$ is evaluated in the base $\tilde{E}_1$ in canonical forms (313), (314); the matrix $rot\ \Theta_{1t}$ – by (259) or (497). Quasipolar representation (176A), (177A) is used for inferring the general law of summing multistep motions or most general homogeneous rotations in the spherical trigonometry of $\langle Q^{n+1}\rangle$, identical up to radius-parameter R to the spherical non-Euclidean geometry of the hyperspheroid. As main result, we obtain the following.

The canonical and polar forms of Quasi-Euclidean homogeneous transformation, in that number, for arbitrary and summarized multistep principal motions:

$$T_{1t} = rot\ \Phi_{12} \cdots rot\ \Phi_{(t-1),t} = rot\ \Phi \cdot rot\ \Theta = rot\ \Theta \cdot rot\ \overset{\angle}{\Phi} = \tag{202A}$$

$$= \left[\begin{array}{c|c} \cos\varphi \cdot \overleftarrow{\mathbf{e}_\sigma \mathbf{e}'_\sigma} + \overrightarrow{\mathbf{e}_\sigma \mathbf{e}'_\sigma} & -\sin\varphi \cdot \mathbf{e}_\sigma \\ \hline +\sin\varphi \cdot \mathbf{e}'_\sigma & \cos\varphi \end{array}\right] \cdot \left[\begin{array}{c|c} [rot\ \Theta]_{n\times n} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \end{array}\right] =$$

$$= \left[\begin{array}{c|c} [rot\ \Theta]_{n\times n} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} \cos\varphi \cdot \overleftarrow{\mathbf{e}_{\overset{\angle}{\sigma}} \mathbf{e}'_{\overset{\angle}{\sigma}}} + \overrightarrow{\mathbf{e}_{\overset{\angle}{\sigma}} \mathbf{e}'_{\overset{\angle}{\sigma}}} & -\sin\varphi \cdot \mathbf{e}_{\overset{\angle}{\sigma}} \\ \hline +\sin\varphi \cdot \mathbf{e}'_{\overset{\angle}{\sigma}} & \cos\varphi \end{array}\right] =$$

$$= \left[\begin{array}{c|c} [rot\ \Theta]_{n\times n} - (1-\cos\varphi) \cdot \mathbf{e}_\sigma \mathbf{e}'_{\overset{\angle}{\sigma}} & -\sin\varphi \cdot \mathbf{e}_\sigma \\ \hline +\sin\varphi \cdot \mathbf{e}'_{\overset{\angle}{\sigma}} & \cos\varphi \end{array}\right] =$$

$$= \left[\begin{array}{c|c} [rot\ \Theta]_{n\times n} - (1-\cos\varphi) \cdot \overleftarrow{\mathbf{e}_\sigma \mathbf{e}'_{\overset{\angle}{\sigma}}} & -\sin\varphi \cdot \mathbf{e}_\sigma \\ \hline +\sin\varphi \cdot \mathbf{e}'_{\overset{\angle}{\sigma}} & \cos\varphi \end{array}\right]$$ (Compare with asymmetric tensor (153A)).

Formulae (202A) give **General law of summing principal rotations (motions** in $\langle Q^{n+1}\rangle$ and on the hyperspheroid, expressed in their canonical forms with respect to the initial unity base $\tilde{E}_1 = \{I\}$. The matrix $rot\ \Phi_{(n+1)\times(n+1)}$ is emanated, for example, by the last element t_{nn} and all the right elements t_{kn} for matrix T in (202A). They permit one to express it in the base $\tilde{E}_1$ in canonical forms (313), (314) with the frame axis in $\langle Q^{n+1}\rangle$ and evaluate scalar and vector trigonometric functions of the angle φ with its directional vector $\mathbf{e}_\sigma$.

The $3D$ case corresponds to $n = 2$, when the canonical structures of matrices $rot\ \Phi_{3\times3}$ and cell $rot\ \Theta_{2\times2}$ are expressed by (313) and (259), but with θ. The complete matrix $rot\ \Theta_{3\times3}$ at $n = 2$ or general $rot\ \Theta_{(n+1)\times(n+1)}$ may be computed also by matrix formula (184A), or through (497)-(499) if $n = 3$ with the frame Euclidean axis $\mathbf{e}_N$ and the sign of θ.

If $n = 2, k = 1, 2$, there hold:

$$\left.\begin{array}{l}\cos\varphi = t_{33}, \quad \sin\varphi = +\sqrt{1-\cos^2\varphi} = ||-\sin\varphi\cdot\mathbf{e}_\sigma||;\ \sin\varphi_k = -t_{k3};\\ \cos\sigma_k = -t_{k3}/\sin\varphi,\ \cos\overset{\angle}{\sigma}_k = t_{3k}/\sin\varphi,\ \mathbf{e}_\sigma = \{\cos\sigma_k\}, \mathbf{e}_{\overset{\angle}{\sigma}} = \{\cos\overset{\angle}{\sigma}_k\}.\\ \cos\theta = \mathbf{e}'_\sigma\cdot\mathbf{e}_{\overset{\angle}{\sigma}} = \mathbf{e}'_{\overset{\angle}{\sigma}}\cdot\mathbf{e}_\sigma,\ \sin\theta = \sqrt{1-\cos^2\theta} > 0 \text{ at } \varepsilon > 0 \text{ and v. v.}\end{array}\right\} \quad (203A)$$

Besides, if $n = 3$, then we use formulae (499): $\overrightarrow{\mathbf{r}_N}(\theta_{13}) = \mathbf{e}_{\overset{\angle}{\sigma}} \otimes \mathbf{e}_\sigma = \pm\sin\theta_{13}\cdot\overrightarrow{\mathbf{e}_N}$. It is similar to (153A–155A) in $\langle P^{n+1}\rangle$ on the abstract hyperbolic-spherical analogy from Ch. 6.

Scalar final results do not change under the mirror permutation of particular motions. It leads only to the substitution in (202A): $T \to T^\star$ with $\Theta \to -\Theta$, $\mathbf{e}_\sigma \to \mathbf{e}_{\overset{\angle}{\sigma}}$.

The specific matrix $T^\star$ in (185A) with contrary ordering of partial motions ($T^\star \neq T'$, as $\Phi \neq \Phi'$, but $\Phi = -\Phi'$) has the general structure, gotten from T with $\mathbf{e}_\sigma \leftrightarrow \mathbf{e}_{\overset{\angle}{\sigma}}$:

$$T^\star = rot\ \Phi_{23}\cdot rot\ \Phi_{12} = rot\ \overset{\angle}{\Phi}\cdot rot\ (-\Theta) = rot\ (-\Theta)\cdot\ rot\ \Phi = \{rot\ (-\Theta)\cdot T\cdot rot(-\Theta)\}$$

$$= \left[\begin{array}{c|c}[rot\ (-\Theta)]_{2\times2} - (1-\cos\varphi)\cdot\mathbf{e}_{\overset{\angle}{\sigma}}\mathbf{e}'_\sigma & -\sin\varphi\cdot\mathbf{e}_{\overset{\angle}{\sigma}}\\ \hline +\sin\varphi\cdot\mathbf{e}'_\sigma & \cos\varphi\end{array}\right]. \quad (204A)$$

T and $T^\star$ are connected by *simple* transposing in original complex binary base (271), where due to (277) they both are Hermitianly symmetric (see at beginning of this Chapter).

Theorem. *In general, any polysteps non-collinear spherical rotations $rot\ \Phi_{1t}$ in $\langle Q^{n+1}\rangle$ or motions on hyperspheroid are represented as spherical one and single orthospherical shift.*

Such interpretation of law (202A) for summing spherical rotations (motions) is confirmed in the quasi-Euclidean space, for example, by the fact, that $rot\ \Theta$ is revealed in the base $\tilde{E}_{1s} = rot\ \Phi_{1t}\cdot\tilde{E}_1$ by polar decomposition in (181A). In the Chapter, laws of hyperbolic geometry motions, established in Chs. 5A and 7A, were transformed sometimes by us very simply by inverse hyperbolic-spherical analogy (323) $i\Gamma \to \Phi$ into spherical ones! And then polar representation (183A) was inferred in analogous form of the quasi-Euclidean tensor trigonometry with the use of analogy (322) $-i\Phi \leftrightarrow \Gamma$. Between two types of geometries and tensor trigonometries, we used the abstract analogy $\Phi \leftrightarrow -i\Phi \leftrightarrow \Gamma \leftrightarrow i\Gamma \leftrightarrow \Phi$ entirely.

* * *

First steps in creating hyperbolic non-Euclidean geometry were made by J. H. Lambert [36] and F. A. Taurinus [38]. Lambert assumed its geometric analogy on a hypothetic sphere of an imaginary radius iR with relations $\varphi \to i\gamma$ and $\gamma = \gamma(\varphi)$, and revealed exactly the angular defect in the hyperbolic triangle. Taurinus established on the sphere first formulae of its planimetry and proved that in its triangle the sum of angles less than $\pi/2$. Later F. Klein [48] and H. Minkowski [65] proved that this hypothetical geometric object is the upper complex hyperboloid II. Nicolai Lobachevsky [40, 41] and János Bolyai [42] created independently this first non-Euclidean geometry in sufficiently full forms by the Euclidean axiomatic method. Unfortunately the Lobachevsky–Bolyai plane and space on the whole are unvisual for men, in contrast to the Lambert's imaginary sphere (as upper hyperboloid II).

"Everything must be made as simple as possible. But not simpler." – Albert Einstein

Chapter 9A

Real and observable by us space-time in the gravity field [1]

In present we can state, under enough logical previous and modern arguments, that, indeed, Tensor Trigonometry with its differential and integral parts since 2004 [15] is applicable for simplest correct studying and description of relativistic motions in the Poincaré – Minkowski space-time of the Nature in the presence of gravitation and in parallel with the simplest trigonometric explanations of all STR- and GR-effects and paradoxes. For this we apply mathematical–physical analogy (sect. 12.3) between physical acceleration and intensity of the gravitational field on the basis of the classical Newtonian Principle of Equivalency, with introducing the so-called *accelerational and gravitational cosines* as such equivalent factors of two specific time dilations. Note, that they do not relate to the well-known Minkowski time dilation from velocity (Ch. 3A). Here are the factors from acceleration g_a and intensity g_f.

If Poincaré life had not ended so early – at the age of 58 (in 1912), then, perhaps, he would have continued to develop his new relativistic theory of space-time and in the gravitational field along the same path, especially since in his pioneering article [63] he predicted the possibility of the existence of gravitational waves, i. e., without unnecessarily bending space-time, but due to additional bending light rays to Newtonian optical reasons – see below in (208A). Before the creation of hyperbolic non-Euclidean geometry, in fact on the surface with its inherent curvature, it never occurred to anyone to take light rays for as if a priori straight lines. In GTR, the Einsteinian mixing straight lines and light rays into one concept occurred, but as geodesics in the curvilinear pseudo-Riemannian space-time.

The historical merit that the inertia of any massive object is created by the Mass of the Universe as a whole belongs to Ernst Mach [55] – eminent physicist and philosopher-positivist of Science. The mechanism of action of this fantastic Mach hypothesis remained unclear for a long time. And Albert Einstein in his GTR refused it with all the Galilean inertial systems. The Mach System, associated with the Center of Mass of the Universe, specified a priori the unique inertial System of Galileo generally for space-time and relative to it all other Galilean systems. But in 1964, finally, the Higgs theory appeared [82], which explained that, during development of the Universe with formation of its Mass as a whole, the latter produces the specific Higgs field, created the Galileo's inertia of any matter as the necessary force of the Nature. Moreover, just like in the flat space-time by Poincaré – Minkowski, at any point and in any direction of this Higgs field in the Universe, the inertia depends only on the mass of an object, in accordance with the Galileo's Law! Then, it is the real space-time by Poincaré – Minkowski is combined with the homogeneous and isotropic material Higgs field of the Universe! This corresponds to conditions of the Noether's Theorem for acting the Law of Energy–Momentum conservation. According to the Newton's Equivalence Principle, inertial and gravitational mass are identical, and this fact has been repeatedly and accurately confirmed, starting with Newton's own experience. Consequently, with the Newton's theory of gravitation, but taking into account the finite speed "c" of the wave-like propagation of gravity, due to Poincaré himself in [63], in fact since June 1905 such new relativistic space-time was introduced! The term "uniform rectilinear motion" in the Higgs Theory has been revived in this relativistic space-time. The so-called "ether", rejected also by Einstein, factually returned in the Universe as the material medium of the Higgs field, but under other name. Poincaré and Lorentz never rejected the material medium of the Universe, and in their works it appeared under the term "ether" accepted at that time. So, the great chemist Dmitri Mendeleev has placed the "ether" in the zero cell of the fundamental "Periodic Table of Chemical Elements", which he had discovered on March 1, 1869.

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[1] The chapter 9A had before discussional character up to this 3-rd edition of the book.

The new essential renovation of the real space-time conception is realizing from 1964 [82], by the very eminent now Peter Higgs, within the framework of the Standard Model for the set of elementary particles, put forward a revolutionary theory, that during the formation of the Universe, according to the Big Bang Theory of the eminent physicist George Gamow, at the stage when its full Mass appears, the latter creates in the Universe a certain material field with its most massive quantum particle "boson". It is the Higgs field creates "inertia", as the fundamental force of the Nature under such its well-known name. The inertia acts on any massive object proportionally to its mass (as its charge), but iff this object deviates from the uniform and rectilinear motion in the field, i. e., due to Galileo's Law. This Higgs theory was strictly confirmed with the discovery of the Higgs boson in 2012 at the Hadron Collider in the Switzerland. And he was deservedly awarded the Nobelean Prize.

We hope that the brief explanations above help to understand to our readers, why the author, since first publication of Tensor Trigonometry in 2004 [15], develops it together with the many various applications of this new math subject in the Theory of Relativity, namely, in the 4D pseudo-Euclidean Poincaré – Minkowski space-time. But, unlike a number of very aggressive apologists of GTR-curved space-time, the author did not impose own author's point of view on the other men, as should be in the Free Science, developing this direction, in accordance with own self independent scientific approach. Besides, we take into account the first theory with flat relativistic space-time in a gravitational field by Nathan Rosen, who was Einstein's assistant at Princeton. However, Albert Einstein did not prevent him from daring in other direction! But the book author believes that theories with curving space-time may mapping only observational space-time through as if a gravitational lens of the gravitation field, but any real calculations can be true only in the Poincaré – Minkowski space-time. Our approach is a good compromise that does not destroy the harmony, but excludes the positivism in real assessments of the relativistic world events. Unfortunately, aggressive behavior of specific apologists of a curved space-time resists such a peace-loving point of view and continues to secretly and persistently hinder its popularization.

The *Special Theory of Relativity* (STR) formulates the Laws of relativistic movement of the matter both in inertial and in uninertial coordinate systems under abstract condition that gravitation is supposed to be absent – see, for example, in [76]. The absolute motion takes place in the macroworld and the microworld and does not depend on a nature of active forces. In Chs. 1A–7A, we used tensor trigonometry for describing Laws of the relativistic motion in clear trigonometric forms. In June 1905, Henri Poincaré made the super revolutionary step: he introduced the *imaginary* time axis with scale coefficient "c" identical to constant speed of light in far Cosmos. With the use of this innovation, he suggested the idea of the united complex-valued space-time with its pseudo-Euclidean metric based on the group nature of its coordinates' transformations, named by him as Lorentzian ones [63, 76, p. 107]. However this genius idea of Poincaré was ignored and not estimated by contemporaries (besides by Hendric Lorentz himself). In 1909, Hermann Minkowski suggested the realificated variant of the pseudo-Euclidean space-time above [65; 76, p. 41], but without reference to the Poincaré fundamental works. He has introduced in the relativistic theory the notions of isotropic cone, time-like and space-like intervals, proper time, time dilation and many others. (*We can only guess what the relationship was between German and French scientists at that time!*)

* * *

While elaborating the GTR [69], Albert Einstein paid attention to empiriocritical Mach's regards on the celestial mechanics uniting dynamics and gravitation, especially on the Law of Gravitational and Inertial Masses Identity. So, gravitational mass does not depend on substance nature, this was established by I. Newton and confirmed with high precision by L. Eötvös. This Principle of Equivalence holds in classical and relativistic forms, but no one has established experimentally: whether this Principle applies to moving mass "m" or not? We'll use below such kinematic full mass "m" of the Mercury with non-moving mass "M_0" of the Sun in our trigonometric representation of the Mercury perihelion *relativistic* shift.

For more convincing concept of GTR, A. Einstein had proposed the General Principle of Relativity, instead Galilean-Poincaré, in which all Laws of Nature have covariant forms in any free moving frames of reference (*but only in the frame's origin!?*). For its realization, he introduced in addition the General Principle of Equivalence of inertia and gravitation. This led to curving 4D space-time of GTR. Such bend relates not only to time-arrows, but and to the geometry of 3D Euclidean subspace with its geometric material objects!?

Obviously, this Principle and this definition of an inertial system as freely falling in space completely contradict the Higgs theory. This time aggressive apologists of GTR remain silent!

Another explanation of both these masses identity Law is closer to Mach's approach. So, for a body M, the Newtonian force of attraction is caused by *active* gravitational action of other material objects, while the force of inertia is caused by *passive* gravitational influence of the whole Mass in the Universe M and now due to the Higgs theory. In such interpretation, the 2nd Newtonian Law of mechanics complements naturally his the Law of Gravitation. To get for M their geometric influences in $\tilde{E}_m$ of $\langle\mathcal{P}^{3+1}\rangle$, we pass from its inner acceleration to its proportional analog in (81A) – the local pseudo-Euclidean curvature K of a world line:

$$-F_{(\imath)} = F_{(a)} = m_0 g = m_0 c^2/R_K = E_0/R_K \to R_K = E_0/F_{(\imath)},\ K = F_{(\imath)}/E_0 < 0. \quad (205A)$$

Here:
F is the inner (i. e., applied in the current base $\tilde{E}_m$) active force causing bending trajectory of the *absolute motion* of M in $\langle\mathcal{P}^{3+1}\rangle$ with the pseudo-Euclidean curvature K;
$F_{(\imath)}$ is the passive force of inertia counteracting to F in $\tilde{E}_m$;
m_0 and E_0 are the own mass and the own energy of the material point;
R_K is the radius of instantaneous absolute pseudo-Euclidean curvature K of the world line at the point M in $\langle\mathcal{P}^{3+1}\rangle$;
c is the constant module of 4-*pseudovelocity* of M in $\langle\mathcal{P}^{3+1}\rangle$ introduced by Henri Poincaré.

Energetic gravitational form (205A) of the 2nd Newtonian Law is in accordance (and it is necessary) with his 1st and 3rd ones, where, in particular, $F = F_{(a)}$ or $F = F_{(f)}$ as the force of gravitation::

$$F = 0 \leftrightarrow g = 0 \leftrightarrow K = 0\ (1st),\ F = -F_{(\imath)}\ (3rd). \quad (206A.)$$

In this Chapter, *we'll bond both kinds of the matter Higgs inertia, caused by acting either physical acceleration $g_{(a)}$ (1) or gravity-intensity $g_{(f)}$ (2) with two equivalent local cosine time dilations. That is, the inertia in both kinds and the local time dilations in both kinds are bonded in $\langle\mathcal{P}^{3+1}\rangle$.* The first case is illustrated at Figure 3A, Ch. 5A, with trigonometric and physical formulae. See the following development in relations (209A), (210A).

From "energetic formula" (205A), $E_0 = -F_{(\imath)} \cdot R_K$, as an inertial torque of the passive force $F_{(\imath)}$, causes local pseudo-Euclidean rotation of a world line ($F = 0 \leftrightarrow R_K = \infty$). For each body absolutely moving with general acceleration (in extreme cases, as parallel or normal to velocity), such "gravitational interpretation" of inertia as in (205A) means that $F_{(\imath)}$ is the centripetal force always directed towards the instantaneous center of a pseudocircle (either of a hyperbola or of a normal circle), and namely $F_{(\imath)}$ curves world lines in $\langle\mathcal{P}^{3+1}\rangle$. Recall, that as long ago as in the 15-th century Nicholas of Cusa (Nicolaus Cusanus) noted: "The Universe is a sphere, and its Center is everywhere!"

With results, gotten preliminary in Ch. 7A in (131A-III) and (145A), for a point M of summations in $\langle\mathcal{P}^{3+1}\rangle$, we have got the Euclidean Rules with Pythagorean theorems for summing orthogonal hyperbolic angular differentials, curvatures and inner accelerations, in that number, collinear and normal ones. Indeed, at $\cos\varepsilon = \pm 1$, we have from (124A):

$$\sinh^2\gamma_{13} = \sinh^2\gamma_{12} + \sinh^2\gamma_{23} + 2\cdot\sinh^2\gamma_{12}\cdot\sinh^2\gamma_{23} + 2\cdot\cosh\gamma_{12}\cdot\sinh\gamma_{12}\cdot\cosh\gamma_{23}\cdot\sinh\gamma_{23} \Rightarrow$$

$$\Rightarrow \sinh\gamma_{13} = \sinh\gamma_{12}\cdot\cosh\gamma_{23} \pm \cosh\gamma_{12}\cdot\sinh\gamma_{23} \Rightarrow \gamma_{13} = \gamma_{12} \pm \gamma_{23}.$$

At $\gamma_{12} \to 0$ and $\gamma_{23} \to 0$, we obtain $d\gamma_{13} = d\gamma_{12} \pm d\gamma_{23} \to k_{13} = k_{12} \pm k_{23} \to g_{13} = g_{12} \pm g_{23}$.

But at $\cos\varepsilon = 0$, we have fo independent two steps and three steps sine summaation in $\langle\mathcal{P}^{3+1}\rangle$:

$$\sinh^2\gamma_{13} = (\cosh\gamma_{23}\cdot\sinh\gamma_{12})^2 + \sinh^2\gamma_{23} = (\cosh\gamma_{12}\cdot\sinh\gamma_{23})^2 + \sinh^2\gamma_{12}.$$

At $\gamma_{12}\to 0$ and $\gamma_{23}\to 0$, we obtain: $d\gamma_{13}^2 = d\gamma_{12}^2 + d\gamma_{23}^2 \to k_{13}^2 = k_{12}^2 + k_{23}^2 \to g_{13}^2 = g_{12}^2 + g_{23}^2$.

By analogy up to Eucldean dimension 3 in $\langle\mathcal{P}^{3+1}\rangle$, we have:

$$\sinh^2\gamma_{14} = \sinh^2\gamma_{12} + \sinh^2\gamma_{23} + \sinh^2\gamma_{34} +$$

$$+\sinh^2\gamma_{12}\cdot\sinh^2\gamma_{23} + \sinh^2\gamma_{12}\cdot\sinh^2\gamma_{34} + \sinh^2\gamma_{23}\cdot\sinh^2\gamma_{34} + \sinh^2\gamma_{12}\cdot\sinh^2\gamma_{23}\cdot\sinh^2\gamma_{34} =$$

$$= (\cosh\gamma_{34}\cdot\cosh\gamma_{23}\cdot\sinh\gamma_{12})^2 + (\cosh\gamma_{34}\cdot\sinh\gamma_{23})^2 + \sinh^2\gamma_{34} =$$

$$= (\cosh\gamma_{12}\cdot\cosh\gamma_{23}\cdot\sinh\gamma_{34})^2 + (\cosh\gamma_{12}\cdot\sinh\gamma_{23})^2 + \sinh^2\gamma_{12}.$$

In its turn, at $\gamma_{12}\to 0$, $\gamma_{23}\to 0$ and $\gamma_{34}\to 0$, we obtain:
$d\gamma_{14}^2 = d\gamma_{12}^2 + d\gamma_{23}^2 + d\gamma_{34}^2 \to k_{14}^2 = k_{12}^2 + k_{23}^2 + k_{34}^2 \to g_{13}^2 = g_{12}^2 + g_{23}^2 + g_{34}^2$.

If a material point M is subjected to simultaneous actions of a few of active forces with different directions (only three may be independent), then forces and generated by them inner accelerations are summarized as 3-vectors in Euclidean subspace $\langle\mathcal{E}^3\rangle^{(m)}$ of $\langle\mathcal{P}^{3+1}\rangle$:

$$\mathbf{F} = \sum_{j=1}^{t}\mathbf{F}_j = \sum_{j=1}^{t} m_0\cdot\mathbf{g}_j = m_0\cdot\mathbf{g} = m_0 g\cdot\mathbf{e} \quad\to\quad \mathbf{g} = \sum_{j=1}^{t}\mathbf{g}_j,\ \mathbf{k} = \sum_{j=1}^{t}\mathbf{k}_j = K\cdot\mathbf{e}. \quad (207A.)$$

Therefore in $\langle\mathcal{P}^{3+1}\rangle$ with fully compatible both the Higgs inertia and Newton gravity fields, we fix the cardinal difference of this non-relativistic Law of summations proportional inner characteristics of motions and regular curves from the relativistic Law of summing velocities in STR, and the same characteristic in the curving space-time of GTR by Einstein.

In Ch. 10A, we'll obtain all Relative and Absolute Pythagorean Theorems for summing in $\langle\mathcal{P}^{2+1}\rangle$ and in $\langle\mathcal{P}^{3+1}\rangle$ all inner curvatures, accelerations and hyperbolic or spherical angular differentials in the most common forms.

Since the *admissible spherical curvature* has similar properties, then for the radius of curvature of light's way, the additive optical Newtonian formula acts with the same property:

$$1/R_1 + 1/R_F = 1/R_2, \quad\to\quad k_1 + k_F = k_2\ (\mathbf{e}_\alpha = \text{const}), \qquad (208A)$$

where R_F is the focal distance of a lens or a mirror, it is either negative, or positive. It is the **first formula for summation of curvatures** applied repeatedly at summation points of a certain light ray along optical axis, each time for trigonometric admissible curvatures!).

In STR, from the point of view of a *Galilean-inertial* Observer N_j in the Euclidean subspace $\langle\mathcal{E}^3\rangle^{(j)}$, any *accelerated* frame of reference $\bar{E}_m$, as an *instantaneous base*, preserves formally for his estimations the inertiality in $\langle\mathcal{P}^{3+1}\rangle$: i. e., $\bar{E}_m \in \langle\bar{E}_j\rangle$. This fact was used in Chs. 5A–7A. However, for an accelerated Observer N_m, situated in the current Euclidean subspace $\langle\mathcal{E}^3\rangle^{(m)}$, its frame of reference, noted further as $\bar{\mathbf{E}}_m$, is Galilean-uninertial one with respect to $\langle\bar{E}_j\rangle$! Thus we have the *relativistic dualism* and two ways (simplest and complex) for describing accelerated movement in $\langle\mathcal{P}^{3+1}\rangle$. Such a dualism was considered, for example, in [105, p. 121-128]. In $\bar{\mathbf{E}}_m = \{\bar{\mathbf{x}}, c\tau\}$ coordinates are curvilinear. Mapping $\bar{E}_j \leftrightarrow \bar{\mathbf{E}}_m$ is isomorphism. Specific examples of such isomorphism were given in Ch. 5A and 6A as descriptions of the same hyperbolic motions in inertial and uninertial coordinates with mapping pro-generated time-like hyperbola into other curves – as if a time-like catenary and as if a space-like tractrix with one common internal argument γ along all these curves. The connection between the coordinates in the bases $\bar{E}_m$ and $\bar{\mathbf{E}}_m$, is expressed also by a smooth function, that is why differentials $d(c\tau)$ and $d\bar{x}_k$ in $\bar{\mathbf{E}}_m = \{\bar{\mathbf{x}}, c\tau\}$ are homogeneous linear functions depending on $dx_k^{(m)}$ and $d(c\tau^{(m)})$ in $\bar{E}_m = \{\mathbf{x}^{(m)}, c\tau^{(m)}\}$, this is equivalent to the one-valued connection of differentials as $d\bar{\mathbf{u}} = V_{(i)}^{-1}d\mathbf{u}^{(m)}$.

The arc of a world line at a point M, as invariant scalar element in $\langle\mathcal{P}^{3+1}\rangle$, may be evaluated by these two ways, either in $\bar{E}_m$, or in $\bar{\mathbf{E}}_m$:

$$[d(c\tau)]^2 = [d\mathbf{u}^{(m)}]'\cdot I^{\pm}\cdot d\mathbf{u}^{(m)} = d\bar{\mathbf{u}}'\cdot\{V'_{(i)}\cdot I^{\pm}\cdot V_{(i)}\}\cdot d\bar{\mathbf{u}} = d\bar{\mathbf{u}}'\cdot G^{\pm}_{(i)}\cdot d\bar{\mathbf{u}}.$$

The matrix of local linear transformation $V_{(i)}$ is uniquely determined by this general congruent representation of the *metric tensor of inertia* (see also in sect. 11.1):

$$G^{\pm}_{(i)} = R'\cdot D^{\pm}\cdot R = (\sqrt{D^{\oplus}}\cdot R)'\cdot I^{\pm}\cdot(\sqrt{D^{\oplus}}\cdot R) = V'_{(i)}\cdot I^{\pm}\cdot V_{(i)}.$$

Thus, *the initial metric of the basis space of events is preserved under passage into its accelerated bases.* In the flat Minkowskian space-time $\langle\mathcal{P}^{3+1}\rangle$, applying Gaussian curvilinear coordinates with respect to $\tilde{\mathbf{E}}_m$ for inner analysis of accelerated motions formally leads to the use of Ricci tensor calculus with conservation of topology. So, in uninertial and inertial bases, differentials of their coordinates for the same arc are affine-connected, this connection is determined by variable tensor $G^{\pm}_{(i)}$ in the Minkowskian space-time (the so-called *metric tensor of inertia*). The tensor acts as a certain function of all coordinates of an arbitrary point M. *It is important that tensor of Riemannian–Christoffelian curvature for* $G^{\pm}_{(i)}$ *is zero here, as this basis space-time is flat.* In an accelerated frame of reference, bending the coordinate grid takes place just *relatively to Observer* N_m. He is situated always in the center of his own instantaneous base $\tilde{\mathbf{E}}_m$. But *Galilean-inertial* Observer N_j notices no bend of coordinates $\mathbf{x}$ and $\overrightarrow{ct}$ with respect to the instantaneous frame of reference $\tilde{E}_m$ wherever N_m is in $\tilde{E}_m$. In particular, a rod moving with acceleration together with Observer N_m is seen by N_j as rectilinear, since for the Observer at any points of $\tilde{E}_m$ the metric tensor is $I^{\pm}$. However, *uninertial* Observer N_m in $\tilde{\mathbf{E}}_m$ can see the exactly same rod in $\tilde{E}_j$ bent. This relativistic effect is *observable*! There are no additional mechanical stretches in this rod merely *seemed* bent, as the same *active inner forces* may be expressed in any inertial frames of reference (due to the common scale of a dynamometer in $\tilde{E}_j$). They are identical as absolute characteristics in $\langle\mathcal{P}^{3+1}\rangle$. The metric tensor $G^{\pm}_{(i)}$ is used for representing the quadratic form of a metric interval in the basis space as the scalar product of differentials. Such tensor is determined also in terms of a linear element $\tilde{\mathbf{u}}$ differentials:

$$[dl]^2 = d\mathbf{u}'_{con} \cdot d\mathbf{u}_{cov} \equiv d\mathbf{u}'_{cov} \cdot d\mathbf{u}_{con} = d\mathbf{u}'_{con}\frac{d\mathbf{u}_{cov}}{d\mathbf{u}_{con}}d\mathbf{u}_{con} = d\mathbf{u}'_{con}G_{(i)}d\mathbf{u}_{con} \equiv$$

$$\equiv d\mathbf{u}'_{cov}\frac{d\mathbf{u}_{con}}{d\mathbf{u}_{cov}}d\mathbf{u}_{cov} = d\mathbf{u}'_{cov}\hat{G}_{(i)}d\mathbf{u}_{cov}, \quad \hat{G}_{(i)} = G^{-1}_{(i)}.$$

In accelerated $\tilde{\mathbf{E}}_m$, one have distorted Minkowski geometry, variable $G^{\pm}_{(i)}$, and the zero tensor of the Riemannian-Christoffelian curvature (sect. 11.1). Christoffelian symbols in $\tilde{\mathbf{E}}_m$ play a role of the *tensor* acceleration.

If gravitation is present, then N_j in $\tilde{E}_j$ fixes the distortion of $\tilde{E}_m$ too with the metric tensor $G^{\pm} = \{g_{kl}\}$ (i. e., if till the 2-nd order of approximation to the real distortion!), as N_j and N_m are divided by a field. The cardinal reason for such distortion is that in cosmic space there is only one tool of estimating geometric and temporal parameters of GTR: it is a light ray between an object and its external Observer (on the Earth N_j in a weak field). This light ray, due to changes in the potential of the field on a light's path, is subjected to corresponding Soldner's [79] and additional Einstein's [69] bends. The idea of accepting rays of light as straight lines or geodesics in GTR) in cosmic space was taken by Einstein from the experiment of the great Carl Gauss with his students (as a head of the astronomical observatory in Göttingen) with measuring the sum of the angles of a triangle formed by three mountain peaks. It were necessary to solve the dilemma: either what is observed and measured using light rays should be taken for reality (a positivist approach), or the same should not always be considered as real assessment of the present, but only as its mapping with possible distortion of real local data (an objectivist approach). Einstein accepted the first point of view, as a result of which, a curvature of the single space-time $\langle\mathcal{R}^{3+1}\rangle$ of GTR with its time arrow and *geometric objects* arose. Then for N_j in the field, the tensor of Riemann–Christoffel curvature becomes non-zero in GTR. The dualism in description of the same motion by N_j and N_m was essentially widen: now two scalar products are one-valued functions one of another. In space-time $\langle\mathcal{R}^{3+1}\rangle$ there is no such deviations of light rays, because in it these rays are straight lines. In the space-time $\langle\mathcal{P}^{3+1}\rangle$ both deviations of light rays are fixed with respect to its pseudo-Euclidean straight lines in $\tilde{E}_j$.

Similar dualism takes place in the *bimetric theories of gravitation* (BMT) with metric tensors $I^{\pm}$ of the Minkowskian space-time and $G^{\pm}$ of the pseudo-Riemannian space-time. They do not full refuse of the Minkowski space-time, as GTR, and use it in different degree. The first BMT was constructed, in the USA, in 1940–1975 by Nathan Rosen [78], who was an assistant and colleague of Albert Einstein, in that number at the Princeton University! In Rosen variant of BMT, metric tensor $I^{\pm}$ describes in $\langle\mathcal{P}^{3+1}\rangle$ as in STR the inertial part connected with the absolute matter motion. The tensor of energy–momentum for a field of gravitation is evaluated, it characterizes this field by $G^{\pm}$, which determines $\langle\mathcal{R}^{3+1}\rangle$ with the pseudo-Riemannian geometry for observations of such relativistic movements in a weak field. Under translation into $\langle\mathcal{R}^{3+1}\rangle$ by Observer on the Earth, the time slows down; but *geometric parameters are as if distorted, as real kinetic distortion of material objects is impossible.* We have *paradox* in BMT like apparent optical curving a light picture seen through a lens, where $G^{\pm}$ is a *gravitational lens for* $\langle\mathcal{R}^{3+1}\rangle$ *as the lensed space-time*! This term is used in Astronomy, when cosmic objects are observed on the Earth through a strong field of gravitation [97].

In the USSR, in 1984-1987 the group of physicists from the Moscow St. University headed by academician Anatoly Logunov constructed the relativistic theory of gravitation (RTG), as a kind of BMT [104], used the same two metric tensors with dividing inertia and gravitation unlike GTR. Gravitation is regarded to the tensor *physical field* in $\langle\mathcal{P}^{3+1}\rangle$ generated by the tensor of energy-momentum for all kinds of matter including fields. The motions equations were formulated in the *effective Riemannian space-time*, generated by tensor $G^{\pm}$ of this field.

The Riemannian binary space has some internal local geometry. Its geometry has a differential character, defined through the set symmetric metric tensor of its space, as the function of a point element. But the Riemannian geometry as a whole differs significantly from homogeneous geometries, such as quasi- and pseudo-Euclidean ones, in which the concepts of *group of motions*, *freedom of motion of figures*, and *topological properties* are of particular importance. For the Riemannian space as a whole with its indefinite topology, the notion of "embeddability" with respect to the Euclidean superspace does not make any sense. This causes the uncertainty for it of the minimum dimension of the enveloping superspace n_{min}. But if we restrict ourselves to the study of any topologically affine-equivalent domain of the Riemannian m-dimensional space, then the value of n_{min} is determined entirely by its local differential-geometric properties. The symmetric tensor of $\langle\mathcal{R}^m\rangle$ contains a maximum of $k = m \cdot (m+1)/2$ independent functional scalar elements g_{ij} in all its cells. Hence, the domain $\mathcal{D}$ of the Riemannian m-space is embeddable in flat $\langle\mathcal{E}^k\rangle$ without changing internal geometry. This was inferred strictly by E. J. Cartan [108]. Consider an analytical definition of $\mathcal{D}$ in the superspace $\langle\mathcal{E}^n\rangle$, where $n \geq k$, with its Cartesian base through $n \times 1$-radius-vector $\mathbf{u}$ with m degrees of freedom of translations. Let each degree of freedom $\mathbf{u}$ corresponds to the Gaussian curvilinear coordinate v_i of the Riemannian m-space. Then there is an exact map $\mathbf{v(u)} \leftrightarrow \mathbf{u(v)}$. Hence, at an each point $\mathbf{v}$ of $\mathcal{D}$ in $\langle\mathcal{R}^m\rangle$ there exists $n \times m$ Jacobi matrix $d\mathbf{u}/d\mathbf{v}$ $(n > m)$ as 1-st derivative of $\mathbf{u}$ in $\mathbf{v}$.

The internal geometry of $\mathcal{D}$ is defined through the homomultiplication as the $m \times m$ metric tensor of $\langle \mathcal{R}^m \rangle \subset \langle \mathcal{E}^n \rangle$:

$$\mathbf{dv}' \cdot G^+ \cdot \mathbf{dv} = \mathbf{du}' \cdot \mathbf{du} \;\Leftrightarrow\; G^+ = \left\{\frac{\mathbf{du}}{\mathbf{dv}}\right\}' \cdot \left\{\frac{\mathbf{du}}{\mathbf{dv}}\right\},\; det\; G^+ \neq 0\; (\mathbf{v}, \mathbf{u} \in \mathcal{D}).$$

For $\langle \mathcal{R}^{3+1} \rangle$, due to A. Friedman in 1961 [109], there is $10D$ space of embedding $\langle \mathcal{P}^{9+1} \rangle$, and then

$$\mathbf{dv}' \cdot G^\pm \cdot \mathbf{dv} = \mathbf{du}' \cdot I^\pm \cdot \mathbf{du} \;\Leftrightarrow\; G^\pm = \left\{\frac{\mathbf{du}}{\mathbf{dv}}\right\}' \cdot I^\pm \cdot \left\{\frac{\mathbf{du}}{\mathbf{dv}}\right\},\; det\; G^\pm \neq 0\; (\mathbf{v}, \mathbf{u} \in \mathcal{D}).$$

For the functional independence of all k elements of the symmetric metric tensors, it is necessary that the inequality $n \geq k$ holds. *In the case of an equal sign, this independence is realized only with the affine topology of the given Riemannian space.* Otherwise, they are connected by some parameters. So, Cartesian coordinates of a sphere are connected by its radius R. For $n > k$, the analogue of Gaussian Egregium Theorem allows to lower the order of embedding of a bounded domain of the Riemannian m-space to at least $n_{min} = k$ using bending. By this way, an isomorphic translation of the motions described in k-dimensional pseudo-Euclidean space, but within m-dimensional pseudo-Riemannian space embedded in it, is carried out. For the observational pseudo-Riemannian space-time $\langle \mathcal{R}^{3+1} \rangle$, it is $n_{min} = 10$. $\langle \mathcal{P}^{9+1} \rangle$ can be a flat space-time for complete mapping motions in a gravitational field by Observer in a weak field. (See more in 2004 [15, p. 290-293] and further in 2011 [107].)

* * *

For simplest kinds of gravitational fields, it is possible to use our trigonometric approach with the *Newtonian Principle of Equivalence* as $-g_{(s)} = g_{(a)} \equiv g_{(f)}$. The hyperbolic motion in $\langle \mathcal{P}^{3+1} \rangle$ (see in Ch. 5A), produced by the uniformly accelerated rectilinear movement in the time under an action of a constant tangential inner acceleration $g_{(a)}$, is as if physically equivalent to the hyperbolic motion under (only!) an action of a static gravitational field with the field intensity $g_{(f)}$ (as the rectilinear movement in the time) – both beginning from the origin of the common base $\tilde{E}_1$ (with $x_0 = 0,\ t_0 = \tau_0 = 0$). Another simplest pseudoscrewed motion in $\langle \mathcal{P}^{3+1} \rangle$ (see in last Ch. 10A), produced by the uniformly accelerated circular movement in the time under an action of a constant centripetal inner acceleration $g_{(a)}$, is as if physically equivalent to the pseudoscrewed motion under (only and one-times!) an action of a spherically symmetric gravitational field with the field intensity $g_{(f)}$ from an astronomic object (as the circular planetary movement in the time) – both with the common origin of the base $\tilde{E}_1$ (with $x_0 = 0,\ t_0 = \tau_0 = 0$). For a correct comparison of the local coordinate time t (usually on the Earth) and the local time in the motion and in the field τ, we chose for them as now adopted the local standard atomic clocks. Below at estimations of the potentials, where $g \approx$ const and $v << c$, for both rotations we use $E \approx m_0 v^2/2 = J_0 w^2/2 = m_0 (rw)^2/2$.

$$\left.\begin{aligned}
d\frac{d(ct)}{d(c\tau)} &= d\cosh\gamma_{(a)} = g_{(a)}d\chi/c^2 = F_{(a)}d\chi/(m_0\cdot c^2) = dE_{(a)}/E_0 = d(P_E)/c^2,\\
\frac{d(ct)}{d(c\tau)} &= \cosh\gamma_{(a)} = 1 + g_{(a)}\chi/c^2 = 1 + \Delta E_{(a)}/(m_0\cdot c^2) = 1 + \Delta(P_E)/c^2 > 1,\\
\frac{d(ct)}{d(c\tau)} &= \cosh\gamma_{(a)} = 1 + (rw_{(a)})^2/2c^2 = 1 + E_{(a)}/(m_0\cdot c^2) = 1 + (P_E)/c^2 > 1.
\end{aligned}\right\} \qquad (209A)$$

$$\left.\begin{aligned}
d\frac{d(ct)}{d(c\overset{\bullet}{\tau})} &= d\cosh\gamma_{(f)} = g_{(f)}d\chi/c^2 = F_{(f)}d\chi/(m_0\cdot c^2) = dE_{(f)}/E_0 = d(-P_G)/c^2,\\
\frac{d(ct)}{d(c\overset{\bullet}{\tau})} &= \cosh\gamma_{(f)} = 1 + g_{(f)}\chi/c^2 = 1 + \Delta E_{(f)}/(m_0\cdot c^2) = 1 + \Delta(-P_G)/c^2 > 1,\\
\frac{d(ct)}{d(c\overset{\bullet}{\tau})} &= \cosh\gamma_{(f)} = 1 + (rw_{(f)})^2/2c^2 = 1 + E_{(f)}/(m_0\cdot c^2) = 1 + (-P_G)/c^2 > 1.
\end{aligned}\right\} \qquad (210A)$$

In accordance with the classical Newton's Equivalence Principle, we introduced in (209A) *accelerational* and in (210A) *gravitational hyperbolic cosines* in result of acting inner force $F = m_0 g$ on a body with inner acceleration in direction of pseudonormal, causing equivalent cosine time dilation $dt \Rightarrow d\tau$ always in direction of tangent to a world line (see in Ch. 10A).

We established on these extreme examples of motions – hyperbolic and pseudoscrewed, that a transition to proper time $dt \Rightarrow d\tau$ requires proportional expenditure of energy, here mechanical or gravitational, with increasing potential. It is the transition to proper time needs in the increase of energy's level (as potential), with respect to time in immovable state or in inertially moving frame of reference in (209A) and (210A). For arbitrary motion, our inferring is generalized by decomposition of the inner acceleration g onto tangential and normal accelerations, with respect to the current velocity vector $\mathbf{v}$, in accordance with the Absolute 3D Euclidean Pythagorean theorem (145A) – see it strictly in (229A), Ch. 10A. Therefore the true cause of matter inertia in the real space-time $\langle \mathcal{P}^{3+1} \rangle$ is a transition to proper time $dt \Rightarrow d\tau$, which is appeared in the process of accelerated/decelerated motions with energetic expenditures! This statement corresponds to the Higgs theory of inertia [82].

For instance, the direct and reverse hyperbolic motions need in equal expenditures of energy – see at Figure 3A. It is the transition to proper time $dt \Rightarrow d\tau$ is felt by us or perceived by instrument as the inertia! Such translation of the time dilation in the inertia is realized in the Higgs field with the definite energetic expenditure!

The gravitational time dilation was predicted by Einstein in 1907 [73], but as local one! From (210A) we get the *Einsteinian gravitational time dilation*, however at $c = \text{const}$ (!):

$$\frac{d(c\overset{\bullet}{\tau}_1)}{d(c\overset{\bullet}{\tau}_2)} = \frac{d\overset{\bullet}{\tau}_1}{d\overset{\bullet}{\tau}_2)} = [1 + (-P_2)/c^2]/[1 + (-P_1)/c^2] \approx 1 + [(-P_2) - (-P_1)]/c^2. \qquad (211A)$$

However such time dilation is evaluated up to now by decreasing electromagnetic radiation frequency, usually by oscillations frequency of photons. While, locally in $\langle \mathcal{P}^{3+1} \rangle$, photons get farer from the smaller negative potential $-P_2$ to the bigger negative potential $-P_1$, their kinetic energy $h\nu$ and frequency ν decrease due to overcoming negative $[(-P_2) - (-P_1)]$, but with increasing energy's level (as potential) in the field with $-P_1$. Therefore the Einsteinian gravitational time dilation has a pure quantum–mechanical nature under conserving $\langle \mathcal{P}^{3+1} \rangle$!

If $P_1 = 0 = \max$, then $\overset{\bullet}{\tau}_1 = t^{(1)}$ is also non-relativistic time of N_1 on the Earth, and (!) the Newtonian potential of M gives us rather precise estimation in the near-Solar region [73]:

$$\frac{d(ct)}{d(c\overset{\bullet}{\tau})} = \cosh \gamma_{(f)} = 1 + \frac{(-P_G)}{c^2} \approx 1 + \frac{fM}{r}/c^2 = 1 + g_{(f)}r/c^2 > 1. \qquad (212A)$$

Hence, "gravitational twins paradox" with $g_{(f)}$ is possible in addition to $g_{(a)}$ in STR, Ch. 5A.

At free accelerated motion in space-time under acting gravitation, we obtain exactly twice time dilation from two factors $g_{(t)}$ and $g_{(f)}$ – factually Newtonian and Einsteinian at only equivalence of inertial and gravitational masses. One must choose – either additional local curving of a world line of the free moving object M from $g_{(f)}$ in the Minkowski space-time or, according to Einstein [69], equivalence local curving of space-time with its transformation into the pseudo-Riemannian space-time with the sign indefinite metric tensor. According to our tensor trigonometric approach, we chose first variant with STR in $\langle \mathcal{P}^{3+1} \rangle$.

Free Science allows freedom of choice! Who believe it should be controlled hold back it!

Up to now we dill with massive particles or body with the relativistic mass in moving $m = m_0 \cdot \cosh \gamma$. The same cosine coefficient leads to the time dilations in (209A) and (210A). In the following, we'll dill with the so-called massless particle, mainly, as a photon. The term "massless" means only their zero mass as if in absence of motion. According to the Planck-Einstein formula for massless particles, for example, for a photon, the kinetic energy of its motion is equal to $E_L = h\nu$, i. e., it is defined only by the frequency ν of its oscillation during motion. And for them the concepts in (209A) *accelerational* and in (210A) *gravitational hyperbolic cosines* are as if not acting. Thus, instead (212A), for massless particles with a photon, we must adopt, that

$$\frac{d(ct)}{d(c\overset{\bullet}{\tau})} = \frac{\overset{\bullet}{\nu}}{\nu} = \frac{\lambda}{\overset{\bullet}{\lambda}} = 1 + \frac{(-P_G)}{c^2} \approx 1 + \frac{fM}{r}/c^2 > 1. \quad (\nu \cdot \lambda = \overset{\bullet}{\nu} \cdot \overset{\bullet}{\lambda} = c = \text{const}) \qquad (213A)$$

A photon, as the Newtonian corpuscle of light, was introduced again in XX cent. by Albert Einstein to interpret due to the Quantum mechanics dualism the Laws of photoeffect by Alexander Stoletov (in 1888-90). Evaluate Newtonian, but with STR (1), and refractional (2) approaches to revealing complete deflection of a light ray near the Sun. Let that a photon of mass m (in moving) moves with respect to an astronomical mass M at velocity $\mathbf{v} = \mathbf{c}$ under angle ε to the radius-vector $\mathbf{r}$ from M barycenter. By the Newtonian Laws with STR, there holds:

$$\mathbf{F} = F \cdot \mathbf{e}_\beta = [(f \cdot M \cdot m_L)/r^2] \cdot \mathbf{e}_\beta = m_L g \cdot \mathbf{e}_\beta = [(m_L \cdot c^2)/R] \cdot \mathbf{e}_\beta = \overset{\perp}{F} \cdot \mathbf{e}_\nu + \overline{\overline{F}} \cdot \mathbf{e}_\alpha. \quad (214A)$$

$$\overset{\perp}{F} = \sin\varepsilon \cdot [(f \cdot M \cdot m_L)/r^2] = (m_L \cdot c^2)/\overset{\perp}{R} = E_L/\overset{\perp}{R} = h\nu/\overset{\perp}{R}\,, \quad (214A-I)$$

$$\overline{\overline{F}} = \cos\varepsilon \cdot [(f \cdot M \cdot m_L)/r^2] = (m_L \cdot c^2)/\overline{\overline{R}} = \frac{d(m_L \cdot c)}{d\tau} = \frac{dP_L}{d\tau} = \frac{dE_L}{dl} = \frac{d(h\nu)}{dl}. \quad (214A-II)$$

Two orthoprojections of the inner force F, acceleration g and curvature $K = 1/R$, as normal and tangential, are summarized by the Pythagorean Theorems as above and generally in last Ch. 10A. Since $M >> m$, then a photon at each moment of time receives some total differential of movement in $\langle\mathcal{E}^3\rangle$ around the Sun. F tangential projection causes acceleration/deceleration of the light particle along vector-velocity $\mathbf{c}$. For the photon, it merely increases or decreases its energy E_L and oscillation frequency during motion at $c =$ const. Hence, this projection (with very small change of mass m) does not influence on the *Newtonian* normal spherical deviation of a light. Hyperbolic curving is also absent at $c =$ const as the scale coefficient to time by Poincaré. Contrary, F normal projection, as a centripetal force, causes the *Newtonian bend* of the light ray with its local normal radius.

Note that the trajectory of this light ray is extremely stretched due to the high velocity c of light. For the simple trigonometric approach, this makes it possible to construct a special current right triangle with a constant leg b opposite the spherical angle ε between the vectorial speed of light $\mathbf{c}$ and the radius-vector $\mathbf{r}$, directed as field's intensity $g_{(f)}$. From (I) we have: $1/\overset{\perp}{R} = \sin\varepsilon(fM)/(rc)^2$. In the right triangle [79; 69; 75, p. 351-355], the leg $b =$ const is the distance between barycenter of M and the intersection point of this light ray two asymptotes: $b \approx r \cdot \sin\varepsilon \approx \min(\mathrm{r})$, the extremely stretched arc of this light ray and second leg is $l \approx r \cdot \cos\varepsilon$; then $1/\overset{\perp}{R} = \sin^3\varepsilon(fM)/(bc)^2$.

With (I) this light ray bend is expressed in the differential and integral forms as follows:

$$d\delta_I = dl/\overset{\perp}{R} \approx d(-r \cdot \cos\varepsilon)/\overset{\perp}{R} = b\; d(-\cot\varepsilon)/\overset{\perp}{R} = [fM/(bc^2)] \cdot \sin\varepsilon\; d\varepsilon = [-P(\varepsilon)/c^2]\; d\varepsilon > 0,$$

$$\delta_I \approx [fM/(b \cdot c^2)] \cdot \int_0^\pi \sin\varepsilon\; d\varepsilon = 2fM/(b \cdot c^2) = 2 \cdot (-P_{min})/c^2.$$

With (II), the photons in this light ray itself along the vector $\mathbf{c}$, till the middle way point, receive the energy, and after middle way point, give back it as $\pm h\Delta\nu$, with preserving their initial energy (!) Just this Newtonian estimation δ_I was obtained by Einstein in 1911 [76, p. 202] at $c =$ const, but as often for him, without references to predecessors. So, Johann von Soldner was historically first, who evaluated it in 1801 [79; 97, p. 7] following to the Newton's gravitational and corpuscular theories. Moreover, Isaac Newton forecasted discovery of this effect for his light corpuscles in 1704!

In 1915, Einstein evaluated GTR correction for a light ray bend in a spherically symmetric gravitation field using the Tensor Calculus, *with decreasing* c *in the field*. New value was proved to be twice larger. To estimate in $\langle\mathcal{P}^{3+1}\rangle$ this 2-nd term, we use the *mathematical analogy* of light propagation in the optic medium with variable refraction index and in the gravitational field with variable potential [75, p. 308]. But we took into account the variable frequency of photons in parallel projection (II), causing by the change of photons kinetic energy $h\nu$ from the variable potential, and constancy of the light speed with relation $c = \nu \cdot \lambda =$ const. The oscillations frequency of photons ν increases in the 1-st part of its trajectory and decreases in the 2-nd with the same relation for $\pm\Delta\nu$, with the corresponding changes of their waves length $\mp\Delta\lambda$. The *angle of incidence* is ε, if $\varepsilon \le \pi/2$, the angle of incidence is $(\pi - \varepsilon)$ if $\varepsilon > \pi/2$. With (II) and the Snellius Law (1626), this is interpreted as the *additional to the classic Soldner's bend* of a light ray towards the barycenter of M:
$\sin\varepsilon/\sin(\varepsilon - d\delta_{II}) = \frac{\nu + d\nu}{\nu}$, $\varepsilon \le \pi/2$; $\sin(\pi - \varepsilon)/\sin(\pi - \varepsilon + d\delta_{II}) = \frac{\nu - d\nu}{\nu}$, $\varepsilon > \pi/2 \;\rightarrow$

$\rightarrow d\delta_{II} = \pm d\nu/\nu = \overset{\perp}{dc}/c = \overset{\perp}{g}\; d\tau/c = \overset{\perp}{g}\; dl/c^2 = dl/\overset{\perp}{R} = d\delta_I$. Here 1-st differencial of the deviation of the vector $\mathbf{c}$ with the light ray is orthogonal to it and hence has above corresponding notation.

Under finally been accepted by physicists condition in the field $\nu \cdot \lambda = c =$ const, we have $d\nu/\nu = d(c/\lambda)/(c/\lambda) = -d\lambda/\lambda$. The refractional spherical deviation δ_{II} in the gravitational field relates only to oscillating time particles moving near light, including De Broglie ones! For a ray along the central axis from the Star to the barycenter of mass M, there is no gravitational refraction at all (how for an optical spherical lens!) as the normal to world line deviating projection of the gravitation force $\mathbf{F}$ is zero in this case with $\varepsilon = 0$.

We got with (214A-I and II) twice deviation of a light ray from the Sun potential changes:

$$\delta = \delta_I + \delta_{II} = 4fM/(bc^2) = 4(-P_{min}/c^2) = 4(-P_S/c^2)\cdot(r/b) \quad \text{– under } c = \text{const!} \quad (215A)$$

1-st curving is caused by variable normal $g_{(f)}$, 2-nd curving is caused by variable P_G.

The photon's momenta vectors $\mathbf{P}_0$ and $\mathbf{p}$ change only direction. The work of positive or negative parallel projections turns in positive or negative changes of its kinetic energy $\pm\Delta h\nu$. The normal positive or negative parts of photon's energy changes relate to the *Newtonian part* also in accordance with the Law of Energy Conservation:
$dE = \pm(-P_{min})\cdot\sin\varepsilon\, d\varepsilon\cdot h\nu_0/c^2$, $\Delta E_{max} = (-P_{min})\cdot h\nu_0/c^2$, $h\nu_{max} = h\nu_0[1+(-P_{min})/c^2]$.

We may add to the *Poincaré Principle of Relativity in* $\langle\mathcal{P}^{3+1}\rangle$*, but with a field of gravity*:
The gravitational potential in any world point cannot be determined by the value of speed of light $c = \nu\lambda$ measured locally by some manner. Scalar speed of light c in the cosmic vacuum is equal to the Poincaré scale coefficient for time.

A very far Observer in a weak field perceives the same local events in a strong field as if in distorted space-time $\langle\mathcal{R}^{3+1}\rangle$ with bivalent metric tensor up to 2-nd order of approximation.

Consider another, *but as if GR-effect* – the "red shift" of the Sun radiation spectrum, predicted in 1913 by Albert Einstein. Though it was predicted first in 1783 by John Michell in his letter to the London Royal Society [81] on the basis of the Newton's corpuscular and gravitation theories! It is caused by slowing-down of all electromagnetic oscillations from the Sun surface due to there very strong negative potential [75, p. 346]. Due to (212A-II), we have: $\lambda > \dot{\lambda}$, $(\nu\cdot\lambda = c)$. Let's pay attention to the fact that the assessment of this effect is confirmed precisely on the Earth with the atomic clocks, i. e., in a weak gravitational field! The "red shift" was precisely affirmed on the Earth in 1959 by R. Pound and Jt. Rebka with the use of Mössbauer's effect [96]. Though the difference of two potentials was very small.

We interpret "red shift" by the *energetic part* of our conception *without normal refraction* (at $\varepsilon = 0$). The photons or other massless particles, under negative acting of gravitation, get decreasing of their kinetic energy $E = h\nu$ with increasing of their light waves length $\lambda = c/\nu = h/(E/c) = h/p$ for an Observer of this radiation on the Earth. For massless particles at $v = c$, there holds $E = pc = mvc = mc^2$ as here $E_0 = 0$. (For a body, we have *equivalent* decreasing of total energy E and $pc = mvc$ (Ch. 5A) with increasing of De Broglie waves length $\lambda = h/p = h/(mv)$.) Then this effect for the Sun radiation is explained by us on the basis of the Newtonian gravitation accompanied by the quantum-mechanical approach:

$$E_L = h\nu = m_Lc^2 = \dot{E}_L - (-P_S)\cdot m_L = h\,\dot{\nu} - (-P_S)\cdot m_L < h\,\dot{\nu} \Rightarrow \nu < \dot{\nu},\ \lambda > \dot{\lambda}, \quad (216A)$$

where $m_L = h\nu/c^2$ is the Planck–Einstein formula for the mass of a moving photon; $\dot{\nu}$, $\dot{\lambda}$ are local values on the Sun surface; ν, λ are values on the Earth. The *energetics approach*, with full executing the Law of Energy Conservation, were first noted by the eminent physicist (progenitor of the matrix quantum mechanics) Max Born [74]. He did not develop this idea and rested Einsteinian GTR-interpretation of this effect. Recall also (see more in the end of Chs. 12 and 7A), that relation $E = mc^2$ *for the light's energy, as a kind of electromagnetic radiation*, was discovered in 1900 by Henri Poincaré in one from many his pioneer articles [62].

Indeed, due to this Law, while photons get farer from the Sun to the Earth, its kinetic energy and frequency decrease due to overcoming the negative Sun potential in direction to the Earth. Without the *Doppler effect*, suppose initially that speeds of light near them are: $\dot{c} = \dot{\nu} \cdot \dot{\lambda}$ – on the Sun, and $c = \nu \cdot \lambda =$ on the Earth at $h =$ const.
From (216A), we obtain result: $\nu = c/\lambda \ < \ \dot{c} \ / \ \dot{\lambda} = \dot{\nu}$. Further we have only two variants:
(1) $\lambda > \dot{\lambda} \Rightarrow \dot{c} = c$ – it is correct variant, the effect "red shift" is fixed on the Earth;
(2) $\lambda = \dot{\lambda} \Rightarrow \dot{c} > c$ – it is incorrect variant. (The variant $\dot{c} < c$ is absent in (216A) at all!)
One must choose either the correct variant (1), or choose even the non-existing incorrect variant (2) and *in the "red shift" theory refuse of the Law of Energy Conservation*. We chose variant (1). It corresponds to strictly inferred relation (216A). The photons on the Sun in its strong gravitational field have the *initial* frequency by the local atomic clocks on the Sun and the wave length (as those radiated on the Earth or without gravitation at all). When they achieve the Earth, this radiation has less its frequency by the local atomic clocks on the Earth proportionally to decreasing of photons energy and more its wave length, according to variant (1) with the "red shift". Interchange a source of radiation and Observer. Due to the Principle of Relativity, Observer in the strong field will see inverse "violet shift". Both shifts of De Broglie waves length must take place also for massive or massless particles. *In essence, this effect relates to Newtonian theories with the Quantum Mechanics, but no to Relativity!*

It is usually believed that the third GR-effect "the Mercury perihelion *relativistic* shift" is not explained in frame of Newtonian theories with STR, and can be interpreted only by GTR in the strange form: "*It is GTR-equations' solution*". Our its simplest tensor trigonometric explanation with immediate physical interpretation in $\langle \mathcal{P}^{3+1} \rangle$ is based on three STR cosine time dilations (Ch. 3A, 5A), with their doubling as in the equivalent *accelerational and gravitational hyperbolic cosines* in (209A) and (210A), in that number, for relativistic rotations. For estimation of this effect, we adopt the next. (1) The motion of the Mercury is almost circular. (2) In two rotational formulae from (209A) and (210A), we use now the values of kinetic energy $E \approx mv^{*2}/2 = Jw^{*2}/2 = m(rw^*)^2/2$ as approximated well to relativistic values, instead classical. On the orbit of the Mercury, we obtain in rotational parts of (209A) and (210A) three STR cosine dilations: one by translation to the relativistic mass $m = \cosh\gamma \cdot m_0$ and two by translation to the proper velocity in the item v^{*2}, where $v^* = \cosh\gamma \cdot v$. With (206A), (209A), (210A), they lead to the summary time dilation by six factors $k_E = \cosh\gamma - 1$ under approximation $\cosh^6\gamma - 1 \approx 6(\cosh\gamma - 1)$ at $v/c << 1$.
With respect to a time in a weak field as on the Earth, for the orthospherically planetary rotated Mercury at our correction above in (209A) and (210A) in the base $\tilde{E}_1$ and without hyperbolic bending of its world line, the perihelion is shifting orthospherically with coefficient $k = 3 \times 2\pi r$, *that up to now nobody physically understanded* ! Estimate this relativistic shift of the Mercury perihelion in one its revolution, in such our interpretations at $v/c << 1$:

$$\Delta = +T \cdot 6\ (\cosh\gamma - 1) \cdot \frac{d\alpha}{dt} = \frac{6 \cdot 2\pi r}{v} \cdot (w_\alpha^* - w_\alpha) \approx \frac{12\pi r}{v} \cdot \frac{\sinh^2\gamma}{2} \cdot w_\alpha = \frac{6\pi r}{c^2} \cdot v^* \cdot w_\alpha^* =$$

$$= \frac{6\pi r}{c^2} \cdot \frac{v^{*2}}{r} = \frac{6\pi r}{c^2} \cdot \overset{\perp}{g}_{(f)} = \frac{6\pi r}{c^2} \cdot \frac{fM}{r^2} \approx 6\pi \cdot \frac{fM}{rc^2} = 6\pi \cdot \frac{(-P_G)}{c^2} > 0. \qquad (217A)$$

We got the well-known and confirmed formula for this effect that accumulates over time. With such approach, it is not necessary to reduce the local speed of light and to bend Minkowski space-time, but only to dilate time by six factors k_E (99A). This positive orbital orthospherical shift is expressed in $\tilde{E}_1$ in the normal plane of Euclidean rotations $\langle \mathcal{E}^2 \rangle_N$ as if together with negative Thomas precession (172A) around the instantaneous axis $\mathbf{e}_\mu \equiv \mathbf{e}_\alpha \times \mathbf{e}_\nu$ perpendicularly to the orbit. The eccentricity of the Mercury orbit gives only an astronomical opportunity us to observe this perihelion shift. The *average radius* $\overline{r}$ is calculated through the well-known connection $\overline{r} = a(1 - e^2)$ with a big semiaxis of the exactly elliptical orbit.

Albert Einstein evaluated this additional shift of the Mercury perihelion by the so called "exact formula" (inferred above), but in the frame of GTR with curving by gravitation space-time, in his articles [72, 69]. For objectivity, it should be noted, that Einstein took the well-known in that time formula by Paul Gerber again without reference, published twice in 1898 and 1902, which has explained the Mercury perihelion's shift very well, but from the non-relativistic arguments [99]. Einstein has expressed the opinion that such physical formula would be impossible to derive strictly as the *exact solution* from the GTR equations. However in 1916, in the frame of the Einsteinian GTR, the World War I veteran Karl Schwarzschild introduced dilation of coordinate time into proper one [100; 75, p. 326, 348], and, in his new coordinates, realized such "exact formula". In Chs. 5A and 6A, we showed that translation $dt \to d\tau$ leads in Theory of Relativity to the loss of polysteps principal operations $\langle roth\,\Gamma_k\rangle$. Such approach is artificial as if for a necessary known result, as was often in GTR infers.

Note, thanks to the mathematically identical sixfold dilation of time in (217A), we proved that in Newton's Law of Universal Gravitation, both gravitational masses must not only be equivalent to the inertial masses, but also be relativistic, i. e., with their own cosine factors!

For executing the Law of Energy conservation, we must adopt, that energetic expenditure on this Mercury perihelion *positive relativistic shift in time* is compensated by the Sun, close enough to the Mercury. Contrary, the electron in the Thomas precession has no energetic compensation. Then the Thomas precession is caused physically also by negative one-times kinematic energy rebound with the factor k_E in (99A) and (172A) of the increased relativistic energy of rotation on orbit with the same factor k_E due to $w^*_\alpha > w_\alpha$ under translation to proper time. And, thus, this rebound restores acting of the Law of Energy conservation!

In our STR-interpretation of Δ, in accordance with Einstein's wishes in the Epigraph to Chapter, it is seen that the dissonance $\delta = w^*_\alpha - w_\alpha = (\cosh\gamma - 1)\cdot w_\alpha$ is the quintessence of our formulae (217A) and (172A), which moves with the plus sign the Mercury perihelion and with the minus sign in (172A) the Thomas precession, and $k_E = \pm(\cosh\gamma - 1) = \pm\Delta E/E_0$ is an energetics factor in them. The dissonance arises from the fact that both these rotations with close velocities act in adjacent Euclidean planes at small inclination γ between them.

Our explanations of GR-effects are in accordance with the Principle of Correspondence by Niels Bohr! So, transferring to non-relativistic time and ignoring the gravitational refraction, we return to the Newtonian theory. We are not at all satisfied with the notorious approach to explaining GR-effects with camouflaging formulations like: "*it is equations' solution*" (similar to abstract fantasies). Theory of Relativity in its original sense with the group mathematical approach by Poincaré–Lorentz is the rigorously determined and exact science.

There is an undeniable fact: GR-effects in the Solar system are fixed by Observers on the Earth in a weak field of gravitation, *but occur in a strong field of gravitation near the Sun*. Therefore, their full description must have dualism from two points of view as in BMT. But GTR gives only single interpretation [75, p. 346-356], as seen by Earth Observers without taking into account that local information must reach him through decreasing gravity field. Such *positivist interpretation* inevitably leads *to violation of the Law of Energy Conservation*.

The historical statement of David Hilbert as the first author of GTR motions equations [70] (1917) becomes: "I assert ... that for the general theory of relativity, i. e., in the case of general invariance of the Hamiltonian function, ... corresponding to the energy equations in orthogonally invariant theories do not exist at all. I could even take this circumstance as the characteristic feature of the general theory of relativity." [71]. This has not been recognized by physical community for a long time. This violation is caused by that GTR space-time do not contain the ten-parametric group of motions (presenting in $\langle\mathcal{P}^{3+1}\rangle$), due to its pseudo-Riemannian space-time bent in a field of gravity – see in [105, p. 163]. That is why, D. Hilbert, yet in the beginning of 1915, put the task for famous colleague Emmy Noether in Göttingen: to find conditions for fulfilling this Law of Nature. And in 1915, she proved the fundamental Theorem of mathematical physics, connected the Integral Law of Energy and Momentum Conservation for motions with parameters of a space-time symmetry [102].

However the *general* pseudo-Riemannian space-time is non-homogeneous and non-isotropic. Therefore this fundamental classical Law of Nature cannot hold in it. The curved space-time cannot have even constant curvature, as it depends on hierarchical casual mass distribution.

In 2004, with publication in Russia of 1st edition of our "Tensor Trigonometry" [15], at this time the eminent English mathematician, physicist and GTR philosopher Roger Penrose, professor of Mathematics at the University of Oxford, wrote the similar in his book [98]:

"We seem to have lost those most critical conservation laws of physics, the laws of conservation of energy and momentum! In fact, there is a more satisfactory perspective on energy-momentum conservation, which refers also to certain curved space-times $\mathcal{M}$ as well as to Minkowski space ... These conservation laws hold only in a space-time for which there is the appropriate symmetry, given by the Killing vector $\mathbf{k}$. Nevertheless, they do not really help us in understanding what the fate of the conservation laws will be when gravity itself becomes an active player. We still have not regained our missing conservation laws of energy and momentum, when gravity enters the picture." Anything to add to these clear words!

Soviet academician Vladimir Fock proved that predictions of GTR concerning GR-effects in the Solar system are ambiguous [77]. They depend on coordinate conditions. By the cause, Einstein considered GR-effects as if they are in a weak stationary gravitational field in fact embedded into the Minkowskian space-time $\langle \mathcal{P}^{3+1} \rangle$ [105, p. 156–165]. Such an artificial approach did not fix this problem. Numerous strange attempts to combine GTR without group approach and the Quantum Mechanics with group approach, including many years Einsteinian himself, have not yielded any results and do them similar squaring a circle (but here as a hyperbola). The main reason of this lies in the positivist essence of the GTR, which combines the real and the observable into one whole. If return to the Poincaré – Minkowski space-time, then this problem can be solved quite naturally, as was in the well-known Pole Dirac approach to the Quantum Mechanics [101], but now together with the Higgs field.

The fix-idea of Einstein's GTR [69] is expressed by the General Principle of Relativity as his Postulate: *All physical Laws in free arbitrary moving frames of reference* $\tilde{\mathbf{E}}_m$ *must have locally standard forms determined by metric tensor* $I^{\pm}$ (as if in all $\tilde{E}_k$ of STR). Strictly speaking, this Postulate is a hypothesis and relates only to zero point of $\tilde{\mathbf{E}}_m$, while it is not confirmed convincing enough, so, by experiments with a free horoscope in a cosmic orbit. STR is valid in GTR only in locally tangent $\langle \mathcal{P}^{3+1} \rangle$, hence the Mach's base $\tilde{E}_0$ was refused by Einstein. Although he did not turn away from the Mach's positivism [55). That is why, GTR cannot be realized in the material Higgs field with its Galileo inertia! Thus, in GTR all frames of reference free-moving in presence of gravitation became equivalent. This was expressed in his well-known extreme, but scientifically honest statement on the equal rights of Kopernik and Ptolemy Solar systems. Indeed, in Einsteinian curved space-time, it is so. In flat Minkowski space-time it is not so! Unfortunately, the very aggressive behavior of specific apologists of a really curved space-time still resists for other logical points of view in the sphere of scientific publications and they continue to make from Albert Einstein as if the new Ptolemy how in the middle Ages. So, recall wise saying of Einstein himself: "Um mich für meine Autoritäts Verachtung zu bestrafen, hat mich das Schicksal selbst zu einer Autorität gemacht!" (*To punish me for my contempt for authority, fate made me an authority myself!*) The first confirmation of GTR curved space-time with a lot of sensational noise around this event was that astronomers confirmed the twice bend of a light beam close enough to the Sun during its eclipse. Nobody remembered that a light beam is bent, passing through an optically non-homogeneous medium due to the Snellius Law under the influence of electromagnetism, which determines the refractive index, and this may be also in addition to its Soldner's bending. Nobody had previously thought to curve our space under acting the electromagnetic field to explain such a bend of the light beam. Those who accept everything remotely observed and measured as an exact reality are committed to positivism, although this pseudo-scientific philosophy with its apologists as if remained in the 19th century.

The observed and real space-time cannot be perceived identically on the astronomical scale of the Universe, if only because the information about variously distant space objects arrives to the Earth at different time intervals. So far, no one has managed to make this picture seem us simultaneous. However, by our opinion, the most unacceptable thing in the GTR is such, that in gravity field its real distortion propagates not only to the time, which is quite natural even due to STR, but and to the geometric parameters of material objects in the Universe?! Similar a misconception was once held by some relativists regarding reality of the Lorentz contraction (see in Ch. 4A). The enthusiasm with the empty project of voyages through "wormholes-tunnels" in the Universe, with a renovated and now cosmic perpetuum mobile, and many other baseless GTR-fantasies are an usual pseudo-scientific PR-populism!

These unanswered by the GTR questions are answered clarity and unambiguously by BMT–theories with two metric tensors. The first BMT (as if with metric tensors $I^{\pm}$ of basis $\langle\mathcal{P}^{3+1}\rangle$ and $G^{\pm}$ of observable $\langle\mathcal{R}^{3+1}\rangle$ till the 2-nd order of approximation to the possible distortions), was created by Nathan Rosen, *the Einstein's assistant and clouse colleague!* [78]. Correct physical conclusions can be drawn as true only from local and not observed data. That is why, $\langle\mathcal{P}^{3+1}\rangle$ exists really in BMT, but with its accompanied observed lensed mapping as $\langle\mathcal{R}^{3+1}\rangle$, i. e., with acting the great Mach Principle [55]! Conception of BMT, by historic roots, rises on the Hegel dialectic spiral [57] to teachings of great thinkers of the Past: I. Kant with his Postulate on the basic role of the Euclidean space in the real world [56] and, of course, I. Newton with his Postulate on the absolute space and time [54]. These notions were united by H. Poincaré in 1905 [63] and by H. Minkowski in 1909 [65] in the *absolute space-time* $\langle\mathcal{P}^{3+1}\rangle$ of the Nature. BMT may interpret the pseudo-Riemannian space-time as observable lensed Minkowski space-time. Then all motions and events have place really in basis $\langle\mathcal{P}^{3+1}\rangle$, what gives compatibility with the *Principles of Correspondence, Causality, Uniqueness*, with the Law of momentum-energy conservation according to the Noether Theorems [102], and with the Laws of Quantum mechanics (as in STR)! But in order to close this problem, it is necessary to abandon the existing up to now positivist approach to General Relativity and theoretically to separate the real and observed pictures of the Universe. Such dualism of BMT approach may be used in explicit description of relativistic motions in space-time under the field of gravitation: firstly, as real ones in $4D$ Minkowskian space-time, and, secondly, as observable ones in $\langle\mathcal{R}^{3+1}\rangle$, or even in the $10D$ space-time $\langle\mathcal{P}^{9+1}\rangle$ (see above), with the use of the Tensor Trigonometry. (See more about the last idea in [109], [15], [107]).

In passing, we note that BMT leads to the affine topology of the Nature space-time with properties of *endlessness and infinity.* Ones argue so: an infinite space-like part of this 4D world must have due to the H. Olbers' paradox (1826), the light night sky, contrary to the finite world of radius R. But the mathematical infinity of $\langle\mathcal{P}^{3+1}\rangle$ does not mean the infinity of world's matter. It may be limited. How apologists of the finite space-time can place in it the endless time-arrow without violating the determinism? According to H. Poincaré, this time-arrow is imaginary, which revealed by him a pseudo-Euclidean nature of our space-time.

A dual opinion on the "Black Holes" in Big Cosmos from points of view of descriptions in $\langle\mathcal{P}^{3+1}\rangle$ and in observed $\langle\mathcal{R}^{3+1}\rangle$ deserves a brief explanation. So, these objects were predicted in 1783 by John Michell on the basis of Newtonian Theories and later they have considered in details by the great Laplace [81]. The smaller and very larger "Black Holes" can be formed accordingly by some enough massive tight Star and in the center of some very massive Galaxy, including our Milky Way. Such "Black Holes" are surrounded by their theoretical horizon of events. And what is happening beyond this horizon, no one knows, but purely theoretical it is possible to look there. For massive tight Stars, the Michell's radius of such "Black Hole" is equal $r = fM/c^2$, even in according to the Newtonian Theories. The so-called Schwarzschild radius for the Einsteinian "Black Hole" is twice more, i. e., as $r = 2fM/c^2$ [100]. This dual opinion is explained by the equivalent influence of *accelerational and gravitational hyperbolic cosines* from (209A) and (210A), as we noted above for similar doubling the relativistic Mercury perihelion shifts and consider further in last Ch. 10A.

The Hubble Law in its 1-st ancestral form $\Delta\lambda/\lambda = -\Delta\nu/\nu = -\Delta h\nu/h\nu = Hl/c = Ht$, with author's interpretation of the constant H, only connects the relative light's "red shift" and the distance l or "light time" t till a Galaxy. Later, from discovery in 1929, it was used for confirmation of the Theory of Expanding Universe by Alexander Friedmann *(and later by others with acceleration?!)*. But this Law have another logical interpretation of the eminent astrophysics F. Zwicky in 1929 (introduced concept of "black matter"). So, this "red shift" may express the lack of the photons energy proportionally to their long way from a Galaxy to the Earth due to permanent loss of their energy (like a certain cosmic "friction"). Then the photons lose energy with decreasing frequency and increasing wave length at $c =$ const. And as a result of such interpretation, a need in the so called *dark energy* to justify the hypothesis of the Universe expansion *with its acceleration* is absent. At analyzing of this red shift, the light coming from the galactic cloud from billions of Stars, as something average with a uniform scale of local time, should not be considered, of course, exactly as a beam of light from the Sun or other single Stars. Though, for the book author, a *pulsating model* of the Universe (expansion-contraction) is more preferable, since in it matter does not disappear anywhere and does not come from anywhere, under its conservation. The strange courage is striking when some hypotheses relating even to the Universe and its hyper-remote objects are easily turned by their apologists into the final theories that are not subject to doubt!

A priori a certain geometry of the real space-time in the large was not here discussed. For our opinion, the complete knowledge of its global structure, in principle, cannot be achieved. Illusions of complete knowledges in Mathematics were broken by the Gödel's Theorems. But in Theoretical Physics, the idea about transcendent nature of all the Universe is not yet understood. Moreover, in our time, any physicist-relativist must decide on the main dilemma: either to accept again the great Principle of Relativity by Galileo – Poincaré, formulated fully at the beginning of the 20th century, compatible with the new Higgs theory of matter inertia (even with the Rosen's BMT under two metric tensors), or to continue to stubbornly adhere to General Principle of Relativity by Einstein (1916), incompatible with the Higgs theory, as well as with the Quantum Mechanics; and what is even worse: to continue to impose the latter in new scientific publications and in the educational process. However a concept of the entire Universe curved by the global gravity was not confirmed by numerous long time astronomical observations, beside of curving the light rays propagation.

Since the 1-st edition of our monograph in 2004 [15], in final Chapter 10A, we apply widely the Poincaré–Minkowski space-time (but now combined with the Higgs field), using our tensor trigonometric approach, added by its differential and integral parts, for analysis of motions along any world lines and regular curves in pseudo- and quasi-Euclidean spaces.

As a result, it is possible to adopt reasonably the following important inferences.

If we consider various relativistic motions exclusively locally as if in the real physical space-time including a gravitational field, but with the real Minkowski or complex Poincaré space-time, where $c = dx^{(k)}/dt^{(k)} =$ const, then it is possible, with fairly high degree of accuracy (as was shown above), to study and describe these motions with their kinematic and dynamic characteristics at a local level directly in such basis space-time without distortions.

Thus, relativistic motions in $4D$ Minkowski space-time $\langle \mathcal{P}^{3+1} \rangle$ have the four absolute geometric and physical parameters with relative ones in the $3D$ Euclidean subspace $\langle \mathcal{E}^3 \rangle$ and scalar projection onto the time-arrow $\overrightarrow{ct}$. Absolute motion is mapped by a world line in $\langle \mathcal{P}^{3+1} \rangle$ in pseudo-Cartesian coordinates with admitted values of its slope to the time-arrow. A world line has important feature as its *dynamical* character with 4-velocity $\mathbf{c}$ of Poincaré. This enable us to determine all absolute and relative geometric and physical parameters of the motion along it of a body or a particle. After full confirmation of the Higgs Theory with the Mach Principle in 1964–2012, the Tensor Trigonometry became simplest, clear, well-understanding and all-around mathematical instrument for homogeneous and isotropic spaces, perfect hypersurfaces with non-Euclidean geometries, and the Theory of Relativity!

"Poincaré, genie égal à Gauss, et aussi universel."
– Jean Dieudonné, fondateur du groupe Bourbaki

Chapter 10A

Differential tensor trigonometry of world lines and curves

According to Hermann Minkowski [65], each material point M, including barycenter of a body, is permanently absolutely moving along its world line in the homogeneous and isotropic space-time $\langle \mathcal{P}^{3+1} \rangle$ at $n = 3, q = 1$ as realificated isomorphism of the original complex-valued Poincaré space-time $\langle \mathcal{Q}^{3+1} \rangle_c$. We may analyze a curved world line with an increase in its complexity from $n = 1$ till $n = 3$ $(q = 1)$ for rectilinear, flat and spatial relativistic movements. The world line is a *geometric invariant* of Lorentzian continuous transformations of the pseudo-Cartesian bases, and it is a regular curve with local 4×1 radius-vector $\mathbf{r}(c\tau)$. The inexorable absolute motion, limited by the slope of a world line to the time-arrow below of the light line, ensures its regularity. Physically its trajectory is a locally oriented proper time-arrow $\overrightarrow{c\tau}$ of object or particle M. The scalar integral value of proper time along a world line does not depend on a pseudo-Cartesian base too. By their slope $d\mathbf{r}$ – Figure 2A, the world lines relate only to the internal cavity of the light cone. For descriptivety and visuality, we analyze world lines with pseudo-Cartesian bases $\tilde{E}_1 = \langle \mathbf{x}, \overrightarrow{ct} \rangle$ and $\tilde{E}_m = \langle \mathbf{x}^{(m)}, \overrightarrow{ct} \rangle$. In $\tilde{E}_1$, their inclination corresponds, due to specific tangent–tangent analogy, to the visual spherical angle φ_R : $\tanh \gamma \equiv \tan \varphi_R$. In a neighborhood of its point M, the world line with its orientation and configuration is completely determined by four absolute scalar and relative 4-vector differential-geometric parameters in $\langle \mathcal{P}^{3+1} \rangle$. The scalar parameters are invariants under continuous Lorentzian transformations. Such construction gives us opportunity for using Frenet–Serret approach to the differential theory of regular curves in the 3D Euclidean space [21], when they are supposed to be embedded namely *in the homogeneous and isotropic space of its fixed dimension for unique results.*

All angular parameters of motion along a world line – hyperbolic angle γ of motion with its direction $\mathbf{e}_\alpha$ are defined initially through the radius-vector of a world point on it:

$$\left.\begin{array}{l}
\mathbf{r}^{(1)}(c\tau) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ ct \end{bmatrix} = \begin{bmatrix} \mathbf{x}(c\tau) \\ ct(c\tau) \end{bmatrix}, \;\; \mathbf{i}_\alpha = \mathbf{i}_\alpha(c\tau) = \mathbf{i}(\gamma, \mathbf{e}_\alpha) = \dfrac{d\mathbf{r}}{d(c\tau)} = \begin{bmatrix} \sinh\gamma \cdot \mathbf{e}_\alpha \\ \cosh\gamma) \end{bmatrix}; \\
\mathbf{sinh}\,\gamma = \dfrac{d\mathbf{x}}{d(c\tau)} = \sinh\gamma \cdot \mathbf{e}_\alpha = \dfrac{\mathbf{v}^*}{c}, \;\; \mathbf{tanh}\,\gamma = \dfrac{d\mathbf{x}}{d(ct)} = \tanh\gamma \cdot \mathbf{e}_\alpha = \dfrac{\mathbf{v}}{c} = \mathbf{sinh}\,\gamma / \cosh\gamma; \\
\gamma = \operatorname{arsinh} \dfrac{\sqrt{dx_1^2 + dx_2^2 + dx_3^2}}{d(c\tau)} = \operatorname{artanh} \dfrac{\sqrt{dx_1^2 + dx_2^2 + dx_3^2}}{d(ct)} > 0, \text{ as } d(ct) > 0; \\
\mathbf{e}_\alpha = \{\cos\alpha_k\}, \; k = 1,2,3; \;\; \cos\alpha_k = \dfrac{dx_k}{||d\mathbf{x}||}; \;\; \eta_\gamma^* = \dfrac{d\gamma}{d\tau}, \; w_\alpha^* = \dfrac{d\alpha}{d\tau} = \text{–angular velocities.}
\end{array}\right\}$$

In particular, the so-called *uniform* absolute motions $\mathbf{r} = \mathbf{r}(c\tau)$ are of especial interest. Among them, the physically most important are the following three types:
the uniform rectilinear movement at $\gamma = \text{const}$, $\mathbf{e}_\alpha = \mathbf{const}$ (Chs. 1A–4A);
the uniformly accelerated rectilinear movement at $\eta_\gamma^* = \text{const}$, $\mathbf{e}_\alpha = \mathbf{const}$ (Ch. 5A);
the circular movement with velocities $v^* = \text{const}$ and $w_\alpha^* = \text{const}$ at $\gamma = \text{const}$.

In $\langle \mathcal{P}^{3+1} \rangle$ with tensor $\{I^\pm\}$ (17A), we introduced in Ch. 5A measureless trigonometric 4×4 tensor of motion $\mathit{roth}\ \Gamma^{(m)} = F(\gamma, \mathbf{e}_\alpha)$ (100A), as the pseudoorthogonal bivalent symmetric tensor, on the basis of rotations (348) and (362), (363) for applications in Theory of Relativity. It determines along a world line the current local base $\tilde{E}_m^{(4)} = \mathit{roth}\ \Gamma^{(m)} \cdot \tilde{E}_1$, and its local hyperbolic inclination Γ with the local Euclidean orientation $\mathbf{e}_\alpha$ in $\tilde{E}_1$.

This tensor is defined at the current point M in the base $\tilde{E}_1 = \{I\}$ by canonical structures (362) or (363). The change $d\Gamma$ causes locally the change of hyperbolic inclination as arc of the hyperbolic rotation $d\gamma$ and the change of spherical orientation $\mathbf{e}_\alpha$ as possible arcs of the orthospherical rotations $d\mathbf{e}_\alpha = \mathbf{d}\alpha_{1,2,3}$ for a curve in $\langle\mathcal{P}^{3+1}\rangle$ with $\langle\mathcal{E}^3\rangle_{(m)}$. Hence any world line can have at its point M maximum four *intrinsic vector-parameters* of orders up to 4, completely defined its local configuration in $\langle\mathcal{P}^{3+1}\rangle$. The pseudo-Euclidean integral length of a world line $\overrightarrow{c\tau}$ is counted conventionally from the initial point O with its differential $d\mathbf{r}$. It is an internal argument for a world line. In the theory of relativity, speed of an absolute motion of a material point M along a world line is defined as the time-like 4×1-*velocity* introduced in first by Henri Poincaré in 1905 with his homogeneous 4 space coordinates:

$$\left.\begin{aligned} &\mathbf{c}(c\tau) = c\cdot\frac{d\mathbf{r}}{d(c\tau)} = \frac{d\mathbf{r}}{d\tau} = \frac{d\overrightarrow{c\tau}}{d\tau} = c\cdot\mathbf{i}(c\tau) = c\cdot\mathbf{i}_\alpha, \\ &\mathbf{c}'(c\tau)\cdot I^{\pm}\cdot\mathbf{c}(c\tau) = ||\mathbf{c}(c\tau)||^2_P = -c^2 = \text{const}. \end{aligned}\right\}\quad (\ \vec{\mathbf{c}} = c\cdot\mathbf{i}_\alpha\ ) \qquad (218A)$$

It may be also represented in $\langle\mathcal{P}^{3+1}\rangle$ as the 4×1 radius-vector $R = i\mathbf{c}$ of the hyperboloid II. Its pseudomodule "c" is the constant normalizing scale multiplier to time-axis, introduced by H. Poincaré in 1905 [63] for isotropy, homogeneity and metric properties of $\langle\mathcal{P}^{3+1}\rangle$ (Ch. 1A). Other proper parameters, in term of proper time τ along a world line, mean the following:
$\mathbf{r}(c\tau)$ is a 4×1-radius-vector of the point M of a world-line in the base $\tilde{E}_1 = \{I\}$,
$\mathbf{i}(c\tau) = \mathbf{i}_\alpha$ is a unity 4-vector along proper time arrow $\overrightarrow{c\tau}$ which may be interpreted as
1) the 4-tangent to a world line as $\mathbf{r}(c\tau)$, 2) the 4-th column of tensor of motion $roth\,\Gamma^{(m)}$,
3) the time-like 4×1 radius-vector (146A) of the unity hyperboloid II.

Since in homogeneous coordinates of Poincaré, with the scaling coefficient c for times $t^{(k)}$, the light ray is expressed in bases $\tilde{E}_k$ as $\Delta x^{(k)} = \Delta[ct^{(k)}]$, then the consequence immediately follows: $\Delta x^{(k)}/\Delta t^{(k)} = c = \text{const}$. (Hence, it is excess Einsteinian STR Principle of equality of light speed "c" in all $\tilde{E}_k$ [67].) Though the constancy of c, as result of measurements on the Earth and in near cosmos, is merely a hypothesis, which cannot be inferred and spread into the whole Universe and onto the global world time. Perhaps, it is more important than the answer to the still debatable question: "Is it necessary bending space-time or not?"

Coordinate 3-velocity $\mathbf{v}$ is a *tangent cross projection* (Ch. 4A) of the 4-velocity $\mathbf{c}$ into $\langle\mathcal{E}^3\rangle$. Its *sine projection* is a *proper* 3-velocity $\mathbf{v}^*$. (Both velocities have Euclidean direction $\mathbf{e}_\alpha$.) Its *cosine projection* onto $\overrightarrow{ct}$ is a *scalar supervelocity of the time t stream* $c^* = \cosh\gamma\cdot c$ for given angle γ of motion in the base $\tilde{E}_1 = \{I\}$. The 4-velocity $\mathbf{c}$ of a particle or a body can be changed only in its spatial directions: hyperbolic γ with respect to the time-arrow and/or spherical $\mathbf{e}_\alpha$ with respect to the Euclidean subspace. This takes place whenever *inner force* $\overrightarrow{\mathbf{F}}$ acts on them. For any material objects (an electron, a down, a star, and so one) independently on their mass the pseudomodule of 4-velocity of their *absolute motion* in $\langle\mathcal{P}^{3+1}\rangle$ is the constant c. All these arguments are summarized in the following assertion. *Any material body is permanently absolutely moving in the Minkowski space-time $\langle\mathcal{P}^{3+1}\rangle$ along own world line as its current time-arrow $\overrightarrow{c\tau}$ with the motion 4-pseudovelocity $\mathbf{c} = c\cdot\mathbf{i}$ having the constant c and the directional pseudounity 4-vector $\mathbf{i}$, which is constant only for uniform rectilinear physical movement of the body iff no any inner force is applied to it.*

In philosophy, such an assertion means the so called *perpetual matter movement.*

The Postulate is based on the original notions introduced by Poincaré and Minkowski as 4-velocity $\mathbf{c}$ and a world line in space-time as a trajectory of the absolute motion of the body M. With it we connect main dynamic physical characteristics: the own 4-momentum $\mathbf{P}_0 = m_0\vec{\mathbf{c}}$, the real momentum $\mathbf{p} = m\mathbf{v} = m_0\mathbf{v}^*$ and the total momentum $P = mc$. See them in Chs. 5A and 7A, where they were connected by the pseudo-Euclidean Absolute Pythagorean Theorem in $\langle\mathcal{P}^{3+1}\rangle$. All measured physical values relate to their projections from a world line onto $\overrightarrow{ct}$ and into $\langle\mathcal{E}^3\rangle$. They are changed iff the direction of $\mathbf{i}$ is changed!

The General Postulate by Poincaré–Minkowski gives us to do the important infers.

1. It allows to consider world lines not only geometrically, but and physically as the time nature world trajectories with absolute local kinematic and dynamic characteristics of the body M in the metric space-time $\langle \mathcal{P}^{3+1} \rangle$, and evaluate additionally its relative independent geometric and physical characteristics of orders till 4 in a certain pseudo-Cartesian base $\tilde{E}_k$.

2. It gives simple explanation to a nature of the permanent matter movement as stream of proper time $c\tau$ along a world line, and vice versa! They both move with 4-pseudovelocity $\mathbf{c}$.

3. It mathematically explains either hyperbolic, or orthospherical (under hyperbolic sine and cosine slopes) partial distortions of a world line in $\langle \mathcal{P}^{3+1} \rangle$ under physical factors acting on a particle or barycenter of a body. Indeed, due to constant pseudomodule of $\mathbf{c}$, its vector derivative along a world line is permanently pseudo-Euclidean orthogonal to $\mathbf{c}$ (or $\mathbf{i}$):

$$\mathbf{c}'(c\tau) \cdot I^{\pm} \cdot \mathbf{c}(c\tau) = -c^2 = \text{const} \ \Rightarrow \ \mathbf{c}'(c\tau) \cdot I^{\pm} \cdot \left[c \cdot \frac{d\mathbf{c}(c\tau)}{d(c\tau)} \right] = \mathbf{c}'(\tau) \cdot I^{\pm} \cdot \left[\frac{d\mathbf{c}(\tau)}{d\tau} \right] =$$

$$= \mathbf{c}'(\tau) \cdot I^{\pm} \cdot \mathbf{g}(\tau) = \mathbf{c}'(c\tau) \cdot I^{\pm} \cdot \mathbf{g}(c\tau) = c \cdot \mathbf{i}'(c\tau) \cdot I^{\pm} \cdot \mathbf{p}(c\tau) \cdot g = 0. \tag{219A}$$

We obtain zero scalar product of the time-like 4-vector $\mathbf{c}$ with its space-like 4-vector-derivative $\mathbf{g}$, though such pseudoorthogonality holds with new other 4-vector-derivatives of higher orders up to 4 in $\langle \mathcal{P}^{3+1} \rangle$. In result of successive orthogonal differentiation of unity vectors along a world line, we should obtain all four unity vectors (of its curvatures and proportional accelerations with scalar parameters) orthogonal to each other. Similar idea was realized in the Frenet-Serret theory of regular curves in Euclidean space $\langle \mathcal{E}^3 \rangle$ [14, 21].

We also note that the pseudoorthogonal characteristics, *as 4-vectors* $\mathbf{c}$ and $\mathbf{g}$, differ here from the orthogonal Euclidean 3-projections in that, they contain non-zero fourth scalar time projections. Before in Chs. 5A, 7A, 8A we dealt with similar absolute notions, but they were *by pure Euclidean 3-vectors*, i. e., they were expressed in the instantaneous base $\tilde{E}_m$ under zero value of fourth time projections – see (97A), (145A) (161A), (198A). Now we mean them as more general absolute concepts in full form *as 4-vectors* with their scalar modulus characteristics too, for example, in the original base $\tilde{E}_1$. The concepts which include only spatial or only temporal components, provided that both of them are non-zero, are considered as relative ones. For example, the theorems expressed by formulae (145A), (198A) were absolute, but relations of type (135A), (163A), (192A) give the relative theorems.

Continuing (219A), in the neighborhood of a point M along a world line, in result of free 1-st pseudoorthogonal differentiation of the tangent $\mathbf{i}(c\tau)$ in $c\tau$ not only within the *osculating pseudoplane* to a curve, we get *total* scalar and 4-vector characteristics of the 2-nd order as 4-pseudocurvature $\mathbf{k}$ (with radius $R_K = 1/\mathcal{K}$) and inner 4-acceleration $g = c^2 \cdot K$, introduced by us in (79A) and (161A) *as 3-vector*, with their common unity pseudo-Euclidean vector of the instantaneous pseudonormal $\mathbf{p}_\beta$ and common internal Euclidean direction $\mathbf{e}_\beta$:

$$\mathcal{K}_\beta(c\tau) = 1/R_K^{(m)} = g_\beta(c\tau)/c^2, \tag{220A}$$

$$\mathbf{k}_\beta(c\tau) = \mathcal{K}_\beta(c\tau) \cdot \mathbf{p}_\beta(c\tau) = [g_\beta(c\tau)/c^2] \cdot \mathbf{p}_\beta(c\tau) = \mathbf{g}_\beta(c\tau)/c^2. \tag{221A}$$

Define the *order of embedding* ζ of a world line as the least dimension $\zeta = k+1$ of the pseudo-Euclidean subspace $\langle \mathcal{P}^{k+1} \rangle$ of the space-time $\langle \mathcal{P}^{3+1} \rangle$ containing the whole curve. All the possible values of this order are $\zeta \in \{1, 2, 3, 4\}$ at $k = 0, 1, 2, 3$. So, if $\zeta = 1$ $(k = 0)$, then this enveloping subspace is the straight time-arrow $\overrightarrow{ct}$ as itself. This is a relatively immovable voyage in time along a straight world line with the same pseudovelocity c. A flat world line has $\zeta = 2$ $(k = 1)$. This corresponds to accelerated rectilinear movement. A twisted world line has order ζ as 3 or 4 corresponding to order of the line curvature 2 or 3. The order $k = \zeta - 1$ is the minimal dimension of the Euclidean subspace $\langle \mathcal{E}^k \rangle$, where a trajectory of physical movement is represented as Euclidean orthoprojection of absolute motion in $\langle \mathcal{P}^{k+1} \rangle$.

Unity principal tangent $\mathbf{i}_\alpha(c\tau)$ to a world line (Figure 2A(3)) is the primary vector characteristic of a curve $\mathbf{r}(c\tau)$ – see in Ch. beginning. It is produced in the pseudo-Cartesian base $\tilde{E}_1$ by the unambiguous hyperbolically orthogonal differentiation (218A) of radius-vector $\mathbf{r}(c\tau)$ in $d\gamma$ exactly along a world line in the space-time $\langle\mathcal{P}^{3+1}\rangle$ (as a regular curve):

$$\left\{\frac{d\mathbf{r}(c\tau)}{d\gamma}\right\}_\alpha = \mathbf{i}_\alpha(c\tau) = \begin{bmatrix} \sinh\gamma_i \cdot \mathbf{e}_\alpha \\ \cosh\gamma_i \end{bmatrix} = roth\ \Gamma_i \cdot \mathbf{i}_1 = roth\Gamma_i \cdot \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}, \qquad (222A)$$

where $\mathbf{i}_1$ is center (146A) of the unity hyperboloid II (Figure 4) and the unity 4-vector of $\overrightarrow{ct}$; $roth\ \Gamma_i = F(\gamma_i, \mathbf{e}_\alpha)$ is here tensor of motion (100A), with frame axis $\overrightarrow{ct}$, bonding $\mathbf{i}_1$ and $\mathbf{i}_\alpha$.

From here, at constant $\mathbf{e}_\alpha$, we have the 1-st differential of hyperbolic motion, considered in Ch. 5A. If to do this differentiation along a world line more free as non-collinear one, we must use in addition the lateral differential orthospherical rotation, and both motions must be in correspondence with the 1-st two-step metric normal form (132A) on a hyperboloid II. What is more, the time-like tangent $\mathbf{i}_\alpha(c\tau)$ of a such world line is simultaneously both a pseudonormal as 4×1 radius-vector (146A) of hyperboloid II and 4-vector of a time-like tangent to the locally conjugated hyperboloid I, where only one geodesic hyperbola can pass through a point M. In the following similar bonds help us till the final differentiation along a world line, when they will close all the cycle, and here's why.

Let's pre-attach to a world line with $d\gamma_i \neq 0$ at M the *concomitant movable conjugate unity hyperboloids* I and II (see at Figure 4) so, that they may be determined locally by four current pseudoorthogonal each to other unity basis vectors of a world line – tangent $\mathbf{i}_\alpha$, pseudonormal $\mathbf{p}_\alpha$ and two binormals (as the *hyperboloidal model*). Our idea is to connect for trigonometric descriptivety as one to one the 1-st metric form of a world line with the 1-st metric forms of two unity hyperboloids (132A, 133A). We'll find these metric forms with their basis unity vectors in process of sequential differentiations along a world line. This will interrupt the process of differentiation in final, as it should be in the type of theory. *In second*, we must connect this system of four pseudoorthogonal basis vectors with the existing system of four basis vectors-columns in our tensor of motion (100A) in form (362).

The principal and free-valued characteristics $\mathbf{k}_\alpha$ and $\mathbf{k}_\beta$ are produced with the 1-st differentiations in $c\tau$ along a world line with one and two degrees of freedom (at $\zeta \geq 3, k \geq 2$), logically accompanied with the concomitant hyperboloid II:

$$\left.\begin{aligned} \left\{\frac{d\mathbf{i}_\alpha(c\tau)}{d(c\tau)}\right\}_\alpha &= \mathcal{K}_\alpha(c\tau) \cdot \begin{bmatrix} \cosh\gamma_i \cdot \mathbf{e}_\alpha \\ \sinh\gamma_i \end{bmatrix} = \mathcal{K}_\alpha(c\tau) \cdot \mathbf{p}_\alpha(c\tau) = \mathbf{k}_\alpha(c\tau) = \frac{\mathbf{g}_\alpha}{c^2}, \\ \frac{d\mathbf{i}_\alpha(c\tau)}{d(c\tau)} &= \mathcal{K}_\beta(c\tau) \cdot \begin{bmatrix} \cosh\gamma_p \cdot \mathbf{e}_\beta \\ \sinh\gamma_p \end{bmatrix} = \mathcal{K}_\beta(c\tau) \cdot \mathbf{p}_\beta(c\tau) = \mathbf{k}_\beta(c\tau) = \frac{\mathbf{g}_\beta}{c^2}. \end{aligned}\right\} \qquad (223A)$$

Unity space-like 4-vectors $\mathbf{p}_\alpha$ and $\mathbf{p}_\beta$ are principal and free pseudonormals to a world line. Derivatives in γ: $\mathbf{i}'_\alpha = \mathbf{p}_\alpha \leftrightarrow \mathbf{p}'_\alpha = \mathbf{i}_\alpha$; $\mathbf{i}_\alpha$ and $\mathbf{p}_\alpha$ at change of curve slope either converge or diverge. First expression in (223A) is the tensor trigonometric *pseudoanalog of the 1-st Frenet–Serret formula*, but second expression must reveal the binormal in the normal plane. All free vectors $\mathbf{p}_\beta$ are *pseudoorthogonal* to $\mathbf{i}_\alpha$ (222A), $\mathbf{p}_\alpha$ is pure *hyperbolically orthogonal* to $\mathbf{i}_\alpha$. We have $\cos\varepsilon = \mathbf{e}'_\beta\mathbf{e}_\alpha = \mathbf{e}'_\alpha\mathbf{e}_\beta$. From the condition of pseudoorthogonality for $\mathbf{i}_\alpha$ and $\mathbf{p}_\beta$, we obtain the connection of positive angles γ_p and γ_i:

$$\{\tanh\gamma_p = \cos\varepsilon \cdot \tanh\gamma_i \leftrightarrow \ \tanh\gamma_p = \cos\varepsilon \cdot v_i/c\} \to \gamma_p < \gamma_i (\gamma \in [0,\infty), \varepsilon \in [0;\pi]). \quad (224A)$$

If $\mathbf{e}_\beta = \mathbf{e}_\alpha$, then $\mathbf{i}_\alpha$ and $\mathbf{j}_1 = \mathbf{p}_\alpha$ determine conjugate points on the hyperboloids I and II in (146A), (149A) and at Figure 4. If $\mathbf{e}_\nu \perp \mathbf{e}_\alpha$ in $\langle\mathcal{E}^2\rangle^{(m)} \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle^{(m)} \equiv \langle\mathbf{v}, \mathbf{g}^*\rangle^{(m)}$, then $\mathbf{j}_2 = \mathbf{p}_\nu$ is a *binormal* (i. e., a pseudonormal with its minimal pure *Euclidean norm* at $\cos\varepsilon = 0, \gamma_p = 0$, see bottom point on II). *Recall also very useful decomposition* (137A):

$$\mathbf{e}_\beta = \cos\varepsilon \cdot \mathbf{e}_\alpha + \sin\varepsilon \cdot \mathbf{e}_\nu, \text{where } \ \varepsilon \in [0;\pi], \quad (\mathbf{e}'_\beta \cdot \mathbf{e}_\alpha = \cos\varepsilon, \ \ \mathbf{e}'_\beta \cdot \mathbf{e}_\nu = \sin\varepsilon, \ \ \mathbf{e}'_\alpha \cdot \mathbf{e}_\nu = 0).$$

Proportional space-like 4-vectors $\mathbf{k}_\beta$ and $\mathbf{g}_\beta = c^2 \cdot \mathbf{k}_\beta$ in (233A) are directed inside region of concavity of a world line arc $d^2\mathbf{r}^{(m)}$ out center O of the osculating hyperbola – see at Figure 2A(3): $\cos\varepsilon > 0$ for accelerations ($g_\beta > 0$), $\cos\varepsilon < 0$ for decelerations ($g_\beta < 0$). If $\cos\varepsilon = \pm 1$, then the Euclidean projection of $\mathbf{g}$ is parallel to $\mathbf{v}$ (movement is rectilinear). If $\cos\varepsilon = 0$, then the Euclidean projection of $\mathbf{g}$ gives no increment to $||\mathbf{v}||$ and leads to world line bend towards $\mathbf{e}_\nu$, i. e., Euclidean orthogonally to the curve (movement is centripetal).

Further, for beginning, we consider in particular the instantaneous space-like geometric and physical characteristics from (223A) with their decompositions into pair of orthogonal projections along a world line in the space-time $\langle\mathcal{P}^{3+1}\rangle$, expressed in the base $\tilde{E}_1 = \{I\}$ and in the current base $\tilde{E}_m = roth\,\Gamma_i^{(m)} \cdot \tilde{E}_1 = \{roth\,\Gamma_i^{(m)}\} = \{F(\gamma_i, \mathbf{e}_\alpha)\}$. We'll do in two stages these orthogonal decompositions: at the 1-st stage, of relative Euclidean items on the relative Euclidean *sine normal plane of curvature* given by 3-vectors as $\langle\mathcal{E}^2\rangle_{Ns}^{(m)} \equiv \langle\mathbf{e}_\alpha^{(\mathbf{m})}, \mathbf{e}_\nu^{(1)}\rangle$; and, at the 2-nd stage, of the intrinsic characteristics on the *real Euclidean sine normal plane*, given here by 4-vectors as $\langle\mathcal{E}^2\rangle_{Ns\,II}^{(m)} \equiv \langle\mathbf{p}_\alpha^{(m)}, \mathbf{b}_\nu^{(1)}\rangle$ in the *first partial 3D space-time* $\langle\mathcal{P}^{2+1}\rangle_{II} \equiv \{\langle\mathcal{E}^2\rangle_{Ns}^{(m)} \boxtimes \overrightarrow{ct}\} \equiv \{\langle\mathcal{E}^2\rangle_{Ns}^{(m)} \boxtimes \overrightarrow{y^{(m)}}\}$ (at $\zeta = 3$), where $\mathbf{p}_\alpha$ is a unity 4-vector of the *principal pseudonormal* with $\mathbf{e}_\alpha$, $\mathbf{b}_\nu$ is a unity 4-vector of the *sine binormal* with $\mathbf{e}_\nu$. The *total pseudocurvature* $\mathbf{k}_\beta$ in (223A) is also decomposed into tangential and normal ones.

In the Minkowski space-time $\langle\mathcal{P}^{3+1}\rangle$ with metric tensor $\{I^\pm\}$ (17A) (or in the isomorphic to it Poincaré complex space-time) at $\zeta \geq 3$, due to (223A) with the use of (137A), we execute the first two-steps differentiation along a world line with orthogonal decomposition of the 4-vector of a free pseudocurvature $\mathbf{k}_\beta$ and revealing all relative and absolute characteristics:

$$\mathbf{k}_\beta(c\tau) = \frac{d\mathbf{i}_\alpha(c\tau)}{d(c\tau)} = \frac{d\gamma_p}{d(c\tau)} \cdot \begin{bmatrix} \cosh\gamma_p \cdot \mathbf{e}_\beta \\ \sinh\gamma_p \end{bmatrix} = \frac{d\gamma_p}{d(c\tau)} \cdot \mathbf{p}_\beta(c\tau) = \mathcal{K}_\beta(c\tau) \cdot \mathbf{p}_\beta(c\tau) \equiv \qquad (225A-I)$$

$$\equiv \frac{d\gamma_i}{d(c\tau)} \cdot \begin{bmatrix} \cosh\gamma_i \cdot \mathbf{e}_\alpha \\ \sinh\gamma_i \end{bmatrix}_\alpha + \begin{bmatrix} \sinh\gamma_i \cdot \frac{d\mathbf{e}_\alpha}{d(c\tau)} \\ 0 \end{bmatrix}_\gamma^{(1)} = \frac{d\gamma_i}{d(c\tau)} \cdot \begin{bmatrix} \cosh\gamma_i \cdot \mathbf{e}_\alpha \\ \sinh\gamma_i \end{bmatrix}_\alpha + \begin{bmatrix} \sinh\gamma_i \cdot \frac{d\alpha_1}{d(c\tau)} \cdot \mathbf{e}_\nu \\ 0 \end{bmatrix}_\gamma^{(1)} =$$

$$= \mathcal{K}_\alpha(c\tau) \cdot \begin{bmatrix} \cosh\gamma_i \cdot \mathbf{e}_\alpha \\ \sinh\gamma_i \end{bmatrix}_\alpha + \mathcal{K}_\nu(c\tau) \cdot \begin{bmatrix} \mathbf{e}_\nu \\ 0 \end{bmatrix}_\gamma^{(1)} = \mathcal{K}_\alpha(c\tau) \cdot \mathbf{p}_\alpha(c\tau) + \mathcal{K}_\nu(c\tau) \cdot \mathbf{b}_\nu(c\tau) \equiv$$

$$\equiv \frac{d\gamma_p}{d(c\tau)} \cdot \begin{bmatrix} \cosh\gamma_p \cdot \mathbf{e}_\beta \\ \sinh\gamma_p \end{bmatrix} = \frac{d\gamma_p}{d(c\tau)} \cdot \left\{ \begin{bmatrix} \cos\varepsilon \cdot \cosh\gamma_p \cdot \mathbf{e}_\alpha \\ \sinh\gamma_p \end{bmatrix} + \begin{bmatrix} \sin\varepsilon \cdot \cosh\gamma_p \cdot \mathbf{e}_\nu \\ 0 \end{bmatrix}^{(1)} \right\} =$$

$$= \mathcal{K}_\beta(c\tau) \cdot \begin{bmatrix} \cosh\gamma_p \cdot \mathbf{e}_\beta \\ \sinh\gamma_p \end{bmatrix} = \mathcal{K}_\beta(c\tau) \cdot \mathbf{p}_\beta(c\tau) = \overline{\overline{\mathcal{K}_\beta^\times}} \cdot \mathbf{p}_\alpha(c\tau) + \overset{\perp}{\mathcal{K}_\beta^\star} \cdot \mathbf{b}_\nu(c\tau) = \overline{\overline{\mathbf{k}_\beta^\times}}(c\tau) + \overset{\perp}{\mathbf{k}_\beta^\star}(c\tau).$$

Below we use intuitive understandable notations beginning from the general curvature $\mathcal{K}_\beta$:

$$\left.\begin{aligned}
&\mathcal{K}_\beta = \frac{d\gamma_p}{d(c\tau)} = \frac{g_\beta}{c^2};\;\; \mathcal{K}_\beta^\circ = \sinh\gamma_p \cdot \mathcal{K}_\beta = \mathcal{K}_\alpha^\circ = \sinh\gamma_i \cdot \mathcal{K}_\alpha;\;\; \mathcal{K}_\alpha = \frac{d\gamma_i}{d(c\tau)} = \frac{g_\alpha}{c^2};\\
&\mathcal{K}_\beta^\star = \cosh\gamma_p \cdot \mathcal{K}_\beta = \frac{g_\beta^\star}{c^2},\quad \overline{\overline{\mathcal{K}_\beta^\times}} = \sqrt{\cos^2\varepsilon \cdot \cosh^2\gamma_p - \sinh^2\gamma_p} \cdot \mathcal{K}_\beta = k_p \cdot \mathcal{K}_\beta = \mathcal{K}_\alpha;\\
&\overline{\overline{\mathcal{K}_\beta^\star}} = \cosh\gamma_p \cdot \cos\varepsilon \cdot \mathcal{K}_\beta = \cosh\gamma_p \cdot \overline{\mathcal{K}}_\beta = \cosh\gamma_i \cdot \mathcal{K}_\alpha = \mathcal{K}_\alpha^\star = \frac{\overline{\overline{g_\beta^\star}}}{c^2} = \frac{g_\alpha^\star}{c^2};\\
&\overset{\perp}{\mathcal{K}_\beta} = \frac{\overset{\perp}{g_\beta}}{c^2},\quad \overset{\perp}{\mathcal{K}_\beta^\star} = \frac{\overset{\perp}{g_\beta^\star}}{c^2} = \cosh\gamma_p \cdot \sin\varepsilon \cdot \mathcal{K}_\beta = \mathcal{K}_\nu = \sinh\gamma_i \cdot \frac{d\alpha_1}{d(c\tau)} = \frac{v_i^\star w_{\alpha_1}^\star}{c^2} = \frac{g_\nu}{c^2}.\\
&\underline{\mathcal{K}_\beta^2 = \mathcal{K}_\beta^{\star\,2} - \mathcal{K}_\beta^{\circ\,2} = \overline{\overline{\mathcal{K}_\beta^\star}}^2 + \overset{\perp}{\mathcal{K}_\beta^\star}{}^2 - \mathcal{K}_\beta^{\circ\,2} = \overline{\overline{\mathcal{K}_\beta^\times}}^2 + \overset{\perp}{\mathcal{K}_\beta^\star}{}^2 = \mathcal{K}_\alpha^2 + \mathcal{K}_\nu^2 = \mathcal{K}_R^2\,(\overline{\overline{d\gamma_p}}^2 + \overset{\perp}{d\gamma_p}{}^2).}
\end{aligned}\right\} \quad (225A-II)$$

Equaling under $I^\pm$ paired summands, we get next relations with $\varrho > \varepsilon$: $d\gamma_p^2 = \cosh^2\gamma_p\, d\gamma_p^2 - \sinh^2\gamma_p\, d\gamma_p^2 =$
$= \underline{(\cos^2\varepsilon \cdot \cosh^2\gamma_p\, d\gamma_p^2 + \sin^2\varepsilon \cdot \cosh^2\gamma_p\, d\gamma_p^2)} - \sinh^2\gamma_p\, d\gamma_p^2 = \underline{(\cosh^2\gamma_i\, d\gamma_i^2 + \sinh^2\gamma_i\, d\alpha_1^2)} - \sinh^2\gamma_i\, d\gamma_i^2 =$
$= \underline{d\gamma_i^2 + \sinh^2\gamma_i\, d\alpha_1^2} = \underline{(\cos^2\varepsilon \cdot \cosh^2\gamma_p - \sinh^2\gamma_p)\, d\gamma_p^2 + (\sin^2\varepsilon \cdot \cosh^2\gamma_p)\, d\gamma_p^2} = \underline{\cos^2\varrho\, d\gamma_p^2 + \sin^2\varrho\, d\gamma_p^2} > 0.$

Surprisingly, but we get two identical decompositions of $d\gamma_p$ – pseudo-Euclidean and Euclidean (with underline for Relative and Absolute Theorems), the latter corresponds to 1-st metric form (132A) of hyperboloid II ! *This paradox* relates to hypotenuses of right triangles only in the external cavity of isotropic cone at $n \geq 2$.

$\mathbf{p}_\alpha = \begin{bmatrix} \cosh\gamma_i \cdot \mathbf{e}_\alpha \\ \sinh\gamma_i \end{bmatrix}$ is a principal pseudonormal, as a unity vector of the principal pseudocurvature $\mathcal{K}_\alpha$, and $\mathbf{k}_\alpha = \mathcal{K}_\alpha \cdot \mathbf{p}_\alpha$ is a 4-vector of the principal (collinear) pseudocurvature; $\mathbf{b}_\nu = \begin{bmatrix} \mathbf{e}_\nu \\ 0 \end{bmatrix}$ is a space-like *sine binormal*, as the unity vector of the *normal curvature* $\mathcal{K}_\nu$, $\mathbf{k}_\nu = \mathcal{K}_\nu \cdot \mathbf{b}_\nu$, situated contrary to the angle γ_i, is a 4-vector of the sine normal curvature; $\frac{||d\mathbf{e}_\alpha||_E}{d\tau} \cdot \frac{d\mathbf{e}_\alpha}{||d\mathbf{e}_\alpha||_E} = \frac{d\alpha_1}{d\tau} \cdot \mathbf{e}_\nu = w^*_{\alpha_1} \cdot \mathbf{e}_\nu$ is a proper orthospherical angular velocity of $\mathbf{e}_\alpha$.

We use asterisk for proper items, star and circle for cosine and sine projections!

The Relative Pythagorean theorem follows from space-like part of (225A) in 3-vector and quadric scalar forms. It acts in the sine normal plane $\langle\mathcal{E}^2\rangle^{(m)}_{Ns} \equiv \langle \mathbf{e}^{(m)}_\alpha, \mathbf{e}^{(1)}_\nu\rangle$ for these three proportional characteristics as their orthoprojections into the Cartesian subbase $\tilde{E}^{(3)}_1$ at $\gamma \in [0, \infty)$, $\varepsilon \in [0; \pi]$, using (225A) with (137A) and confirming preliminary (162A), (163A):

$$\left\{\begin{array}{l} \cosh\gamma_p\, d\gamma_p \cdot \mathbf{e}_\beta = \cosh\gamma_p\, (\cos\varepsilon\, d\gamma_p \cdot \mathbf{e}_\alpha + \sin\varepsilon\, d\gamma_p \cdot \mathbf{e}_\nu) = \cosh\gamma_i\, d\gamma_i \cdot \mathbf{e}_\alpha + \sinh\gamma_i\, d\alpha_1 \cdot \mathbf{e}_\nu, \\ \cosh^2\gamma_p\, d\gamma_p^2 = \cosh^2\gamma_p\, (\cos^2\varepsilon\, d\gamma_p^2 + \sin^2\varepsilon\, d\gamma_p^2) = \cosh^2\gamma_p\, [(\overline{\overline{d\gamma_p}})^2_E + (\overset{\perp}{d\gamma_p})^2_E] = \\ = \cosh^2\gamma_i\, d\gamma_i^2 + \sinh^2\gamma_i\, d\alpha_1^2; \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} \mathcal{K}_\beta \cdot \cosh\gamma_p \cdot \mathbf{e}_\beta = \mathcal{K}^\star_\beta \cdot \mathbf{e}_\beta = \cos\varepsilon \cdot \mathcal{K}^\star_\beta \cdot \mathbf{e}_\alpha + \sin\varepsilon \cdot \mathcal{K}^\star_\beta \cdot \mathbf{e}_\nu = \overline{\overline{\mathcal{K}^\star}}_\beta \cdot \mathbf{e}_\alpha + \overset{\perp}{\mathcal{K}^\star_\beta} \cdot \mathbf{e}_\nu = \\ = \mathcal{K}_\alpha \cdot \cosh\gamma_i \cdot \mathbf{e}_\alpha + \sinh\gamma_i \cdot \frac{d\alpha}{d(c\tau)} = \mathcal{K}^\star_\alpha \cdot \mathbf{e}_\alpha + \frac{v^\star_i \cdot w^\star_{\alpha_1}}{c^2} \cdot \mathbf{e}_\nu = \mathcal{K}^\star_\alpha \cdot \mathbf{e}_\alpha + \mathcal{K}_\nu \cdot \mathbf{e}_\nu = \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} = \mathbf{k}^\star_\beta = \overline{\overline{\mathbf{k}^\star_\beta}} + \overset{\perp}{\mathbf{k}^\star_\beta} = \mathbf{k}^\star_\alpha + \mathbf{k}_\nu, \\ (\mathcal{K}^\star_\beta)^2 = (\overline{\overline{\mathcal{K}^\star_\beta}})^2 + (\overset{\perp}{\mathcal{K}^\star_\beta})^2 = (\mathcal{K}^\star_\alpha)^2 + (\mathcal{K}_\nu)^2; \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} \cosh\gamma_p \cdot \mathbf{g}_\beta = \mathbf{g}^\star_\beta = \overline{\overline{\mathbf{g}^\star_\beta}} + \overset{\perp}{\mathbf{g}^\star_\beta} = \cosh\gamma_i \cdot g_\alpha \cdot \mathbf{e}_\alpha + v^\star_i w^\star_{\alpha_1} \cdot \mathbf{e}_\nu = g^\star_\alpha \cdot \mathbf{e}_\alpha + g_\nu \cdot \mathbf{e}_\nu, \\ \cosh^2\gamma_p \cdot g_\beta^2 = {g^\star_\beta}^2 = (\overline{\overline{g^\star_\beta}})^2 + (\overset{\perp}{g^\star_\beta})^2 = \cosh^2\gamma_i \cdot g_\alpha^2 + (v^\star_i w^\star_{\alpha_1})^2 = {g^\star_\alpha}^2 + g_\nu^2. \end{array}\right\} \quad (226A)$$

$$\mathcal{K}_\beta \cdot \sinh\gamma_p = \mathcal{K}_\alpha \cdot \sinh\gamma_i \Leftrightarrow \boxed{\sinh\gamma_p\, d\gamma_p = \sinh\gamma_i\, d\gamma_i} \to d\gamma_p/d\gamma_i > 1. \quad (227A)$$

$$\Rightarrow \quad \cosh\gamma_p \cdot \cos\varepsilon\, d\gamma_p = \cosh\gamma_p\, \overline{\overline{d\gamma_p}} = \cosh\gamma_i\, d\gamma_i \Rightarrow \;\; \cos\varepsilon = 1 \leftrightarrow \gamma_p = \gamma_i, \;\cos\varepsilon = 0 \leftrightarrow \gamma_p = 0;$$

$\gamma_p/\gamma_i < 1$ – see in (224A), $\Rightarrow \;\gamma_p < \gamma_i\ (v_p < v_i),\ \gamma_i = 0 \leftrightarrow \gamma_p = 0;\ d\gamma_p > \overline{\overline{d\gamma_p}} > d\gamma_i\ \{\gamma \in [0, \infty)\}$.

From (225A)–(227A), we obtain the *Absolute Euclidean Pythagorean theorem* with the *1-st mobile trihedron* $\hat{E}^{(3)}_m = \langle \mathbf{p}_\alpha,\ \mathbf{b}_\nu,\ \mathbf{i}_\alpha\rangle$ in $\langle\mathcal{P}^{3+1}\rangle$ **under metric tensor** $I^\pm$ (17A-I)! It acts on the Euclidean sine normal plane $\langle\mathcal{E}^2\rangle^{(m)}_{Ns\ II} \equiv \langle \mathbf{p}^{(m)}_\alpha, \mathbf{b}^{(1)}_\nu\rangle$ in 3D $\langle\mathcal{P}^{2+1}\rangle_{II} \equiv \{\langle\mathcal{E}^2\rangle^{(m)}_{Ns} \boxtimes \overrightarrow{ct}\}$ $(\zeta = 3)$. In the right triangle of $\mathbf{i}_\alpha$ rotations, it corresponds to the *angular normal* 1-st metric form (132A) for the concomitant hyperboloid II (!!!), as a *perfect hypersurface* of $\langle\mathcal{P}^{3+1}\rangle$. It is expressed in the universal complete *tensor-vector-scalar ("tvs") form* with own proportional geometric and physical items:

$$\left\{\begin{array}{l} \mathbf{k}_\beta = \mathcal{K}_\beta\ \mathbf{p}_\beta = \overline{\overline{\mathcal{K}^\times_\beta}}\ \mathbf{p}_\alpha + \overset{\perp}{\mathcal{K}^\star_\beta}\ \mathbf{b}_\nu = \mathcal{K}_\alpha\ \mathbf{p}_\alpha + \mathcal{K}_\nu\ \mathbf{b}_\nu, \\ \mathcal{K}_\beta^2 = (\mathcal{K}^\star_\beta)^2 - (\mathcal{K}^\circ_\beta)^2 = (\overline{\overline{\mathcal{K}^\times_\beta}})^2 + (\overset{\perp}{\mathcal{K}^\star_\beta})^2 = \mathcal{K}_\alpha^2 + \mathcal{K}_\nu^2, \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \mathbf{g}_\beta = g_\alpha\ \mathbf{p}_\alpha + g_\nu\ \mathbf{b}_\nu, \\ g_\beta^2 = {g^\star_\beta}^2 - {g^\circ_\beta}^2 = g_\alpha^2 + g_\nu^2, \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} d\gamma_p \cdot \mathbf{p}_\beta = d\gamma_i \cdot \mathbf{p}_\alpha + \sinh\gamma_i\, d\alpha_1 \cdot \mathbf{b}_\nu, \quad (\mathbf{p}'_\alpha \cdot I^\pm \cdot \mathbf{p}_\alpha = +1, \;\; \mathbf{b}'_\nu \cdot I^\pm \cdot \mathbf{b}_\nu = +1) \\ d\gamma_p^2 = d\gamma_i^2 + \sinh^2\gamma_i\, d\alpha_1^2 = \cos^2\varrho\, d\gamma_p{}^2 + \sin^2\varrho\, d\gamma_p{}^2 = \left(\overline{\overline{d\gamma_p}}\right)^2_P + \left(\overset{\perp}{d\gamma_p}\right)^2_E > 0. \end{array}\right\} \quad (228A)$$

Here $d\gamma_p = d\lambda_R/R$, $\varrho > \varepsilon$. By this Egregium Theorem of Differential Tensor Trigonometry (1-st from two hyperbolic), we reduce this mixed motion in the initial $\tilde{E}_1$ along a world line and on II as a perfect surface to the hyperboloidal angular arc as hypotenuse $d\gamma_p$ in the final base $\tilde{E}_m$. Here $\mathbf{g}_\beta$ is a summary 4-acceleration of M, but along hypotenuse $d\gamma_p$ (at velocity $v_p = c \cdot \tanh\gamma_p$). Both are collinear due to (225A). The equation $d\cosh\gamma_p = d\cosh\gamma_i$ infers, that change of time dilation is equal to one in $\tilde{E}_1$, where real velocity v_i acts at a world line.

According to Poincaré simplest approach in 1905–1906 [63, 64] to construction of new relative and absolute dynamical characteristics in the relativistic space-time, based on the classical Newton's mechanics and STR time dilation (Chs. 5A, 7A), we get the relations for a relativistic kinematic capacity of the progressively moving body M. Indeed, the factor $\sinh\gamma_p\ d\gamma_p = \sinh\gamma_i\ d\gamma_i \to d\cosh\gamma_p = d\cosh\gamma_i$ in (227A) causes following equations: $v_p^* \cdot g_\beta = v_i^* \cdot g_\alpha \to v_p^* \cdot m_0 g_\beta = v*_i \cdot m_0 g_\alpha = v_p^* \cdot F_\beta = v_i^* \cdot F_\alpha \to N_{(s)}{}_p^* = N_{(s)}{}_i^*$ in $\tilde{E}_m$ and $\tilde{E}_1$! The values γ_p and $v_p = c \cdot \tanh\gamma_p = \cos\varepsilon \cdot v_i = \cos\varepsilon \cdot c \cdot \tanh\gamma_i$ are calculated by (224A).

Hyperboloidal model (here as top II) is useful for interpretation of relativistic kinematics. We saw this on the numerous examples before. In the given case, due to (228A), the summary velocity v_p or angle γ_p in the final base $\tilde{E}_m$ is less than values of v_i or γ_i in the initial base $\tilde{E}_1$. If v_i or γ_i is zero, then the acceleration g_α and differential $d\gamma_i$ in such an immobile base $\tilde{E}_1$ are become as internal ones with necessary zero g_ν and $d\alpha_1$. From the other hand, the summary acceleration g_β or differential $d\gamma_p$ in the base $\tilde{E}_m$ is bigger than values of g_α and $d\gamma_i$ in the base $\tilde{E}_1$. If the velocity v_p or angle γ_p is zero, then the acceleration g_β and differential $d\gamma_p$ in such an immobile $\tilde{E}_m$ are become as internal ones. In such an immobile base $\tilde{E}_m$, the summary internal acceleration g_β and differential $d\gamma_p$ are decomposed orthogonally into parallel and normal ones, with respect to the velocity v_i or vector e_α; and as $\gamma_i = 0 \leftrightarrow \gamma_p = 0$, then $\gamma_p = \gamma_i = 0$. We get the Local Absolute Euclidean Pythagorean Theorem acting in the sine normal plane $\langle\mathcal{E}^2\rangle_{Ns}^{(m)}$ and given non-completely in (145A), Ch. 7A, now from (228A) as the purely Euclidean case with such an orthogonal decomposition of *internal* $d\gamma_p$ and g_β:

$$\left\{\begin{array}{l} d\gamma_p \cdot \mathbf{e}_\beta = d\gamma_i \cdot \mathbf{e}_\alpha + \sinh\gamma_i\ d\alpha_1 \cdot \mathbf{e}_\nu, \\ \{d\lambda/R\}^2 = d\gamma_p^2 = d\gamma_i^2 + \sinh^2\gamma_i\ d\alpha_1^2 = \\ = (\cos\varepsilon\ d\gamma_p)^2 + (\sin\varepsilon\ d\gamma_p)^2 = (\overline{\overline{d\gamma_p}})_E^2 + (\overset{\perp}{d\gamma_p})_E^2, \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \mathbf{k}_\beta = \overline{\overline{\mathbf{k}_\beta}} + \overset{\perp}{\mathbf{k}}_\beta = \mathbf{k}_\alpha + \mathbf{k}_\nu, \\ \mathcal{K}_\beta^2 = \overline{\overline{\mathcal{K}}}_\beta{}^2 + \overset{\perp}{\mathcal{K}}_\beta{}^2 = (\mathcal{K}_\alpha)^2 + (\mathcal{K}_\nu)^2, \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} \mathbf{g}_\beta = \overline{\overline{\mathbf{g}_\beta}} + \overset{\perp}{\mathbf{g}}_\beta = \cosh\gamma_i \cdot g_\alpha \cdot \mathbf{e}_\alpha + v^* \cdot w_{\alpha_1}^* \cdot \mathbf{e}_\nu = g_\alpha \cdot \mathbf{e}_\alpha + g_\nu \cdot \mathbf{e}_\nu, \\ g_\beta^2 = (\overline{\overline{g_\beta}})_E^2 + (\overset{\perp}{g_\beta})_E^2 = \cosh^2\gamma_i \cdot g_\alpha^2 + (v^* \cdot w_{\alpha_1}^*)^2 = g_\alpha{}^2 + g_\nu{}^2. \end{array}\right. \qquad (229A)$$

Here we inferred in STR the spherically orthogonal decomposition of the general inner acceleration g_β into parallel g_α and normal g_ν ones! If briefly, in (229A), (145A) and (226A), formally we apply also the orthogonal decomposition of as if final here directional vector $\mathbf{e}_\beta$ in the Euclidean subspace of $\langle\mathcal{P}^{3+1}\rangle$ using our useful simple formula (137A) from Ch. 7A.

In addition, we inferred that in transformations above normal sine 3-orthoprojection $\sinh\gamma_i\ d\alpha_1 \cdot \mathbf{b}_\nu$ does not change, since $\mathbf{e}_\nu$ is perpendicular to the direction of motion $\mathbf{e}_\alpha$. We again state the fulfillment of the Herglotz Principle [84] – see it in Chs. 2A and 4A. That is why, for normal projections in $\tilde{E}_1$ we did not use special asterisk as for parallel ones.

Geometrically (228A) corresponds to rotation of tangent $\mathbf{i}_\alpha$ with two degrees of freedom: at complete arc $d\gamma_i$ and at as if cutting arc $d\alpha_1$. Indeed, above we have only its space-like sine projection into $\langle\mathcal{E}^2\rangle$. Although complete $d\alpha_1$ with its cosine and sine projections in $\overrightarrow{ct}$ and $\langle\mathcal{E}^2\rangle$ is a time-like vector sum at the time-like unity normal time-arrow $\mathbf{i}_\nu$. (Also under metric tensor $I^\pm$.) Such cutting is caused by mixing its time-like projection $\cosh\gamma_i\ d\alpha_1$ with the time-like projection $\cosh\gamma_i\ d\gamma_i$ in $d\gamma_i \cdot \mathbf{p}_\alpha$ in (228A). However, at $\gamma_i = \text{const}$, similar mixing is absent, and we can execute as alternative to (225A) two-steps differentiation along a world line with orthogonal decomposition of the complete 4-vector $d\alpha_1 \cdot \mathbf{i}_\nu$ into its sine and cosine orthoprojections. Obviously, such two-steps time-like orthospherical motion must have own trihedron in $\langle\mathcal{P}^{2+1}\rangle$, but (as we shall see) in the central zone of the concomitant hyperboloid I, with the own Absolute pseudo-Euclidean Pythagorean theorem. We'll implement this scenario later for correct construction by Tensor Trigonometry of the time-like pseudoscrewed world line as the 2-nd type of uniformly accelerated motion (in addition to time-like uniform hyperbolic motion in Ch. 5A). In the Frenet-Serret theory of regular curves in $\langle\mathcal{E}^3\rangle$, the peculiarity with mixing in the trihedron by Frenet of the tangent to a curve and its torsion (in TT it is the *orthoprocession* along $\overrightarrow{ct}$) is hushed up by authors of text books in Differential Geometry. Our Tensor Trigonometry in the pseudo- and quasi-Euclidean spaces with frame axis $\overrightarrow{y}$ revealed and eliminated such peculiarity!

According to both theorems (226) and (228A), the geometric meaning of the hyperbolic differential arc $d\gamma$ in normal relation (171A) has become completely clear (with $d\gamma = d\gamma_p$). Physically, it is proportional to the normal acceleration in the Thomas precession. With identical expressions, extracted from 1-st relations in (226A), we give both normal relations, when $\gamma_p = 0$ and when $\gamma_p \neq 0$, as rigorously inferred in the original Euclidean subspace $\langle\mathcal{E}^3\rangle$, produced now by the *tensor differential trigonometry* in $\langle\mathcal{P}^{3+1}\rangle$ under the same tensor $I^{\pm}$:

$$\begin{cases} \cosh\gamma_p \cdot \sin\varepsilon\; d\gamma_p = \overset{\perp}{d\gamma_p} = \sinh\gamma_i\; d\alpha_1 \Rightarrow \cosh\gamma_p \cdot \sin\varepsilon \cdot g_\beta = \overset{\perp}{g_\beta} = v_i^* \cdot w_\alpha^* \; (\gamma_p \neq 0), \\ \sin\varepsilon\; d\gamma_p = \overset{\perp}{d\gamma_p} = \sinh\gamma_i\; d\alpha_1 \Rightarrow \sin\varepsilon \cdot g_\beta = \overset{\perp}{g_\beta} = v_i^* \cdot w_\alpha^* \; (\gamma_p = 0). \end{cases} \quad (230A)$$

Thanks to this normal relation, we may add to the $2D$ Euclidean normal motion in (226A) and (228A) the angular shift $-d\theta$ from (172A) with the Thomas precession in time $-w_\theta$ around the 3-rd normal axis $\mathbf{e}_\mu \equiv \mathbf{e}_\alpha \times \mathbf{e}_\nu$, *with expansion of whole description of differential motions (225A) on the 3D hyperboloid II in complete* $\langle\mathcal{P}^{3+1}\rangle$, *because these additional shift and precession are connected with the space-like cosine projection* $\cosh\gamma_i\; d\alpha_1 \cdot \mathbf{e}_\mu$ *in (173A)!* It is the difference between real arc $d\alpha_1$ and its space-like cosine projection causes the angular defect by Lambert with the physically detected precession by Thomas – see in detail in the end of Ch. 7A and further after (238A). In the beginning of Ch. 7A, we revealed this induced precession in matrix form, but for two-steps non-differential hyperbolic motions.

* * *

In (228A) 3 vectors $\mathbf{p}_\alpha(c\tau)$, $\mathbf{b}_\nu(c\tau)$, $\mathbf{i}_\alpha(c\tau)$ form the right mobile base or the *first 3D trihedron* $\hat{E}_m^{(3)} = \langle\mathbf{p}_\alpha(c\tau),\ \mathbf{b}_\nu(c\tau),\ \mathbf{i}_\alpha(c\tau)\rangle$ in $\langle\mathcal{P}^{2+1}\rangle_{II}$. Generally, in $\langle\mathcal{P}^{3+1}\rangle$, it must be subbase of the *cardinal pseudo-Cartesian base* $\hat{E}_m^{(4)} = \langle\mathbf{j}_1(c\tau),\ \mathbf{j}_2(c\tau),\ \mathbf{j}_3(c\tau),\ \mathbf{i}(c\tau)\rangle$. Differentiating anyone of *orthogonal unity vectors*, for example, $\mathbf{a}_1$ along a world line is reduced to its orthogonal rotation around second vector $\mathbf{a}_2$ with third vector $\mathbf{a}_3$ in a pseudoplane or a plane formed by $\mathbf{a}_1$ and $\mathbf{a}_3$. Then fourth rested basis unity vector $\mathbf{a}_4$ outside this trihedron in $\langle\mathcal{P}^{3+1}\rangle$ must be immobile! A result of this rotation is $\mathbf{a}_3$. This result is equivalent to the result of vector product $\mathbf{a}_1 \times \mathbf{a}_2 = \pm\mathbf{a}_3$ with its right sign. Below, for illustration of this approach, we give these complete Tables with signs for such vector products for two trihedrons in $\langle\mathcal{P}^{3+1}\rangle$ with a frame axis $\mathbf{i}_\alpha$ and $\langle\mathcal{Q}^{2+1}\rangle$ with a frame axis $\mathbf{j}_3$.

for Minkowski space-time $\langle\mathcal{P}^{3+1}\rangle$

	$\mathbf{p}_\alpha$	$\mathbf{b}_\nu$	$\mathbf{i}_\alpha$
$\mathbf{p}_\alpha$	0	$+\mathbf{i}_\alpha$	$+\mathbf{b}_\nu$
$\mathbf{b}_\nu$	$-\mathbf{i}_\alpha$	0	$+\mathbf{p}_\alpha$
$\mathbf{i}_\alpha$	$-\mathbf{b}_\nu$	$-\mathbf{p}_\alpha$	0

1-st trihedron $\hat{E}_m^{(3)} = \langle\mathbf{p}_\alpha, \mathbf{b}_\nu, \mathbf{i}_\alpha\rangle$,

	$\mathbf{p}_\alpha$	$\mathbf{b}_\mu$	$\mathbf{i}_\alpha$
$\mathbf{p}_\alpha$	0	$-\mathbf{i}_\alpha$	$-\mathbf{b}_\mu$
$\mathbf{b}_\mu$	$+\mathbf{i}_\alpha$	0	$-\mathbf{p}_\alpha$
$\mathbf{i}_\alpha$	$+\mathbf{b}_\mu$	$+\mathbf{p}_\alpha$	0

2-nd trihedron $\hat{E}_m^{(3)} = \langle\mathbf{p}_\alpha, \mathbf{b}_\mu, \mathbf{i}_\alpha\rangle$;

in $\langle\mathcal{E}^3\rangle^{(m)} \subset \langle\mathcal{P}^{3+1}\rangle$

	j_1	j_2	j_3
j_1	0	$+j_3$	$-j_2$
j_2	$-j_3$	0	$+j_1$
j_3	$+j_2$	$-j_1$	0

(and in $\langle\mathcal{E}^3\rangle^{(m)} \subset \langle\mathcal{P}^{3+1}\rangle$) .

In final, in general, both trihedrons must form the tetrahedron.

Due to these Tables of the products (in the left one for pseudo-Euclidean rotations, in the right one for orthospherical rotations too in the subspace $\langle\mathcal{E}^3\rangle^{(m)}$ – in the latter, in that number, for the Thomas precession around its axis $\mathbf{b}_\mu$ with velocity w_θ^*), in the upper row we chose the rotated (differentiated) unity vector and in the left column we chose the axis of its rotation in the subspace of rotation. In the intersection, we get the vectorial product. So, for example, we get $\mathbf{i}_\alpha \times \mathbf{p}_\alpha = +\mathbf{b}_\nu$. The mathematical reason for this behavior of signs is that hyperbolic functions preserve their sign during differentiation, while spherical functions change it. The difference in signs of both theories in Euclidean space is eliminated by operation $\mathbf{b}_\nu \leftrightarrow \mathbf{b}_\mu$, due to our chosen strategic plan.

The hyperbolic rotations are described by the sine-cosine functions. Differentiations along the curve as a world-line lead here to the equivalent trigonometric processes $\sinh\gamma \to \cosh\gamma \to \sinh\gamma$... and $\cosh\gamma \to \sinh\gamma \to \cosh\gamma$... for radius-vectors of hyperboloids I and II (Figure 4), where we have sign " + " for both the concave arcs on hyperboloids I and II. For analogous trigonometric version of the Frenet–Serret theory, we obtain such processes for radius-vectors of the *hyperspheroid* (Ch. 8A) with signs variations: $\sin\varphi \to \cos\varphi \to -\sin\varphi$... and $\cos\varphi \to -\sin\varphi \to -\cos\varphi$....

* * *

In the Lagrangian space-time $\langle\mathcal{L}^{3+1}\rangle$, the tangent and the principal normal to a world line are applied, but a pseudonormal does not exist. Non-relativistic decomposition of acceleration at the point M on a world line in the plane $\langle\mathcal{E}^2\rangle_K^{(\mathbf{m})} \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\beta\rangle^{(\mathbf{m})} \equiv \langle\mathbf{v}, \mathbf{g}\rangle^{(\mathbf{m})}$ is performed in the Euclidean-affine space-time $\langle\mathcal{L}^{3+1}\rangle$ (see Ch. 1A), it is the following:

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix}, \qquad \frac{d\mathbf{u}}{dt} = \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = \begin{bmatrix} v\cdot\mathbf{e}_\alpha \\ 1 \end{bmatrix},$$

$$\frac{d^2\mathbf{u}}{dt^2} = \begin{bmatrix} \mathbf{g} \\ 0 \end{bmatrix} = \begin{bmatrix} g\cdot\mathbf{e}_\beta \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{dv}{dt}\cdot\mathbf{e}_\alpha \\ 0 \end{bmatrix} + \begin{bmatrix} v\cdot\frac{d\mathbf{e}_\alpha}{dt} \\ 0 \end{bmatrix} = \begin{bmatrix} \overline{\overline{g}}\cdot\mathbf{e}_\alpha \\ 0 \end{bmatrix} + \begin{bmatrix} \overset{\perp}{g}\cdot\mathbf{e}_\nu \\ 0 \end{bmatrix},$$

$$\text{where} \qquad \overline{\overline{g}} = \frac{dv}{dt} = g\cdot\cos\varepsilon, \quad \overset{\perp}{g} = v\cdot\frac{||d\mathbf{e}_\alpha||_E}{dt} = v\cdot w_\alpha = \frac{v^2}{r} = g\cdot\sin\varepsilon;$$

$$g^2 = (\overline{\overline{g}})^2 + (\overset{\perp}{g})^2, \quad \mathbf{g} = \overline{\overline{\mathbf{g}}} + \overset{\perp}{\mathbf{g}}, \quad \overline{\overline{\mathbf{g}}} \,||\, \mathbf{v}, \quad \overset{\perp}{\mathbf{g}} \perp \mathbf{v} \quad (\overset{\perp}{\mathbf{g}} = \mathbf{g} - \overline{\overline{\mathbf{g}}}).$$

Here $\mathbf{g}(t)$ is decomposed along the direction $\mathbf{e}_\alpha$ of the velocity $\mathbf{v}$ and the orthogonal direction $\mathbf{e}_\eta$ of the principal normal to the curve in the constant Euclidean subspace $\langle\mathcal{E}^3\rangle$ of the Lagrange space-time $\langle\mathcal{L}^{3+1}\rangle$, but with single Pythagorean Theorem!

* * *

Since, at collinear motion, a length of a curve's arc is $dl = K\cdot R$, then the 1-st part of (223A) is pseudoanalog of the 1-st Frenet–Serret formula, gotten by a purely trigonometric alternative way. So, using the hyperbolic angle of motion γ in the osculating pseudoplane, with arc $d\gamma$, we obtain:

$$d\mathbf{i} = \mathbf{p}\,d\gamma \;\Leftrightarrow\; \frac{d\mathbf{i}}{d\gamma} = \mathbf{p} \;\Leftrightarrow\; \frac{d\mathbf{i}}{Rd\gamma} = \frac{d\mathbf{i}}{d(c\tau)} = \frac{d\gamma}{d(c\tau)}\cdot\mathbf{p} = \frac{\mathbf{p}}{R_\gamma} = \mathcal{K}_\gamma\cdot\mathbf{p}. \qquad (231A-I)$$

For regular Euclidean curves with $\mathbf{e}_\alpha = \mathbf{const}$ in its *osculating quasiplane* in the quasi-Euclidean space $\langle\mathcal{Q}^{2+1}\rangle$ with reper axis $\overrightarrow{y}$ (Ch. 8A), using the principal angle φ, with arc $d\varphi$, we obtain, by a purely trigonometric alternative way, quasianalog of the 1-st Frenet–Serret formula:

$$d\mathbf{e} = \mathbf{n}\,d\varphi \;\Leftrightarrow\; \frac{d\mathbf{e}}{d\varphi} = \mathbf{n} \;\Leftrightarrow\; \frac{d\mathbf{e}}{Rd\varphi} = \frac{d\mathbf{e}}{dl} = \frac{d\varphi}{dl}\cdot\mathbf{n} = \frac{\mathbf{n}}{R_\varphi} = \mathcal{K}_\varphi\cdot\mathbf{n}. \qquad (231A-II)$$

* * *

Continuing the previous process, we realize the next two-steps differentiation along a world line, but now as of the principal pseudonormal $\mathbf{p}_\alpha$ to find the remaining motion parameters in the 3D space-time $\langle\mathcal{P}^{(2+1)}\rangle_I$ under contrary to (228A) metric tensor $\{I^\mp\}$ also for descriptivety. Now we'll consider this differentiation logically as accompanied with the concomitant hyperboloid I. For certainty, in the beginning, we take into account only the time-like variant of summary angular motion along a world line (relating namely to the STR), because the rotations of the pseudonormal $\mathbf{p}_\alpha$ give time-like and space-like particular differentials – see preliminary for the hyperboloid-I in (133A-H) and (133A-S) in Ch. 7A.

The principal and free characteristics $\mathbf{i}_\alpha$ and $\mathbf{i}_\kappa$ are produced with the 2-nd differentiations in $c\tau$ along a world line after (223A) with one and two degrees of freedom (at $\zeta = 4$):

$$\left.\begin{aligned} \left\{\frac{d\mathbf{p}_\alpha(c\tau)}{d(c\tau)}\right\}_\alpha &= \mathcal{K}_\alpha(c\tau)\cdot\begin{bmatrix}\sinh\gamma_i\cdot\mathbf{e}_\alpha \\ \cosh\gamma_i\end{bmatrix}_\alpha = \mathcal{K}_\alpha(c\tau)\cdot\mathbf{i}_\alpha(c\tau) = \mathbf{q}_\alpha(c\tau) = \frac{\mathbf{j}_\alpha}{c^2}, \\ \frac{d\mathbf{p}_\alpha(c\tau)}{d(c\tau)} &= \mathcal{Q}_\kappa(c\tau)\cdot\begin{bmatrix}\sinh\gamma_q\cdot\mathbf{e}_\kappa \\ \cosh\gamma_q\end{bmatrix} = \mathcal{Q}_\kappa(c\tau)\cdot\mathbf{i}_\kappa(c\tau) = \mathbf{q}_\kappa(c\tau) = \frac{\mathbf{j}_\kappa}{c^2}. \end{aligned}\right\} \quad (232A)$$

Let's adopt relation as (137A) for new characteristics in (232A) with connection as (224A) from condition of pseudoorthogonality of $\mathbf{p}_\alpha$ and $\mathbf{i}_\kappa$, with a free directive vector $\mathbf{e}_\kappa$ under sine slope to $\mathbf{e}_\alpha$ and 3-rd directive vector $\mathbf{e}_\mu = \mathbf{e}_\alpha \times \mathbf{e}_\nu$ also in the original Euclidean plane::

$$\mathbf{e}_\kappa = \sin\epsilon\cdot\mathbf{e}_\alpha + \cos\epsilon\cdot\mathbf{e}_\mu, \quad \epsilon\in[0;\pi], \quad (\mathbf{e}'_\kappa\cdot\mathbf{e}_\alpha = \sin\epsilon, \quad \mathbf{e}'_\kappa\cdot\mathbf{e}_\mu = \cos\epsilon, \quad \mathbf{e}'_\alpha\cdot\mathbf{e}_\mu = 0). \quad (233A)$$

From condition of pseudoorthogonality for $\mathbf{p}_\alpha$ and $\mathbf{i}_\kappa$, we get relations contrary to (224A):

$$\left.\begin{aligned} &\{\tanh\gamma_i = \sin\epsilon\cdot\tanh\gamma_q \sim \coth\gamma_q = \sin\epsilon\cdot\coth\gamma_i\} \to \gamma_q > \gamma_i(\gamma\in[0,\infty), \epsilon\in[0;\pi]); \\ &\text{at complementary angle}\{\cosh\upsilon_q = \cos\epsilon\cdot\cosh\upsilon_i\} \to \upsilon_q < \upsilon_i(\upsilon\in[0,\infty), \epsilon\in[0;\pi]). \end{aligned}\right\} \quad (234A)$$

At the 2-nd free differentiation in $c\tau$ along a world line, due to (232A) and with the use of (233A), we get *as if the pseudoanalog* of the 2-nd Frenet-Serret formula, with revealing a **space-like** *cosine binormal* $\mathbf{b}_\mu$ and the same principal curvature $\mathcal{Q}_\alpha = \mathcal{K}_\alpha$, but at the *principal tangent* $\mathbf{i}_\alpha$, and now in the *second partial* 3D *space-time* $\langle\mathcal{P}^{2+1}\rangle_I \equiv \{\langle\mathcal{E}^2\rangle^{(m)}_{Nc} \boxtimes \overrightarrow{ct}\} \equiv \{\langle\mathcal{E}^2\rangle^{(m)}_{Nc} \boxtimes \overrightarrow{y^{(m)}}\}$ (also at $\zeta = 3$):

$$\mathbf{q}_\kappa(c\tau) = \frac{d\mathbf{p}_\alpha(c\tau)}{d(c\tau)} = \frac{d\gamma_q}{d(c\tau)}\cdot\begin{bmatrix}\sinh\gamma_q\cdot\mathbf{e}_\kappa\\ \cosh\gamma_q\end{bmatrix} = \frac{d\gamma_q}{d(c\tau)}\cdot\mathbf{i}_\kappa(c\tau) = \mathcal{Q}_\kappa(c\tau)\cdot\mathbf{i}_\kappa(c\tau) \equiv \qquad (235A-I)$$

$$\equiv \frac{d\gamma_i}{d(c\tau)}\cdot\begin{bmatrix}\sinh\gamma_i\cdot\mathbf{e}_\alpha\\ \cosh\gamma_i\end{bmatrix}_\alpha + \begin{bmatrix}\cosh\gamma_i\cdot\frac{d\mathbf{e}_\alpha}{d(c\tau)}\\ 0\end{bmatrix}^{(1)}_\gamma = \frac{d\gamma_i}{d(c\tau)}\cdot\begin{bmatrix}\sinh\gamma_i\cdot\mathbf{e}_\alpha\\ \cosh\gamma_i\end{bmatrix}_\alpha + \begin{bmatrix}\cosh\gamma_i\cdot\frac{d\alpha_2}{d(c\tau)}\cdot\mathbf{e}_\mu\\ 0\end{bmatrix}^{(1)}_\gamma =$$

$$= \mathcal{Q}_\alpha(c\tau)\cdot\begin{bmatrix}\sinh\gamma_i\cdot\mathbf{e}_\alpha\\ \cosh\gamma_i\end{bmatrix}_\alpha + \mathcal{K}_\mu(c\tau)\cdot\begin{bmatrix}\mathbf{e}_\mu\\ 0\end{bmatrix}^{(1)}_\gamma = \mathcal{Q}_\alpha(c\tau)\cdot\mathbf{i}_\alpha(c\tau) + \mathcal{K}_\mu(c\tau)\cdot\mathbf{b}_\mu(c\tau) \equiv$$

$$\equiv \frac{d\gamma_q}{d(c\tau)}\cdot\begin{bmatrix}\sinh\gamma_q\cdot\mathbf{e}_\kappa\\ \cosh\gamma_q\end{bmatrix} = \frac{d\gamma_q}{d(c\tau)}\cdot\left\{\begin{bmatrix}\sin\epsilon\cdot\sinh\gamma_q\cdot\mathbf{e}_\alpha\\ \cosh\gamma_i\end{bmatrix} + \begin{bmatrix}\cos\epsilon\cdot\sinh\gamma_q\cdot\mathbf{e}_\mu\\ 0\end{bmatrix}^{(1)}\right\} =$$

$$= \mathcal{Q}_\kappa(c\tau)\cdot\begin{bmatrix}\sinh\gamma_q\cdot\mathbf{e}_\kappa\\ \cosh\gamma_q\end{bmatrix} = \mathcal{Q}_\kappa(c\tau)\cdot\mathbf{i}_\kappa(c\tau) = \overline{\overline{\mathcal{Q}^\times_\kappa}}\cdot\mathbf{i}_\alpha(c\tau) + \overset{\perp}{\mathcal{Q}^\circ_\kappa}\cdot\mathbf{b}_\mu(c\tau) = \overline{\overline{\mathbf{q}^\times_\kappa}}(c\tau) + \overset{\perp}{\mathbf{q}^\circ_\kappa}(c\tau).$$

Below we use again intuitive understandable notations beginning from the general curvature $\mathcal{Q}_\kappa$!

$$\left.\begin{array}{l}
\mathcal{Q}_\kappa = \frac{d\gamma_q}{d(c\tau)} = \frac{j_\kappa}{c^2};\quad \mathcal{Q}^\star_\kappa = \cosh\gamma_q\cdot\mathcal{Q}_\kappa = \mathcal{Q}^\star_\alpha = \cosh\gamma_i\cdot\mathcal{Q}_\alpha;\quad \mathcal{Q}_\alpha = \mathcal{K}_\alpha = \frac{d\gamma_i}{d(c\tau)} = \frac{j_\alpha}{c^2};\\
\mathcal{Q}^\circ_\kappa = \sinh\gamma_q\cdot\mathcal{Q}_\kappa = \frac{j^\circ_\kappa}{c^2},\quad \overline{\overline{\mathcal{Q}^\times_\kappa}} = \sqrt{\sin^2\epsilon\cdot\sinh^2\gamma_q - \cosh^2\gamma_q}\cdot\mathcal{Q}_\kappa = k_q\cdot\mathcal{Q}_\kappa = \mathcal{Q}_\alpha;\\
\overline{\overline{\mathcal{Q}^\circ_\kappa}} = \sinh\gamma_q\cdot\sin\epsilon\cdot\mathcal{Q}\kappa = \sinh\gamma_q\cdot\overline{\overline{\mathcal{Q}_\kappa}} = \sinh\gamma_i\cdot\mathcal{Q}\alpha = \sinh\gamma_i\cdot\mathcal{K}\alpha = \mathcal{Q}^\circ_\alpha = \frac{\overline{\overline{j^\circ_\kappa}}}{c^2} = \frac{j^\circ_\alpha}{c^2};\\
\overset{\perp}{\mathcal{Q}_\kappa} = \frac{\overset{\perp}{j_\kappa}}{c^2},\quad \overset{\perp}{\mathcal{Q}^\circ_\kappa} = \frac{\overset{\perp}{j^\circ_\kappa}}{c^2} = \sinh\gamma_q\cdot\cos\epsilon\cdot\mathcal{Q}_\kappa = \mathcal{K}_\mu = \cosh\gamma_i\cdot\frac{d\alpha_2}{d(c\tau)} = \frac{c^* w^*_{\alpha_2}}{c^2} = \frac{j_\mu}{c^2}.\\
\underline{\mp\mathcal{Q}^2_\kappa = \pm(\mathcal{Q}^{\circ\,2}_\kappa - \mathcal{Q}^{\star\,2}_\kappa) = (\overline{\overline{\mathcal{Q}^\circ_\kappa}}^2 + \overset{\perp}{\mathcal{Q}_\kappa}{}^2) - \mathcal{Q}^{\star\,2}_\kappa = -\overline{\overline{\mathcal{Q}^\times_\kappa}}^2 + \overset{\perp}{\mathcal{Q}^\circ_\kappa}{}^2 = -\mathcal{Q}^2_\alpha + \mathcal{K}^2_\mu.}
\end{array}\right\}\quad (235A-II)$$

Equaling under $I^\pm$ paired summands, we get next relations at $\eta < \gamma_q$: $\mp d\gamma_q^2 = \mp(\cosh^2\gamma_q\, d\gamma_q^2 - \sinh^2\gamma_q\, d\gamma_q^2) =$
$= \pm\underline{[(\sin^2\epsilon\cdot\sinh^2\gamma_q\, d\gamma_q^2 + \cos^2\epsilon\cdot\sinh^2\gamma_q\, d\gamma_q^2)} - \cosh^2\gamma_q\, d\gamma_q^2] = \underline{(\sinh^2\gamma_i\, d\gamma_i^2 + \cosh^2\gamma_i\, d\alpha_2^2)} - \cosh^2\gamma_i\, d\gamma_i^2 =$
$= \underline{-d\gamma_i^2 + \cosh^2\gamma_i\, d\alpha_2^2} = \pm\underline{[(\sin^2\epsilon\cdot\sinh^2\gamma_q - \cosh^2\gamma_q)\, d\gamma_q^2 + \cos^2\epsilon\cdot\sinh^2\gamma_q\, d\gamma_q^2]} = \underline{\mp(-\sinh^2\eta\, d\gamma_q^2 + \cosh^2\eta\, d\gamma_q^2)}.$

We have two identical decompositions of $d\gamma_q$ – usual and new pseudo-Euclidean, the latter correspond to the 1-st metric form (133A) of the hyperboloid I ! We use also underline for Relative and Absolute Theorems.

The Relative Pythagorean theorem follows from the space-like part of (235A) in its 3-vector and quadric scalar forms acting on the cosine normal plane $\langle\mathcal{E}^2\rangle^{(m)}_{Nc} \equiv \langle\mathbf{e}^{(m)}_\alpha, \mathbf{e}^{(1)}_\mu\rangle$ for three proportional characteristics as their orthoprojections into the Cartesian subbase $\bar{E}^{(3)}_1$ at $\gamma \in [0,\infty)$, $\epsilon \in [0;\pi]$, using (235A) with (233A):

$$\left.\begin{array}{l}
\mathcal{Q}_\kappa\cdot\sinh\gamma_q\cdot\mathbf{e}_\kappa = \mathcal{Q}^\circ_\kappa\cdot\mathbf{e}_\kappa = \sin\epsilon\cdot\mathcal{Q}^\circ_\kappa\cdot\mathbf{e}_\alpha + \cos\epsilon\cdot\mathcal{Q}^\circ_\kappa\cdot\ \mathbf{e}_\mu = \overline{\overline{\mathcal{Q}^\circ}}_\kappa\cdot\mathbf{e}_\alpha + \overset{\perp}{\mathcal{Q}^\circ_\kappa}\cdot\mathbf{e}_\mu =\\
= \mathcal{Q}_\alpha\cdot\sinh\gamma_i\cdot\mathbf{e}_\alpha + \cosh\gamma_i\cdot\frac{d\alpha_2}{d(c\tau)}\cdot\mathbf{e}_\mu = \mathcal{Q}^\circ_\alpha\cdot\mathbf{e}_\alpha + \frac{c^*\cdot w^*_{\alpha_2}}{c^2}\cdot\mathbf{e}_\mu = \mathcal{Q}^\circ_\alpha\cdot\mathbf{e}_\alpha + \mathcal{K}_\mu\cdot\mathbf{e}_\mu.
\end{array}\right\}\Rightarrow$$

$$\Rightarrow\left\{\begin{array}{l}
\mathbf{q}^\circ_\kappa = \overline{\overline{\mathbf{q}^\circ_\kappa}} + \overset{\perp}{\mathbf{q}^\circ_\kappa} = \mathbf{k}^\circ_\alpha + \mathbf{k}_\mu,\\
(\mathcal{Q}^\circ_\kappa)^2 = (\overline{\overline{\mathcal{Q}^\circ_\kappa}})^2 + (\overset{\perp}{\mathcal{Q}^\circ_\kappa})^2 = (\mathcal{Q}^\circ_\alpha)^2 + (\mathcal{K}_\mu)^2,
\end{array}\right\}\left\{\mathcal{Q}_\alpha = \mathcal{K}_\alpha = \frac{d\gamma_i}{d(c\tau)}, \text{but they are time-like and space-like}\right\}\Rightarrow$$

$$\Rightarrow\left\{\begin{array}{l}
\sinh\gamma_q\, d\gamma_q\cdot\mathbf{e}_\kappa = \sinh\gamma_i\, d\gamma_i\cdot\mathbf{e}_\alpha + \cosh\gamma_i\, d\alpha_2\cdot\mathbf{e}_\mu;\\
\sinh^2\gamma_q\, d\gamma_q^2 = \sinh^2\gamma_i\, d\gamma_i^2 + \cosh^2\gamma_i\, d\alpha_2^2 =\\
= \sinh^2\gamma_q\cdot[(\sin\epsilon\, d\gamma_q)^2 + (\cos\epsilon\, d\gamma_q)^2] = \sinh^2\gamma_q\,[(\overline{\overline{d\gamma_q}})^2 + (\overset{\perp}{d\gamma_q})^2],
\end{array}\right\}\Rightarrow$$

$$\Rightarrow\left\{\begin{array}{l}
\mathbf{j}^\circ_\kappa = \overline{\overline{\mathbf{j}^\circ_\kappa}} + \overset{\perp}{\mathbf{j}^\circ_\kappa} = \sinh\gamma_q\cdot\mathbf{j}_\kappa = \sinh\gamma_i\cdot j_\alpha\cdot\mathbf{e}_\alpha + c^*\cdot w^*_{\alpha_2}\cdot\mathbf{e}_\mu = j^\circ_\alpha\cdot\mathbf{e}_\alpha + j_\mu\cdot\mathbf{e}_\mu,\\
j^{\circ\,2}_\kappa = (\overline{\overline{j^\circ_\kappa}})^2 + (\overset{\perp}{j^\circ_\kappa})^2 = \sinh^2\gamma_q\cdot j^2_\kappa = \sinh^2\gamma_i\cdot j^2_\alpha + (c^*\cdot w^*_{\alpha_2})^2 = j^{\circ\,2}_\alpha + j^2_\mu.
\end{array}\right\}\quad (236A)$$

$$\Rightarrow\quad \sinh\gamma_q\cdot\sin\epsilon\, d\gamma_q = \sinh\gamma_q\,\overline{\overline{d\gamma_q}} = \sinh\gamma_i\, d\gamma_i \to \overline{\overline{d\gamma_q}}/d\gamma_i < 1,$$

$[\gamma_q/\gamma_i > 1$ – see in (234A).] $\{\gamma_q > \gamma_i\ (v_q > v_i),\ \gamma_q = 0 \leftrightarrow \gamma_i = 0;\ \overline{\overline{d\gamma_q}} < d\gamma_q < d\gamma_i\}\ \gamma \in [0,\infty);$

$$\mathcal{Q}_\kappa\cdot\cosh\gamma_q = \mathcal{Q}_\alpha\cdot\cosh\gamma_i = \mathcal{K}_\alpha\cdot\cosh\gamma_i,\ \mathcal{Q}_\alpha = \mathcal{K}_\alpha \Leftrightarrow \boxed{\cosh\gamma_q\, d\gamma_q = \cosh\gamma_i\, d\gamma_i} \to d\gamma_q/d\gamma_i < 1.\quad (237A)$$

From (235A)–(237A), we obtain the *Absolute pseudo-Euclidean Pythagorean theorem* with the *2-nd mobile trihedron* $\hat{E}_m^{(3)} = \langle \mathbf{p}_\alpha, \mathbf{b}_\mu, \mathbf{i}_\alpha \rangle$ in $\langle \mathcal{P}^{3+1} \rangle$ **under the same metric tensor** $I^\pm$ (17A-I)! And it acts on the pseudo-Euclidean cosine normal pseudoplane $\langle \mathcal{P}^{1+1} \rangle_{Nc\, I}^{(m)} \equiv \langle \mathbf{i}_\alpha^{(m)}, \mathbf{b}_\mu^{(1)} \rangle$ in 3D $\langle \mathcal{P}^{2+1} \rangle_I \equiv \{ \langle \mathcal{E}^2 \rangle_{Nc}^{(m)} \boxtimes \overrightarrow{ct} \}$ $(\zeta = 3)$. In the right triangle of $\mathbf{p}_\alpha$ rotations, it corresponds to the *angular pseudonormal* 1-st metric form (133A) for the concomitant hyperboloid I (!!!), as a *perfect hypersurface* of $\langle \mathcal{P}^{3+1} \rangle$. It is expressed in the initial base $\tilde{E}_1 = \{I\}$ and final base $\tilde{E}_m$ in the universal complete *tensor-vector-scalar ("tvs") form* with own proportional geometric and physical items:

$$\left\{ \begin{array}{l} \mathbf{q}_\kappa = \mathcal{Q}_\kappa\, \mathbf{i}_\kappa = \overline{\overline{\mathcal{Q}_\kappa^\circ}}\, \mathbf{i}_\alpha + \overset{\perp}{\mathcal{Q}_\kappa^\circ}\ \mathbf{b}_\mu = \mathcal{Q}_\alpha\, \mathbf{i}_\alpha + \mathcal{K}_\mu\, \mathbf{b}_\mu, (\mathcal{Q}_\alpha = \mathcal{K}_\alpha) \\ \mp \mathcal{Q}_\kappa^2 = \pm(\mathcal{Q}_\kappa^{\circ\,2} - \mathcal{Q}_\kappa^{*\,2}) = -(\overline{\overline{\mathcal{Q}_\kappa^\circ}})^2 + (\overset{\perp}{\mathcal{Q}_\kappa^\circ})^2 = -\mathcal{Q}_\alpha^2 + \mathcal{K}_\mu^2; \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \mathbf{j}_\kappa = j_\alpha\, \mathbf{i}_\alpha + j_\mu\, \mathbf{b}_\mu,\ (j_\alpha = g_\alpha) \\ \mp j_\kappa^2 = \pm(j_\kappa^{\circ\,2} - j_\kappa^{*\,2}) = -j_\alpha^2 + j_\mu^2, \end{array} \right\} \Rightarrow$$

$$\left. \begin{array}{l} d\gamma_q \cdot \mathbf{i}_\kappa = d\gamma_i \cdot \mathbf{i}_\alpha + \cosh\gamma_i\, d\alpha_2 \cdot \mathbf{b}_\mu,\ \ (\mathbf{i}'_\alpha \cdot I^\pm \cdot \mathbf{i}_\alpha = -1,\ \ \mathbf{b}'_\mu \cdot I^\pm \cdot \mathbf{b}_\mu = +1) \Rightarrow \\ -d\gamma_q^2 = -d\gamma_i^2 + \cosh^2\gamma_i\, d\alpha_2^2 = -\cosh^2\eta\, d\gamma_q{}^2 + \sinh^2\eta\, d\gamma_q{}^2 = -\left(\overline{\overline{d\gamma_q}}\right)_P^2 + \left(\overset{\perp}{d\gamma_q}\right)_E^2 < 0, \\ +d\gamma_q^2 = -d\gamma_i^2 + \cosh^2\gamma_i\, d\alpha_2^2 = -\sinh^2\eta\, d\gamma_q{}^2 + \cosh^2\eta\, d\gamma_q{}^2 = -\left(\overline{\overline{d\gamma_q}}\right)_P^2 + \left(\overset{\perp}{d\gamma_q}\right)_E^2 > 0. \end{array} \right\} \quad (238A - I, II)$$

Here $d\gamma_q = d\lambda_R/R$, $\eta < \gamma_q$. By this Egregium Theorem of Differential Tensor Trigonometry (2-nd from two hyperbolic), we reduce these mixed motions in $\tilde{E}_1$ along a world line and on I as a perfect surface to hyperboloidal arc if $d\gamma_q^2 < 0$, to ellipsoidal arc if $d\gamma_q^2 > 0$ and to horoline if $d\gamma_q = 0$ as hypotenuses in final $\tilde{E}_m$. Factor $\cosh\gamma_q\, d\gamma_q = \cosh\gamma_i\, d\gamma_i \rightarrow d\sinh\gamma_q = d\sinh\gamma_i$ in (237A) causes equations: $c_q^* \cdot g_\kappa = c_i^* \cdot g_\alpha \rightarrow c_q^* \cdot m_0 g_\kappa = c_i^* \cdot m_0 g_\alpha = c_q^* \cdot F_\kappa = c_i^* \cdot F_\alpha \rightarrow N_{(c)}{}_q^* = N_{(c)}{}_i^*$ in $\tilde{E}_m$ and $\tilde{E}_1$! Values γ_q and $s_q = c \cdot \coth\gamma_q = \sin\varepsilon \cdot c \cdot \coth\gamma_i = \sin\varepsilon \cdot s_i$ are calculated by (234A). But all these items relate to the so-called and hypothetical *Looking Glass of Theory of Relativity* – see below.

Note, in (228A) and (238A), when $\gamma_i \neq const \rightarrow d\gamma_i \neq 0$, *we confirmed our hyperboloidal model for world lines metrics!* They propagate on cases with $d\gamma_i = 0$ too – see further for screwed lines.

We obtain in (236A) the cosine normal acceleration $j_\mu = c^*\ w_{\alpha_2}^*$. Besides, $j_\alpha^{*\,2} = g_\alpha^{*\,2} - g_\alpha^2$ is the acceleration g_α^* excess, which was not explicitly revealed in (226A). We inferred that in (238A) the normal cosine projection of $d\alpha$ does not change, since they with $\mathbf{e}_\mu$ are perpendicular to the principal direction of motion $\mathbf{e}_\alpha$. We again state the fulfillment of the Herglotz Principle [84] – see it in Chs. 2A and 4A and in (228A). That is why, for normal projections in $\tilde{E}_1$, we did not use special sign circle as for parallel ones. For more clarity note, that both parts of $d\gamma_q \cdot \mathbf{i}_\kappa$ at I in (238A) give in the STR the pseudo-Euclidean interior and exterior right triangles in $\langle \mathcal{P}^{(1+1)} \rangle^{(m)} \equiv \langle \mathbf{b}_\mu, \mathbf{i}_\alpha \rangle^{(m)}$. Earlier in (226A) g_α^* (as a leg) and g_α (as a hypotenuse) given only the exterior right triangle with the acute angle γ_i between them, and $g_\alpha^{*\,2} - j_\alpha^{*\,2} = g_\alpha^2 = j_\alpha^2$. In (236A) the parallel accelerations j_α^* (as a leg) and j_α (as a hypotenuse) give the interior right triangle with the obtuse complementary angle υ_i between them. Now we obtain contrary $g_\alpha^{*\,2} - j_\alpha^{*\,2} = j_\alpha^2 = g_\alpha^2$. In both right triangles, j_α^* lies contrary to γ_i. For characterization of hyperboloids I and II, it is necessary to distinguish between their geometry as a whole and the part that relates to Theory of Relativity. We will analyze below the geometric features and more in detail what is related to the theory of world lines.

Theorem (238A) acts at the tangent pseudo-Euclidean hyperplane to concomitant hyperboloid I at slopes of summary motion's arc $d\gamma_q = d\lambda_R/R$ *inside or outside light cone.* Two-steps differentiation (235A) gives rotations of the pseudonormal $\mathbf{p}_\alpha$ with two degrees of freedom: $\mathbf{p}_\alpha \times \mathbf{b}_\mu = +\mathbf{i}_\alpha$ and $\mathbf{p}_\alpha \times \mathbf{i}_\alpha = +\mathbf{b}_\mu$ $(\mathbf{b}_\nu = \text{const})$. Thus, unity $\mathbf{b}_\mu$ and $\mathbf{i}_\alpha$ are here the 2-nd pair of the cardinal pseudo-Cartesian base $\hat{E}_m^{(4)} = \langle \mathbf{p}_\alpha(c\tau),\ \mathbf{b}_\nu(c\tau),\ \mathbf{b}_\mu(c\tau),\ \mathbf{i}(c\tau) \rangle$ as the *movable tetrahedron*, which is rotated around an arbitrary world line in the *entire binary space-time* $\langle \mathcal{P}^{3+1} \rangle$ (see further). The 4-vector $\mathbf{i}_\alpha$ was obtained in (222A) for the sequential two-steps differentiations (225A) and (235A). Each they are realized with two degrees of freedom producing two own specific trihedrons: in (228A) $\hat{E}_m^{(3)} = \langle \mathbf{p}_\alpha(c\tau),\ \mathbf{b}_\nu(c\tau),\ \mathbf{i}_\alpha(c\tau) \rangle$ and in (238A) $\hat{E}_m^{(3)} = \langle \mathbf{i}_\alpha(c\tau),\ \mathbf{b}_\mu(c\tau),\ \mathbf{p}_\alpha(c\tau) \rangle$.

Non-collinear motions in (238A) and on the hyperboloid I with their two differential arcs have own normal relations, in addition to (230). Here is $d\gamma = d\gamma_q$. Since in (238A) $\gamma_i < \gamma_q$, then at $\gamma_q = 0$ we get $\gamma_i = 0$, i. e., physical movement is absent. Hence the Local Absolute pseudo-Euclidean Pythagorean theorem in $\langle \mathcal{E}^3 \rangle^{(1)}$ is absent! That is why, from (235A-II), in the Euclidean subspace $\langle \mathcal{E}^3 \rangle^{(1)}$, when $\gamma_q \neq 0$, we produce on the hyperboloid I in $\langle \mathcal{P}^{2+1} \rangle_I$ its own *normal relations*:

$$\cos\epsilon \cdot \sinh\gamma_q\, d\gamma_q = \overset{\perp}{d\gamma_q} = \cosh\gamma_i\, d\alpha_2 = (d\alpha_2)^* > d\alpha_2\ \ (\gamma_i < \gamma_q). \quad (239A)$$

Note, in the end of Ch. 7A, we established through our simple trigonometric formula (173A) that a true primary reason of the Thomas precession in STR is mathematical "angular dissonance" of a hyperbolic cosine type, having place also in the hyperbolic triangles *on the hyperboloid II and on the Lobachevsky–Bolyai hyperbolic plane* as the Lambert angular defect. By (173A) we have

$$d\theta = d\alpha_1 - (d\alpha_1)^* = d\alpha_1 - \cosh\gamma_i \; d\alpha_1 < 0 \rightarrow d\theta/dt = w_\theta = w_\alpha - w_\alpha^* < 0. \qquad (240A)$$

Differential rotation $d\alpha_1$ of a world line acts in (228A) in the sine normal plane $\langle\mathcal{E}^2\rangle_{Ns}^{(m)} \equiv \langle \mathbf{e}_\alpha^{(m)}, \mathbf{e}_\nu^{(1)}\rangle$ around its instantaneous normal precessing axis $\mathbf{e}_\mu^{(m)}$ sloped locally under $\cosh\gamma_i$ to immobile $\mathbf{e}_\mu^{(1)}$ in the base $\tilde{E}_1$. But how we may describe with (240A) forming the complete Lambert negative angular deviation $-d\theta$, for example, on the hyperboloid II (or in time as the Thomas precession) ? For this we'll use the descriptive process of drawing the triangle on the curvilinear surface of the hyperboloid II continuously and perpendicularly to the vector $\mathbf{e}_\mu$. When we pass along 3 sides of the hyperbolic triangle, the Lambert angular defect is integrated with (240A) along its sides. In the last apex of the hyperbolic triangle, we'll receive the complete angular deviation $\int_0^{\alpha_1} [d\alpha_1 - \cosh\gamma_i(\alpha_1)\; d\alpha_1] = \pi - \alpha_{11}^* - \alpha_{12}^* - \alpha_{13}^* < \pi - \alpha_{11} - \alpha_{12} - \alpha_{13} = 0$. See strictly the prove of the bond of orthospherical shift $d\theta$ with the Lambert angular deviation in (244A-II), Ch. 7A.

However, we obtained above in (236A) the similar normal item $\cosh\gamma_i \; d\alpha_2$, but it is for the second independent cosine normal rotation-motion of a world line, projected in the instantaneous cosine normal plane $\langle\mathcal{E}^2\rangle_{Nc}^{(m)} \equiv \langle \mathbf{e}_\alpha^{(m)}, \mathbf{e}_\mu^{(1)}\rangle$, what is more, around its instantaneous normal axis $\mathbf{e}_\nu^{(m)}$!

* * *

The joint pseudoorthogonality of motions in (228A), (238A) and also (!) of motions on both conjugated Minkowski hyperboloids in $\langle\mathcal{P}^{3+1}\rangle$ are reduced to the equation $\mathbf{p}'_\beta \cdot \{I^\pm\} \cdot \mathbf{i}_\kappa = 0$, which is executed according to (224A) and (234A) iff $\boxed{\mathbf{e}'_\beta \cdot \mathbf{e}_\kappa = \cos\varepsilon \cdot \sin\epsilon}$. Then we have this final equation with conditions of the consistent orthogonality of all four basis vectors, inferred strictly the existence of a complete pseudoorthogonal cardinal base $\hat{E}_m^{(4)}$. But the complete orthogonality of the vectors $\mathbf{e}_\beta$ and $\mathbf{e}_\kappa$ is realized in any of these three cases: (1) $\cos\varepsilon = 0 \rightarrow \mathbf{e}_\beta = \pm\mathbf{e}_\nu$ according to (137A) and (2) $\sin\epsilon = 0 \rightarrow \mathbf{e}_\kappa = \pm\mathbf{e}_\mu$ according to (233A) or full (3) $\cos\varepsilon = \sin\epsilon = 0 \rightarrow \mathbf{e}_\beta = \pm\mathbf{e}_\nu, \mathbf{e}_\kappa = \pm\mathbf{e}_\mu$.

* * *

To realize various alternative motions with the entire angular differential $d\alpha$, essentially in the important case when $d\gamma_i = 0$, we introduce new necessary additional unity vectors. Below put:

$\mathbf{b}_\alpha = \begin{bmatrix} \mathbf{e}_\alpha \\ 0 \end{bmatrix}$ is a cutting 3-rd *space-like binormal* (from complete $\mathbf{p}_\alpha$ and $\mathbf{i}_\alpha$).

$\mathbf{i}_1 = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$ is a cutting *time-like binormal* (from complete $\mathbf{i}_\alpha$ and $\mathbf{p}_\alpha$).

They arise, as the additional unity 4-vectors, when total curvatures of the given principal rotations of time-arrow $\mathbf{i}_\alpha$ and pseudonormal $\mathbf{p}_\alpha$ are decomposed into their spatial and temporal parts.

In addition, for further constructions of screwed curves, we introduce the specific time arrow and pseudonormal, perpendicular to principal ones $\mathbf{i}_\alpha$ and $\mathbf{p}_\alpha$ also presented here for comparison:

$$\mathbf{i}_\nu = \begin{bmatrix} \sinh\gamma_i \cdot \mathbf{e}_\nu \\ \cosh\gamma_i \end{bmatrix}, \;\; \mathbf{p}_\mu = \begin{bmatrix} \cosh\gamma_i \cdot \mathbf{e}_\mu \\ \sinh\gamma_i \end{bmatrix}; \;\; \mathbf{i}_\alpha = \begin{bmatrix} \sinh\gamma_i \cdot \mathbf{e}_\alpha \\ \cosh\gamma_i \end{bmatrix}, \;\; \mathbf{p}_\alpha = \begin{bmatrix} \cosh\gamma_i \cdot \mathbf{e}_\alpha \\ \sinh\gamma_i \end{bmatrix}.$$

* * *

Geometrically (238A) corresponds to rotation of principal pseudonormal $\mathbf{p}_\alpha$ with two degrees of freedom: at complete arc $d\gamma_i$ and at as if cutting arc $d\alpha_2$. Indeed, above we have only its space-like cosine projection into $\langle\mathcal{E}^2\rangle$. Although complete $d\alpha_2$ with its sine and cosine projections in $\overrightarrow{ct}$ and $\langle\mathcal{E}^2\rangle$ is a space-like vector sum at the space-like unity 4-vector of the normal cosine pseudonormal $\mathbf{p}_\mu$. (Also under metric tensor $I^\pm$.) Such cutting is caused by mixing its time-like projection $\sinh\gamma_i \; d\alpha_2$ with the time-like projection $\sinh\gamma_i \; d\gamma_i$ in $d\gamma_i \cdot \mathbf{i}_\alpha$ in (238A). However at γ_i = const, similar mixing is absent, and we can execute as alternative to (235A) two-steps differentiation along a world line with orthogonal decomposition of 4-vector $d\alpha_2 \cdot \mathbf{p}_\mu$ into two trigonometric projections. Obviously, such two-steps space-like orthospherical motion must have own trihedron in 3D $\langle\mathcal{P}^{2+1}\rangle$, however (as we shall see) also in the central zone of the concomitant hyperboloid I, with the own Absolute pseudo-Euclidean Pythagorean theorem. We'll implement this scenario below for correct construction by Tensor Trigonometry of the space-like pseudoscrewed "superlight world line" as the 2-nd type of uniformly accelerated motion (in addition to space-like uniform hyperbolic motion).

* * *

In order to replenish our study of relativistic motions in entire $\langle\mathcal{P}^{3+1}\rangle$, let us remember words of the great Niels Bohr to dared physicists: "Your theory is not correct, as it is not crazy enough!" Then, we may realize the more complete tensor trigonometric presentation of relativistic motions in entire $\langle\mathcal{P}^{3+1}\rangle$, with world lines of two types – usual as they were before and *superlight*, separated there by an isotropic cone. The first are used in the true Poincaré – Minkowski space-time, and the second act in the hypothetical *Looking Glass of Theory of Relativity*, with the well-known and nice voyager Alice (following here to the non-ordinary English writer-mathematician Lewis Carroll). This Looking Glass is realized in entire relativistic or binary geometric $\langle\mathcal{P}^{3+1}\rangle$ and physically beyond the horizon of events as if in another adjacent othersided world. So, for instance, it may be inside the so-called *black hole*, predicted in 1783 by the eminent John Michell [81] only on the basis of the Newtonian Celestial Mechanics. Beside, at our time, in 1962, the well renowned physicist and Pioneer in quantum Tachyon Theory Gerald Feinberg predicted so-called tachyons, as elementary particles that move at speed greater than the constant speed of light c in a vacuum and no-when non-equal really to it. Moreover, up to now nobody asked the sacramental question: "According to what laws and equations of kinematics and dynamics the superlight relativistic motions should be carried out inside the Looking Glass of Theory of Relativity of entire $\langle\mathcal{P}^{3+1}\rangle$?" We may logically adopt, that in this superlight space-time, such laws are developed from differentiation of $\mathbf{p}_\alpha$ along a superlight world line as in (238A). We get the hyperbolic angles of motion $-\upsilon$ off an isotropic cone to $\langle\mathcal{E}^3\rangle$ and complementary to it $-\gamma(-\upsilon)$ as clockwise ones in $\langle\mathcal{P}^{1+1}\rangle$. Coordinate supervelocity is $u = c\cdot\coth\gamma = c\cdot\cosh\upsilon \geq c$ from c till ∞ and proper supervelocity is $u^* = c\cdot\cosh\gamma = c\cdot\coth\upsilon \geq c$ from ∞ till c. Scalar supervelocity of the time t stream is $c^* = c\cdot\sinh\gamma = c\cdot\operatorname{csch}\,\upsilon \geq c$ from ∞ till zero! The arc of a superlight world line is a real valued cosine-sine pseudoinvariant $(dt, d\tau > 0)$. The tensors of motion and deformation with dynamic tensors from Ch. 5A have the tensor angles $-\Upsilon$ or $-\Gamma$ with their structures of types (496) under also constant coefficients m_0c and m_0c^2.

For the next clarity, it is time to consider real localizations of two complete angular arcs $d\gamma$ and all three independent orthospherical arcs $d\alpha_{1,2,3}$ in entire $\langle\mathcal{P}^{3+1}\rangle$. Tangent and pseudonormal, produced by differentiation (222A), (225A), (235A) change along a world line under equivalent action of motion tensor (100A). Two arcs $d\gamma$, primary space-like in (228A) and mutual time-like in (238A), are situated in pseudoplane $\langle\mathcal{P}^{1+1}\rangle_H^{(m)} \equiv \langle\mathbf{p}_\alpha, \mathbf{i}_\alpha\rangle$ of entire $\langle\mathcal{P}^{3+1}\rangle$ presented by analogy with two bonded primary and mutual spherical arcs $d\varphi$ in quasiplane $\langle\mathcal{Q}^{1+1}\rangle_S^{(m)} \equiv \langle\mathbf{n}_\alpha, \mathbf{t}_\alpha\rangle$ at Figure 3 of entire $\langle\mathcal{Q}^{3+1}\rangle$. *The first is the hyperbolic osculating pseudoplane of the hyperbolic curvature* $\mathcal{K}_\alpha$. *The second is the spherical osculating quasiplane of the spherical curvature* $\mathcal{K}_\alpha$. Hyperbolic and spherical angles and differentials act as binary ones too. Both binary differentials $d\gamma$ act symmetrically with respect to isotropic cone in the middle between them – see at Figure 4. They express the hyperbolic identical, but contrary differential rotations of $\mathbf{i}_\alpha$ and $\mathbf{p}_\alpha$, *due to the especial binary structure of our hyperbolic tensor of motion* (100A), with their permanent symmetry to an isotropic cone. It is from here we have their pairwise equality in (225A-II), (235A-II), but as scalar ones. These features has a place in the quasi-Euclidean space for double differentials $d\varphi$ for simultaneous contrary spherical rotations of tangent $\mathbf{t}_\alpha$ and quasinormal $\mathbf{n}_\alpha$ under our spherical tensor of motion (313).

The sine $\mathbf{b}_\nu$ and cosine $\mathbf{b}_\mu$ binormals with their sine and cosine normal curvatures act in the sine and cosine Euclidean normal planes, but with the possible common orthospherical rotation $d\alpha_3$ in the *binormal's Euclidean plane* $\langle\mathcal{E}^2\rangle_B^{(1)} \equiv \langle\mathbf{b}_\nu, \mathbf{b}_\mu\rangle$ – similar to the Cardano gimbal in the Euclidean space $\langle\mathcal{E}^3\rangle$. This plane is spherically orthogonal to the main binormal $\mathbf{b}_\alpha^{(m)}$ in $\langle\mathcal{E}^3\rangle^{(m)}$. And the 3-rd arc $d\alpha_3$ expresses *a non-relativistic free orthospherical rotation* in the binormal's Euclidean plane.

In the 4D pseudo-Euclidean space $\langle\mathcal{P}^{3+1}\rangle$, Euclidean binormal's plane and osculating pseudoplane are pseudoorthogonal and form a direct pseudoorthogonal sum from these relative summands! Each from them is a direct pseudoorthogonal complement to another and is defined by 4×2-lineors $A_1 = |\mathbf{p}_\alpha, \mathbf{i}_\alpha|$ and $A_2 = |\mathbf{b}_\nu, \mathbf{b}_\mu|$ – see them in Ch. 5. We state the additional to (500) and (174A) pseudo- and quasi orthogonal decompositions of both binary spaces into their relative summands:

$$\langle\mathcal{P}^{3+1}\rangle \equiv \langle\mathcal{P}^{1+1}\rangle_H^{(k)} \boxtimes \langle\mathcal{E}^2\rangle_B^{(k)} \equiv \text{CONST.} \tag{241A – I}$$

$$\langle\mathcal{Q}^{3+1}\rangle_c \equiv \langle\mathcal{Q}^{1+1}\rangle_{cH}^{(k)} \boxplus \langle\mathcal{E}^2\rangle_B^{(k)} \equiv \text{CONST.} \tag{241A – II}$$

$$\langle\mathcal{Q}^{3+1}\rangle \equiv \langle\mathcal{Q}^{1+1}\rangle_S^{(k)} \boxplus \langle\mathcal{E}^2\rangle_B^{(k)} \equiv \text{CONST.} \tag{242A}$$

Such properties with (500) and (174A) create a nice trigonometric harmony of these binary spaces!

* * *

Let us note one important property of a world line in $\langle\mathcal{P}^{3+1}\rangle$. Its principal tangent and its pseudonormal are always symmetric with respect to the isotropic cone. The same property relates to concomitant hyperboloids. This property is preserved even during their two-steps differentiations along a world line. So, this should lead to the fact that during two-steps differentiation in (235A), with revealing two basis vectors – the principal tangent and the space-like cosine binormal with cosine curvature at it, in previous (228A) synchronously and in addition to them the principal pseudonormal and the space-like sine binormal with sine curvature at it should appear from (238A). This gives the complete three-steps 1-st metric form for a world line in the usual 4D space-time:

$$\left.\begin{aligned}
&\frac{d\mathbf{i}_\alpha}{d(c\tau)} = \frac{dt\gamma_i}{d(c\tau)}\cdot\mathbf{p}_\alpha + \sin t\gamma_i\cdot\frac{d\alpha_1}{d(c\tau)}\cdot\mathbf{b}_\nu + \cos t\gamma_i\cdot\frac{d\alpha_2}{d(c\tau)}\cdot\mathbf{b}_\mu = \\
&= \sin t\gamma_i\cdot\frac{dt\gamma_i}{d(c\tau)}\cdot\mathbf{i}_1 + \cos t\gamma_i\cdot\frac{dt\gamma_i}{d(c\tau)}\cdot\mathbf{b}_\alpha + \sin t\gamma_i\cdot\frac{d\alpha_1}{d(c\tau)}\cdot\mathbf{b}_\nu + \cos t\gamma_i\cdot\frac{d\alpha_2}{d(c\tau)}\cdot\mathbf{b}_\mu = \\
&= \mathcal{K}_\alpha\cdot\mathbf{p}_\alpha + \mathcal{K}_\nu\cdot\mathbf{b}_\nu + \mathcal{K}_\mu\cdot\mathbf{b}_\mu = \mathcal{Y}_{sin}\cdot\mathbf{i}_1 + \mathcal{X}_{cos}\cdot\mathbf{b}_\alpha + \mathcal{K}_\nu\cdot\mathbf{b}_\nu + \mathcal{K}_\mu\cdot\mathbf{b}_\mu, \\
&\{d\lambda/R\}^2 = d\gamma_i^2 + \sinh^2\gamma_i\ d\alpha_1{}^2 + \cosh^2\gamma_i\ d\alpha_2{}^2 = \\
&= -\sinh^2\gamma_i\ d\gamma_i^2 + \cosh^2\gamma_i\ d\gamma_i^2 + \sinh^2\gamma_i\ d\alpha_1{}^2 + \cosh^2\gamma_i\ d\alpha_2{}^2 \Rightarrow \\
&\Rightarrow\ \mathcal{C}_\mathcal{R}{}^2 = \frac{\eta_\gamma^{*2}}{c^2} + \sinh^2\gamma_i\cdot\frac{w_{\alpha 1}^{*\ 2}}{c^2} + \cosh^2\gamma_i\cdot\frac{w_{\alpha 2}^{*\ 2}}{c^2} = \mathcal{K}_\alpha^2 + \mathcal{K}_\nu^2 + \mathcal{K}_\mu^2; \\
&g_\Sigma\mathbf{p}_\Sigma = g_\alpha\mathbf{p}_\alpha + g_\nu\mathbf{b}_\nu + \jmath_\mu\mathbf{b}_\mu \Rightarrow\ g_\Sigma^2 = (c\eta_\gamma^*)^2 + (v^*w_{\alpha(1)}^*)^2 + (c^*w_{\alpha(2)}^*)^2.
\end{aligned}\right\}\quad (243A, 244A)$$

The item $\cosh\gamma_i$ at $d\alpha_2$ is situated trigonometrically ofside Cayley oval, and they give proportional cosine normal acceleration as also Euclidean pojection. However STR does not impose restrictions onto accelerations, but only on the value of velocity v, besides of the voyager himself! If $s \ll c$, then $\cosh\gamma_i \to 1$ and $\mathcal{K}_\mu \to d\alpha_2/d(c\tau)$ is *as if for rotation of the moving gyroskop* on its world line. We have again complete compatibility with the Principles of Correspondence by Niels Bohr!

And now we may do the following infers: $\mathcal{K}_\alpha, \mathcal{K}_\nu, \mathcal{K}_\mu \neq 0$ is a condition of the $4D$-spatial curves; any two from these curvatures as non-zero is a condition of the $3D$-spatial curves; anyone from them as non-zero is a condition of the flat curves; $\mathcal{K}_\alpha, \mathcal{K}_\nu, \mathcal{K}_\mu = 0$ is a condition of the straight world line.

In the Looking Glass of Relativity, for a *superlight world line* with the cotangent coordinate velocity $s = \coth\gamma_i\cdot c \geq c$ (from Ch. 6A) in external cavity of isotropic cone, we get in the entire 4D space-time or the geometric space $\langle\mathcal{P}^{3+1}\rangle$ the next relations, where as if $\mathbf{p}_\alpha$ and $\mathbf{i}_\alpha$ are exchanged:

$$\left.\begin{aligned}
&\frac{d\mathbf{p}_\alpha}{d(c\tau)} = \frac{dt\gamma_i}{d(c\tau)}\cdot\mathbf{i}_\alpha + \cos t\gamma_i\cdot\frac{d\alpha_2}{d(c\tau)}\cdot\mathbf{b}_\mu + \imath\sin t\gamma_i\cdot\frac{d\alpha_1}{d(c\tau)}\cdot\mathbf{b}_\nu = \\
&= \sin t\gamma_i\ \frac{dt\gamma_i}{d(c\tau)}\cdot\mathbf{i}_1 + \cos t\gamma_i\ \frac{dt\gamma_i}{d(c\tau)}\cdot\mathbf{b}_\alpha + \cos t\gamma_i\cdot\frac{d\alpha_2}{d(c\tau)}\cdot\mathbf{b}_\mu + \imath\sin t\gamma_i\cdot\frac{d\alpha_1}{d(c\tau)}\cdot\mathbf{b}_\nu = \\
&= \mathcal{K}_\alpha\cdot\mathbf{i}_\alpha + \mathcal{K}_\mu\cdot\mathbf{b}_\mu + \imath\mathcal{K}_\nu\cdot\mathbf{b}_\nu = \mathcal{Y}_{cos}\cdot\mathbf{i}_1 + \mathcal{X}_{sin}\cdot\mathbf{b}_\alpha + \mathcal{K}_\mu\cdot\mathbf{b}_\mu + \imath\mathcal{K}_\nu\cdot\mathbf{b}_\nu, \\
&\{d\lambda/R\}^2 = -d\gamma_i^2 + \cosh^2\gamma_i\ d\alpha_2{}^2 + \sinh^2\gamma_i\ d\alpha_1{}^2 = \\
&= -\cosh^2\gamma_i\ d\gamma_i^2 + \sinh^2\gamma_i\ d\gamma_i^2 + \cosh^2\gamma_i\ d\alpha_2{}^2 + \sinh^2\gamma_i\ d\alpha_1{}^2 \Rightarrow \\
&\Rightarrow\ \mathcal{S}_\mathcal{R}{}^2 = -\frac{\eta_\gamma^{*2}}{c^2} + \cosh^2\gamma_i\cdot\frac{w_{\alpha 2}^{*\ 2}}{c^2} + \sinh^2\gamma_i\cdot\frac{w_{\alpha 1}^{*\ 2}}{c^2} = -\mathcal{K}_\alpha^2 + \mathcal{K}_\mu^2 + \mathcal{K}_\nu^2; \\
&g_\Sigma\mathbf{i}_\Sigma = g_\alpha\mathbf{i}_\alpha + \jmath_\mu\mathbf{b}_\mu + g_\nu\mathbf{b}_\nu \Rightarrow\ g_\Sigma^2 = -(c\eta_\gamma^*)^2 + (c^*w_{\alpha(2)}^*)^2 + (v^*w_{\alpha(1)}^*)^2.
\end{aligned}\right\}\quad (245A, 246A)$$

We reduce the arbitrary most general motions in absolute entire $\langle\mathcal{Q}^{3+1}\rangle_c$ and $\langle\mathcal{P}^{3+1}\rangle$, mixed from hyperbolic and orthospherical (under hyperbolic inclinations), again to purely angular ones, along hypotenuse of the right parallelepiped from three legs in $\langle\mathcal{E}^3\rangle^{(m)}$, while preserving the symmetry of tangent and pseudonormal with respect to isotropic cone and under the common metric tensors.

Note, that metric forms (228A), (238A) transform, by abstract analogy, in two-steps quasi-Euclidean ones in quasi-Euclidean space $\langle\mathcal{Q}^{3+1}\rangle$ and on concomitant 3D hyperspheroid, separately from Pole of II and from Equator of I. But this analogy does not relate to three-steps forms, because both hyperboloids do not form one-connected hypersurface, contrary to the hyperspheroid!

In both variants of motions above, their projections onto the frame axis give us the so-called *orthoprocessions* $\mathcal{Y}$ along it with a point of application M. These orthoprocessions move a world line *progressively* parallel to the frame axis $\vec{ct}^{(1)}$ with velocity either $\cosh\gamma_i \cdot c$ or $\sinh\gamma_i \cdot c$. By such a way, we have decomposed even time like and space-like hyperbolic motions in (243A – 246A). This *orthoprocession* moves a world line *progressively* parallel to the frame axis *with hyperbolic shift* $d\gamma$.

Let us add one else possible motion. It is eigen rotation α of the frame axis $\vec{ct}^{(1)}$ or $\vec{y}^{(1)}$ with shift $d\alpha$ or shift in time as the angular velocity w^*_α. It is caused by the change of a world point M orientation, with respect to the frame axis, at rotation of the vector $\mathbf{e}_\alpha$ in normal Euclidean planes! That is why, in result of two-steps differentiations in (228A) and (238A), when $d\gamma_i = 0$, one may have lost the time-like *cosine* and *sine* orthoprojections of $d\alpha$ onto the frame axis as the projective cosine and sine *orthospherical orthoprocessions* along $\vec{ct}^{(1)}$ or in pseudo-Euclidean geometry along $\vec{y}^{(1)}$.

What's more, when $d\gamma_i = 0$ and $d\alpha \neq 0$, we can reveal purely artificially these *cosine* and *sine* orthoprojections of the complete (non-cutting) orthospherical shift $d\alpha$ onto the frame axis $\vec{ct}^{(1)}$ in the 3D pseudo-Cartesian bases $\tilde{E}_1^{(3)}$ of entire $\langle\mathcal{P}^{3+1}\rangle$, furthermore uniting them into complete $d\alpha$, accordingly with its sine and cosine space-like orthoprojections onto the sine and cosine normal Euclidean planes in entire $\langle\mathcal{P}^{3+1}\rangle$. By such a correct and very descriptive manner, we can construct differentially in entire $\langle\mathcal{P}^{3+1}\rangle$ the two complete screw rotations $d\alpha$ of their Euclidean radius r and pseudo-Euclidean radius R_K, with positive and negative signs of the right and left screws shift $d\alpha$ and their sine steep and cosine gentle inclinations γ. Both variants at $d\gamma_i = 0$ ($\gamma_i = \text{const}$) give the two Absolute pseudo-Euclidean Pythagorean theorems for cosine and sine orthospherical curvatures in entire $\langle\mathcal{P}^{3+1}\rangle$ with hypotenuse $\mathcal{C}_R = w^*_\alpha/c$. Their relative projections onto the frame axis $\vec{ct}^{(1)}$ give the own orthoprocessions, introduced above, depending on slopes to $\vec{ct}^{(1)}$ as $\mathcal{Y}_{cos} = \cosh\gamma_i \cdot \mathcal{C}_R$ or as $\mathcal{Y}_{sin} = \sinh\gamma_i \cdot \mathcal{C}_R$. In the hyperbolic case of screw inclinations, with the permanent constant 4-velocity $\mathbf{c}$ of world point M along a world line, two pseudoscrewed motions are natural additions to two hyperbolic motions with slopes above and upper isotropic cone as usual and superlight ones.

It is orthoprocession accompanied by complete rotation $d\alpha$ give screw or pseudoscrew!

We'll consider below briefly two special variants of world lines with $\gamma_i = \text{const} \neq 0$ at $d\gamma_i = 0$.

In the first variant, we'll have 1-st metric form of such a world line with the time-like cosine orthoprocession $\mathcal{Y}_{cos}$ and the space-like sine normal curvature $\mathcal{K}_\nu$, as projections, accompanied by the complete imaginary time-like orthospherical differential $di\alpha$. It is expressed by the *Absolute pseudo-Euclidean Pythagorean theorem* in the interior right triangle from $di\alpha$, cosine orthoprocession and sine normal curvature. Let us split the complete rotation $di\alpha$ onto $\vec{ct}^{(1)}$ as $\cosh\gamma_i\, di\alpha \cdot \mathbf{i}_1$ and into $\langle\mathcal{E}^3\rangle^{(1)}$ as $\sinh\gamma_i\, di\alpha \cdot \mathbf{b}_\nu$. Then the two-steps differentiation of $\mathbf{i}_\alpha$ will consist in the cosine orthoprocession $\mathcal{Y}_{cos} = \cosh\gamma_i \cdot \mathcal{C}_R = \cosh\gamma_i \cdot (iw^*/c)$ and the normal sine curvature $\mathcal{K}_\nu = \sinh\gamma_i \cdot \mathcal{C}_R = \sinh\gamma_i \cdot (iw^*/c)$ with the orthospherical imaginary pseudo-Euclidean rotation $di\alpha$ of the screwed world line with point M at the velocity $di\alpha/d\tau = iw^*_\alpha$ around $\vec{ct}^{(1)}$. The purely Euclidean projected rotation $d\alpha$ acts around 3-rd Euclidean axis $\mathbf{b}_\mu$ in the *sine normal plane* $\langle\mathcal{E}^2\rangle^{(1)}_{N_s} \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\nu\rangle$.

Instead of (228A), with such projecting, in result of the first alternative two-steps differentiation of $\mathbf{i}_\alpha$ in $dc\tau$ under $\gamma_i = \text{const}$ along a world line in $\langle\mathcal{Q}^{3+1}\rangle_c$, we obtain the so-called *tangent time-like pseudoscrew* (with respect to slope to frame axis $\vec{ct}^{(1)}$) with the constant inclination of the curve:

$$\left\{\begin{array}{l} \left\{\dfrac{\mathbf{d}\mathbf{i}_\alpha(c\tau)}{d(c\tau)}\right\}_\gamma = \mathcal{Y}_{cos}\cdot\mathbf{i}_1 + \mathcal{K}_\nu\cdot\mathbf{b}_\nu = \mathcal{C}_R\cdot\mathbf{i}_\nu = \dfrac{w^*_{\alpha_1}}{c}\cdot\mathbf{i}_\nu, \\ -(\mathcal{Y}_{cos})^2 + (\mathcal{K}_\nu)^2 = -(\mathcal{C}_R)^2 = (i\mathcal{C}_R)^2 = -\left(\dfrac{w^*_{\alpha_1}}{c}\right)^2 = \left(\dfrac{iw^*_{\alpha_1}}{c}\right)^2; \end{array}\right\} \Rightarrow \tag{247A}$$

$$\Rightarrow \left\{\begin{array}{l} di\alpha_1\cdot\mathbf{i}_\nu = \cos i\gamma_i\; di\alpha_1\cdot\mathbf{i}_1 + \sin i\gamma_i\; di\alpha_1\cdot\mathbf{b}_\nu, \\ -[d\lambda/R]^2 = -d\alpha_1{}^2 = d(i\alpha_1)^2 = -\cosh^2\gamma_i\; d\alpha_1{}^2 + \sinh^2\gamma_i\; d\alpha_1{}^2. \end{array}\right\} \tag{248A}$$

Such a type of differentiation with as if trihedron $\hat{E}_m^{(3)} = \langle\mathbf{b}_\nu,\ \mathbf{b}_\alpha,\ \mathbf{i}_1\rangle$ leads to summary time-like imaginary $di\alpha$ with unity vector $\mathbf{i}_\nu$ (as normal time arrow) of the world line rotation in $\langle\mathcal{Q}^{3+1}\rangle_c$, which is gotten by orthospherical rotation in the differentiated principal time arrow $\mathbf{i}_\alpha$ of its unity vector of the 3-rd binormal $\mathbf{b}_\alpha$ with $\mathbf{e}_\alpha$ into orthogonal to it the sine binormal with $\mathbf{e}_\nu$:

$$\{rot\ \Pi/2\}_{4\times4}\cdot\mathbf{b}_\alpha = \{rot\ \Pi/2\}_{4\times4}\cdot\begin{bmatrix}\mathbf{e}_\alpha\\0\end{bmatrix} = \begin{array}{|c|c|}\hline \{rot\ \Pi/2\}_{3\times3} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \\ \hline\end{array}\cdot\begin{bmatrix}\mathbf{e}_\alpha\\0\end{bmatrix} = \mathbf{b}_\nu = \begin{bmatrix}\mathbf{e}_\nu\\0\end{bmatrix}.$$

And in the second variant, the two-steps differentiation of $\mathbf{p}_\alpha$ will consist in the time-like sine orthoprocession $\mathcal{Y}_{sin} = \sinh\gamma_i \cdot \mathcal{C}_R = \sinh\gamma_i \cdot (w^*/c)$ and the space-like normal cosine curvature $\mathcal{K}_\mu = \cosh\gamma_i \cdot \mathcal{C}_R = \cosh\gamma_i \cdot (w^*/c)$ with the complete pseudo-Euclidean space-like orthospherical rotation $d\alpha$ of the world line with point M at the velocity $d\alpha/d\tau = w^*_\alpha$ around $\overrightarrow{ct}^{(1)}$. The Euclidean projected part of rotation $d\alpha$ acts as if around axis $\mathbf{b}_\nu$ of the *cosine normal plane* $\langle\mathcal{E}^2\rangle^{(1)}_{Nc} \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\mu\rangle$. It is expressed by the *Absolute pseudo-Euclidean Pythagorean theorem* in the exterior right triangle from complete $d\alpha_2$, sine orthoprocession and cosine normal curvature in the Looking Glass of entire 4D space-time $\langle\mathcal{P}^{3+1}\rangle$.

Instead of (238A), with such projecting, after the second alternative two-steps differentiation of $\mathbf{p}_\alpha$ in $dc\tau$ under $\gamma_i = \text{const}$ along a world line in $\langle\mathcal{Q}^{3+1}\rangle_c$, we get the so-called *cotangent space-like pseudoscrew* (with respect to slope to frame axis $\overrightarrow{ct}^{(1)}$) with the constant inclination of the curve:

$$\left\{\begin{array}{l} \left\{\dfrac{d\mathbf{p}_\alpha(c\tau)}{d(c\tau)}\right\}_\gamma = \mathcal{Y}_{sin}\cdot\mathbf{i}_1 + \mathcal{K}_\mu\cdot\mathbf{b}_\mu = \mathcal{C}_R\cdot\mathbf{p}_\mu = \dfrac{w^*_{\alpha_2}}{c}\cdot\mathbf{p}_\mu = \dfrac{d\alpha_2}{d(c\tau)}\cdot\mathbf{p}_\mu, \\ \\ -(\mathcal{Y}_{sin})^2 + (\mathcal{K}_\mu)^2 = +(\mathcal{C}_R)^2 = \left(\dfrac{w^*_{\alpha_2}}{c}\right)^2 = \left(\dfrac{d\alpha_2}{d(c\tau)}\right)^2; \end{array}\right\} \Rightarrow \qquad (249A)$$

$$\Rightarrow \left\{\begin{array}{l} d\alpha_2\cdot\mathbf{p}_\mu = \sin i\gamma_i\; d\alpha_2\cdot\mathbf{i}_1 + \cos i\gamma_i\; d\alpha_2\cdot\mathbf{b}_\mu, \\ \\ +[d\lambda/R]^2 = d\alpha_2{}^2 = -\sinh^2\gamma_i\; d\alpha_2{}^2 + \cosh^2\gamma_i\; d\alpha_2{}^2. \end{array}\right\} \qquad (250A)$$

In this variant with as if trihedron $\hat{E}^{(3)}_m = \langle\mathbf{b}_\mu,\ \mathbf{b}_\alpha,\ \mathbf{i}_1\rangle$ normal pseudonormal $\mathbf{p}_\mu$ is gotten by orthospherical rotation in the differentiated principal pseudonormal $\mathbf{p}_\alpha$ only its unity vector of the 3-rd binormal $\mathbf{b}_\alpha$ with $\mathbf{e}_\alpha$ into orthogonal to it the cosine binormal with $\mathbf{e}_\nu$ under the cosine slope:

$$\{rot\ \Pi/2\}_{4\times4}\cdot\mathbf{b}_\alpha = \{rot\ \Pi/2\}_{4\times4}\cdot\begin{bmatrix}\mathbf{e}_\alpha\\ 0\end{bmatrix} = \begin{array}{|c|c|}\hline \{rot\ \Pi/2\}_{3\times3} & \mathbf{0}\\ \hline \mathbf{0}' & 1\\ \hline\end{array}\cdot\begin{bmatrix}\mathbf{e}_\alpha\\ 0\end{bmatrix} = \mathbf{b}_\mu = \begin{bmatrix}\mathbf{e}_\mu\\ 0\end{bmatrix}.$$

In (247A), (248A) and (249A), (250A), their pseudo-Euclidean complete angular differential motions $di\alpha$ and $d\alpha$, i. e., upper and below of the isotropic light cone, with physical angular velocities $w^*_\alpha = d\alpha/d\tau$ are displayed in the projective normal sine and cosine Euclidean planes as two progenitor planetary motions, usual and superlight. Moving along two pseudoscrewed world lines (of tangent and cotangent types), they are rotated around $\overrightarrow{ct}^{(1)}$ at the perpendicular time-like time-arrow $\mathbf{i}_\nu$ and at the perpendicular space-like complete pseudonormal $\mathbf{p}_\mu$. *These artificial differentials $di\alpha$ and $d\alpha$ are strained visually along $\mathbf{i}_1$, due to their pseudo-Euclidean metric!*

We get a wonderful and amazing result, consisting in the fact, that the pseudo-Euclidean angular motion, produced identically from the equal planetary angular movement $d\alpha_1$ in time in the sine normal Euclidean plane $\langle\mathcal{E}^2\rangle^{(1)}_{Ns} \equiv \langle\mathbf{b}_\nu, \mathbf{b}_\alpha\rangle$, is displayed entirely in (247A), (248A) from its two sine and cosine orthoprojections into the imaginary time-like motion $di\alpha$ of a pseudoscrewed world line in $\tilde{E}_1$ of the 3D complex quasi-Euclidean space-time $\langle\mathcal{Q}^{2+1}\rangle_{II_s}$ with its imaginary time-arrow $\overrightarrow{ct}^{(1)}$, according to the original approach of Henry Poincaré to creation of the STR in June 1905.

It is a consequence of the fact, that rotation $di\alpha$ around $\overrightarrow{ct}^{(1)}$ is executed at the imaginary perpendicular tangent $\mathbf{i}_\nu$ under its also hyperbolic inclination to $\overrightarrow{ct}^{(1)}$. From (247A) we see that the local time-arrow $\overrightarrow{ct}$ with tangent $\mathbf{i}_\alpha$ to this world line is rotated *entirely* at the imaginary angular differential $di\alpha$ too, but with its space-like Euclidean sine normal part at the 3-vector $\mathbf{b}^{(1)}_\nu$ in the *sine normal plane* $\langle\mathcal{E}^2\rangle^{(1)}_{Ns}$. Therefore similar time-like screwed rotations-motions are accompanied also by the Thomas precession – see above and in Ch. 7A. The Thomas precession acts around the 3-rd normal precessing axis $\mathbf{b}^{(1)}_\mu$ *under complete description of this time-like pseudoscrewed motion in the space-time $\langle\mathcal{Q}^{3+1}\rangle_c$ by Poincaré–Minkowski!* Hence, by such a way, the Thomas precession in time is propagated on any relativistic time-like pseudoscrewed motions – so, as of electrons, sputniks of planets up to big and very big astronomical objects with their relativistic motions of a planetary type (purely spherical and elliptical).

But in (249A), (250A), we revealed a space-like Euclidean *cosine normal part* at the 3-vector $\mathbf{b}^{(1)}_\mu$ in the *cosine normal plane* $\langle\mathcal{E}^2\rangle^{(1)}_{Nc}$, projected from the pseudo-Euclidean complete rotation-motion $d\alpha_2$ of a space-like screwed world line in $\tilde{E}_1$ of the 4D complex quasi-Euclidean space-time $\langle\mathcal{Q}^{3+1}\rangle_c$. However a cosine Euclidean part of this rotation $d\alpha$ does not lead namely to the Thomas precession!

* * *

The resulting relations (247A)–(250A) give us the screwed shape of these curves with kinematic, and it obviously should be repeated as a result of their permanent orthospherical rotation $d\alpha$. Consequently, these relations alone are not sufficient for the overall formation of such curves, since there are no yet some mathematical condition that ensure continuous and smooth connectivity of all their turns – see its below. Then, in result of the integration, we must obtain the pseudoscrewed curves of tangent and cotangent types, i. e., with gentle and steep slopes, and also as right and left turned due to two possible signs of $d\alpha$ directions.

Let us point out another unusual features of these orthospherical rotations along both pseudoscrewed curves with both their differentials $di\alpha_{(1)}$ and $d\alpha_{(2)}$, very important for full understanding their tensor trigonometric arrangement. The fact is that complete angular differentials in relations (247A)–(250A) are of an artificial nature, since they were obtained by combination of two time and space orthoprojections into united one – formally also with two-steps differentiation along a curve. That is why, both these artificial orthospherical rotations have the pseudo-Euclidean nature and metric, and they are situated and act in their 3D pseudo-Euclidean binary spaces. Though their formally equal progenitors $d\alpha$ are situated in their sine and cosine normal Euclidean planes. Hence, a coincidence of scalar forms of these pseudo-Euclidean differentials with the true Euclidean differentials $d\alpha$ in these normal Euclidean planes only means formal equality of these angular differentials – artificial and real. It is such a feature leads to number of the unusual paradoxes of screwed and pseudoscrewed curves with corresponding to them number of characteristics, right triangles with additional Pythagorean theorems, all described below in details!

Note, that these features hold in the spherical case too for the analogical screwed right and left curves in the quasi-Euclidean space of cotangent and tangent types, i. e., with steep and gentle inclinations; and also with conservation of value $d\alpha$ on the curve.

Using locally hyperboloidal model for both types of two pseudoscrewed world lines, we can relate them to the central cylindrical region of the concomitant hyperboloid I – upper and below of the isotropic cone with respect to its central circular zone – an equator of the Euclidean radius $R = r$. This consists on its surface the coincided with them the time-like motion $di\alpha$ and space-like motion $d\alpha$ up to 1-st order of differentiation in $\langle Q^{3+1}\rangle_c$. We use $R = r$ as radius-parameter of this concomitant hyperboloid I and of its central zone and r as the same Euclidean radius of both progenitor planetary movements – usual and superlight!

Thus, above we considered preliminary the main aspects for correct construction of two types screwed world lines and regular curves in the both binary metric spaces.

However using above in (243A)–(250A), and before in (132A), (133A) and (225A), (235A) the complex quasi-Euclidean binary space as 4D space-time by Poincaré, we must add to Chs. 5 and 8A, that not only real-valued quasi-Euclidean spaces, but and complex-valued ones, including $\langle Q^{3+1}\rangle_c$ by Poincaré, have the Euclidean metric tensor and the reflector tensor analogous to one for the pseudo-Euclidean binary space, for which it serves as metric and reflector tensors. *This our mark is necessary for executions of any reflective operations with reflector tensor in these binary spaces.* Its relative complex osculating quasiplane and real-valued bonormal's Euclidean plane form the quasiorthogonal direct sum as the absolute 4D Poincaré – Minkowski space-time in two presentations (241A-II).

Beside similar relative quasiplane and Euclidean binormal's plane form also the quasi-orthogonal direct sum (242A) as the real-valued absolute 4D quasi-Euclidean binary space.

Let us compare these direct pseudo-orthogonal, complex quasi-orthogonal and real-valued quasi-orthogonal sums with the direct sums in general formulae (150), (160), (500), (174A). However it is from the introduced paired Special planes, the sine and cosine orthoprojections of the true complete angular differentials $d\gamma_t$ and $d\alpha$ are realized separately in (243A) in all 1-st metric forms of world lines and regular curves in the binary metric spaces with $q = 1$.

* * *

In (247A), (248A), the complete imaginary differential $di\alpha_{(1)}$ leads to the integrated time-like pseudoscrew $i\alpha_{(1)}$. On the basis, we'll construct this *pseudoscrew* with its true movable trihedron in $\langle\mathcal{P}^{2+1}\rangle_{II}$, the time-like cosine binormal $\mathbf{i}_1$ and the space-like sine binormal $\mathbf{b}_\nu$ acting under cosine and sine slopes to $\overrightarrow{ct}$. This will be a logical completion of our differential tensor trigonometry approach to the theory of world lines developed in $\langle\mathcal{P}^{3+1}\rangle$ and $\langle\mathcal{P}^{2+1}\rangle$.

Let in (228A) $d\gamma_i = 0 \rightarrow \mathcal{K}_\alpha = 0$ with $w^*_\alpha = d\alpha/d\tau$. Physical driving of the pseudoscrew is planetary circular movement in the original Euclidean plane $\langle\mathcal{E}^2\rangle$ in space-time $\langle\mathcal{P}^{2+1}\rangle_{II}$. Such driving alone is not enough to form the full curve, rotated with the time-arrow $\overrightarrow{ct}$ permanently on the angle α, otherwise it will have self-intersections in process of rotation. To avoid this, we reveal an additional *progressive motion* of such a world point M parallel to the frame axis $\overrightarrow{ct}$ under condition of all motions synchronism and continue of the pseudoscrew. It is the *orthoprocession* $\mathcal{Y}_{cos}$ gives this progressive motion in time parallel $\overrightarrow{ct}$ from moving orthoprojection M' of a current world point M and adds to curvature $\mathcal{C}_\mathcal{R}$. The world line as a whole is rotated by space-like $d\alpha$ and moves progressively parallel $\overrightarrow{ct}$. But its general pseudo-curvature $\mathcal{C}_\mathcal{R}$ and its pseudo-radius $R_C = 1/\mathcal{C}_\mathcal{R}$ remain constant. When the curve makes a turn at angle $d\alpha = 1rad$, the point M passes along its arc-segment R_C. Euclidean projection of this segment is opposite to the acute angle γ_i in its vertex. Therefore its length is $r = \sinh\gamma_i \cdot R_C$. The projection of this segment onto $\overrightarrow{ct}$ is $s = \cosh\gamma_i \cdot R_C$. Under the motion $d\alpha = 1rad$, we have a *parametric pseudo-Euclidean right triangle* $\mathbf{A}$ with hypotenuse R_C (the curve arc length) and legs: r (the planetary movement radius) and s (the screw step) with its pseudo-Euclidean Pythagorean theorem $s^2 - r^2 = R_C^2 \rightarrow\ r/s = \tanh\gamma_i \leq 1$. This interior right triangle $\mathbf{A}$ (Ch. 6) ensures the formation of this curve without self-intersection.

The differentiation of the *rotated tangent* $\mathbf{i}_\alpha$, *alternative* to (225A), but now under the constant angle γ_i and rotation α of time arrow $\overrightarrow{ct}$, produces the unity 4-vector $\mathbf{i}_\nu$ with its $\mathbf{e}_\nu$, formed by the spherical shift of the tangent $\mathbf{i}_\alpha$ as $rot\ \Pi/2 \cdot \mathbf{i}_\alpha = \mathbf{i}_\nu$. It is the unity 4-vector of the *normal tangent*, perpendicular to the principal tangent as $\mathbf{i}_\alpha \perp \mathbf{i}_\nu$ since $\mathbf{e}_\alpha \perp \mathbf{e}_\nu$ in $\tilde{E}_1 = \{I\}$. And with the synchronous orthoprocession $\mathcal{Y}$ of the current point M along $\overrightarrow{ct}$ with its supervelocity $c^* = \cosh\gamma_i \cdot c$, we get so the 2-nd *kind of uniform curvilinear motion* as the *pseudoscrew* generated by a circular planetary movement of a body M at $v = c\cdot\tanh\gamma_i$, $d\gamma_i = 0$, $d\alpha/d\tau = w^*_\alpha$, $d\alpha/dt = w_\alpha$ at the metric tensor $I^\mp$, with the new Absolute pseudo-Euclidean Pythagorean theorem in $\langle\mathcal{P}^{2+1}\rangle_{II}$ with sine $\mathbf{b}_\nu$ and cosine $\mathbf{i}_1$ binormals:

$$\left.\begin{aligned}
&\left\{\frac{d\mathbf{i}_\alpha(c\tau)}{d(c\tau)}\right\}_\gamma = \cosh\gamma_i \cdot \frac{w^*_\alpha}{c}\cdot\mathbf{i}_1 + \sinh\gamma_i\cdot\frac{w^*_\alpha}{c}\cdot\mathbf{b}_\nu = \mathbf{y} + \mathbf{k}_\nu = \mathbf{h}_\nu = \\
&= \mathcal{Y}\cdot\mathbf{i}_1 + \mathcal{K}_\nu\cdot\mathbf{b}_\nu = \cosh\gamma_i\cdot\frac{w^*_\alpha}{c}\cdot\begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix} + \sinh\gamma_i\cdot\frac{w^*_\alpha}{c}\cdot\begin{bmatrix}\mathbf{e}_\nu\\ 0\end{bmatrix} = \mathcal{C}_R\cdot\mathbf{i}_\nu = \\
&= \frac{w^*_\alpha}{c}\cdot\mathbf{i}_\nu = \frac{w^*_\alpha}{c}\cdot\begin{bmatrix}\sinh\gamma_i\cdot\mathbf{e}_\nu\\ \cosh\gamma_i\end{bmatrix} \Rightarrow \boxed{\mathcal{C}_\mathcal{R}{}^2 = \mathcal{Y}^2 - \mathcal{K}_\nu^2 = (w^*_\alpha/c)^2 > 0}\,.
\end{aligned}\right\} \qquad (251A)$$

Here: $\mathbf{i}_1$ is the unity 4-vector as the time-like cosine binormal for the *progressive orthoprocession* $\mathcal{Y}$ along rotated time-arrow $\overrightarrow{ct}^{(1)} = s\cdot\alpha\mathbf{i}_1$ at its supervelocity c^*, implemented by the pseudoscrewed motion of object M along a world-line with 4-velocity $\mathbf{c}$ of Poincaré; $\mathbf{h}_\nu = \mathcal{C}_R\cdot\mathbf{i}_\nu$ is a 4-vector of general pseudocurvature directed along a normal tangent $\mathbf{i}_\nu$ of the curve in $\langle\mathcal{P}^{2+1}\rangle_{II}$ to the current point M' as orthoprojection of M onto $\overrightarrow{ct}$,
$\mathcal{Y} = 1/R_Y = \cosh\gamma_i\cdot w^*_\alpha/c$ is a progressive time-like *orthoprocession* of the world line with its point M and its orthoprojection M' along its unity vector $\mathbf{i}_1$ and time arrow $\overrightarrow{ct}^{(1)}$ as a time-like progressive part (!) of the pseudoscrewed motion;
$\mathcal{K}_\nu = 1/R_K = \sinh\gamma_i\cdot w^*_\alpha/c$ is a normal curvature of the world line with its sine binormal $\mathbf{p}_\nu$;
$s = c/w_\alpha = \cosh\gamma_i\cdot c/w^*_\alpha = \cosh\gamma_i\cdot R_C = \cosh^2\gamma_i\cdot R_Y$ is pseudoscrew step at $d\alpha = 1$rad,
$r = R = v/w_\alpha = \sinh\gamma_i\cdot c/w^*_\alpha = \sinh\gamma_i\cdot R_C = \sinh^2\gamma_i\cdot R_K$ is pseudoscrew Euclidean radius, here R is radius-parameter *of the concomitant hyperboloid I*, tangent to it (see above).

Note one else, that the interior right triangle **A** (see about in sect 6.4), introduced above, ensures the formation of this pseudoscrew without self-intersection in the process of continuous motion of a world point M due to its Euclidean rotation and orthoprocession with c^* along $\overrightarrow{ct}$ at $s = r\cdot(c^*/v^*) = r\cdot(c/v) = r\cdot\coth\gamma$. In the Minkowski space-time, this is set independently by Nature itself, since the parametric triangle **A** is similar to the interior right triangle V of the three velocities, where the hypotenuse c is the Poincaré 4-velocity of the point M along a world line and the leg: v^* is the Euclidean proper 3-velocity of the point M and c^* is the scalar time's supervelocity along $\overrightarrow{ct}$ or of M orthoprojection onto $\overrightarrow{ct}$.

In the usual 1-st trihedron, its principal tangent $\mathbf{i}_\alpha$ is here an impotent vector – without curvature ($\mathcal{K}_\alpha = 0$ at $d\gamma = 0$, but $\gamma \neq 0$), although $\mathbf{i}_\alpha(c\tau)$ exists. The normal time-arrow $\mathbf{i}_\nu$ has the curvature $\mathcal{C}_R$, determined by the curvature at the space-like sine binormal $\mathbf{b}_\nu$ and the orthoprocession at the time-like cosine binormal $\mathbf{i}_1$. That's why, for description of the pseudoscrew, we apply as the our artificial trihedron $\hat{E}^{(3)}_m = \langle \mathbf{b}_\nu,\ \mathbf{b}_\alpha,\ \mathbf{i}_1\rangle$ with 3 curvatures! In (251A), in addition, we defined $\mathbf{y} = \mathcal{Y}\cdot\mathbf{i}_1$ (with its unity vector $\mathbf{i}_1$ from this trihedron) as the cosine time-like projection of 4-vector $\mathbf{h}_\nu = \mathcal{C}_R\cdot\mathbf{i}_\nu$ of the general pseudocurvature $\mathcal{C}_R$ and as the time-like vector *orthoprocession* in time (as if time-like "torsion"). It is not a rotation, but it is a purely progressive motion of a world line parallel to $\overrightarrow{ct}^{(1)}$. This orthoprocession in time $\mathcal{Y}$ is a *permanent inherent factor of STR, relating to all world lines*! This pseudoscrew is produced by combination of cosine progressive orthoprocession $\mathcal{Y}$ along $\mathbf{i}_1$ and Euclidean rotation $d\alpha$ around $\mathbf{i}_1$. $\mathcal{Y}\cdot\mathbf{i}_1$ influences on the geometry of world lines and curves, as it strains them along $\mathbf{i}_1$; γ_i affects on r/s, w^*_α affects on s. Such a screw can be a model of a physical centripetal accelerator with these parameters. More generally, a planet or a sputnik is rotated around a star or a big planet on orbit of the Euclidean radius $r = v^*/w^*_\alpha = v/w_\alpha$.

As the extreme example of such screwed motions, we give a pseudoscrewed world line of a photon circular movement around the very massive Star, realized on the isotropic cone with velocities $\mathbf{c} = c\cdot\mathbf{e}_\alpha$ and w^*_α of the radius $r = s = c/w$ at $|v = c,\ r = s\cdot(v/c) = s\cdot\tanh\gamma_i|$, where in the limit: $r/s = \tanh\gamma_i = 1$. Then we see that $w_\alpha = c/r$ is determined only by the radius r of the orbit. Einsteinian photon is rotated at velocities $\mathbf{c}$ and w_α around the Star as a *Black Hole* of radius $r = c/w_\alpha = fM/c^2$ and with the period $T = 2\pi r/c = 2\pi/w^*_\alpha$, predicted in 1783 by John Michell [81] with the use of the Newtonian Theories (!). Recall, that the so-called Schwarzschild's radius for the Black Hole [100] is twice more, but this may be explained by the "gravitational cosine" in (212A), Ch. 9A. It is the case, when there is really no way to check which of the two authors is right more, since this radius is theoretical!

This time-like pseudoscrew (i. e., in the usual space-time) is realized isometrically on its enveloping cylindrical pseudo-Euclidean lateral hypersurface of the Euclidean radius $R = r$. Factually this curve consists from identical repeated artificial time-like differentials $di\alpha$, when the pseudoscrewed curve makes a turn at angle $1\ rad$. Moreover, this cylindrical hypersurface as a fragment is deployed isometrically onto the analogical pseudo-Euclidean plane with translation of a pseudoscrewed world line into straight world line on it. With the tangent to both them central differential cylindrical region of the concomitant hyperboloid I (see above), these three surfaces and equivalent lines on them have the common metric!!!

This time-like pseudoscrewed motion, as a specific $3D$ world line, is realized with the inherent orthoprocession $\mathcal{Y}_{cos} = \cosh\gamma_i\cdot\ w^*_\alpha/c = \cosh\gamma_i\cdot\mathcal{C}_R$ for its progressive part along $\mathbf{i}_1$ and the normal curvature $\mathcal{K}_\nu = \sinh\gamma_i\cdot w^*_\alpha/c = \sinh\gamma_i\cdot\mathcal{C}_R$ for its rotational part in $\langle\mathcal{E}^2_K\rangle$. Necessary quantitative bond of these two partial motions is caused by the fact that both they have the common kinematic factor w^*_α at its each point M under acting of the driving planetary movement. As a result, it has the general 3-pseudocurvature $\mathcal{C}_R$:

$$\mathcal{K}_\nu = 1/R_K = \sinh\gamma_i\cdot\mathcal{C}_R = \sinh\gamma_i\cdot\ w^*_\alpha/c = g_\nu/c^2,\quad \mathbf{k}_\nu = \mathcal{K}_\nu\cdot\mathbf{b}_\nu; \tag{252A}$$

$$\mathcal{Y} = 1/R_Y = \cosh\gamma_i\cdot\mathcal{C}_R = \cosh\gamma_i\cdot\ w^*_\alpha/c,\quad \mathbf{y}_1 = \mathcal{Y}\cdot\mathbf{i}_1. \tag{253A}$$

$$\hat{E}^{(3)}_m = \langle\mathbf{b}_\nu,\ \mathbf{b}_\alpha,\ \mathbf{i}_1\rangle. \tag{254A}$$

The triple $\mathcal{K}_\nu$, $\mathcal{Y}$ (legs), $\mathcal{C}_R$ (hypotenuse) forms *interior right triangle of pseudoscrew* **P** in (251A), where $\mathcal{K}_\nu/\mathcal{Y} = \tanh\gamma_t < 1$ is its constant time-like slope. It is realized in the pseudoplane of general curvature $\langle\mathcal{P}^{1+1}\rangle_\mathcal{C} \equiv \{\mathbf{p}_\nu, \mathbf{i}_1\}$. In addition, on the cylindrical surface, we get the spherically bended interior right triangle **A1** with legs r, s and hypotenuse $R_C = 1/\mathcal{C}_R$, where $r = \sinh\gamma_t \cdot R_C$, $s = \cosh\gamma_t \cdot R_C$. (In **A1** s is coaxial to $\overrightarrow{ct}$.) Then there is invariant $s^2 - r^2 = R_C^2$. *R_C expresses the pseudo-Euclidean length of the pseudoscrew arc at $\alpha = 1$ rad.* The *identical, but flat interior right triangle* **A2** is realized in the same pseudoplane $\langle\mathcal{P}^{1+1}\rangle_\mathcal{Y}$. (In **A2** r is coaxial to $\mathbf{p}_\nu$). Their common straight leg is $s > r$.

As the **geometric paradox of all screws**, we obtain two wonderful right triangles: **P** of pseudoscrew in (251A) and **A** in their two variants above with two pseudo-Euclidean Pythagorean theorems! Their legs are proportional with common coefficient $s/\mathcal{Y} = r/\mathcal{K}_\nu$, they have equal adjacent angles. Hence, both triangles are homothetic. However their hypotenuses are inverse each another as $\mathcal{C}_R = 1/R_C$!!! (This paradox extends to screwed curves in the quasi-Euclidean space $\langle\mathcal{Q}^{2+1}\rangle$ with similar Euclidean Pythagorean theorems!)

This pseudoscrewed world line in the same pseudoplane $\langle\mathcal{P}^{1+1}\rangle_\mathcal{Y}$ generates, in addition, two pseudo-Euclidean right triangles: they are the exterior right triangle **B** and the interior right triangle **C**, with their exterior and interior pseudo-Euclidean Pythagorean theorems.

The exterior right triangle **B** has space-like hypotenuse $R_K = 1/\mathcal{K}_\nu$ (radius of sine curvature under inclination γ_t to $\langle\mathcal{E}^2\rangle$), time-like leg $R_C = \sinh\gamma_t \cdot R_K$ – opposite to γ_t, and space-like leg $b_1 = \cosh\gamma_t \cdot R_K$ – adjacent to γ_t. From triangle **B** we have $R_K^2 = b_1^2 - R_C^2 > 0$.

The interior right triangle **C** has time-like hypotenuse $R_Y = 1/\mathcal{Y}$ (radius of cosine torsion under inclination γ_t to $\overrightarrow{ct}$), time-like leg $R_C = \cosh\gamma_t \cdot R_Y$ – adjacent to γ_t, and space like leg $b_2 = \sinh\gamma_t \cdot R_Y$ – opposite to γ_t. From triangle **C** we have $R_Y^2 = R_C^2 - b_2^2 < 0$.

With first triangle **P**, this *screwed world line has 5 characteristic right triangles*! If the enveloping tangent cylinder with this screwed curve is cut along the central axis $\overrightarrow{ct}$, further to develop it into fragments of the pseudoplane and finally to add these fragments so to coincide windings of this screw, then we get the same but straight world line in the flat pseudoplane. This convincing example demonstrates very clarity, how *minimal* curving the basis flat space, even into the cylindrical space $\langle\mathcal{C}^{2+1}\rangle$, complicates in a *large* extent description of the simplest straight world line with introducing a lot of additional parameters!!!

In cylindrical coordinates, we summarize found parameters of this pseudoscrew with its tangent type till the isotropic light cone, where initially we adopt that $\tanh\gamma = r/s$: $r = R,\ \ x_1 = r\cdot\cos\alpha,\ \ x_2 = r\cdot\sin\alpha,\ \ ct = s\cdot\alpha\ \ (r = v/w_\alpha = \text{const},\ s = c/w_\alpha = \text{const})$.

$$\left.\begin{aligned}
&\sinh\gamma = r/R_C = R_C/R_K \to \sinh^2\gamma = r/R_K = r\cdot\mathcal{K}_\nu, \to r = R = \tanh\gamma\cdot s,\\
&\cosh\gamma = s/R_C = R_C/R_Y \to \cosh^2\gamma = s/R_Y = s\cdot\mathcal{Y}, \to s = \coth\gamma\cdot r;\\
&\to r^2 - s^2 = R_C^2,\ \mathcal{Y}^2 - \mathcal{K}_\nu^2 = \mathcal{C}_R^2 = 1/R_C^2 = 1/R_Y^2 - 1/R_K^2 < 0;\\
&\to b_1 = R_K\cdot\cosh\gamma,\ b_2 = R_Y\cdot\sinh\gamma,\quad b_2/b_1 = \tanh^2\gamma,\\
&\to R_K^2 = b_1^2 - R_C^2 > 0,\ R_Y^2 = R_C^2 - b_2^2 < 0;\ R_Y/R_K = \mathcal{K}_\nu/\mathcal{Y} = \tanh\gamma.
\end{aligned}\right\}\quad (255A)$$

(For superlight pseudoscrew in (249A, 250A) of the cotangent type, we adopt $\coth\gamma = r/s$.)

For the pseudoscrewed motion (for instance, in accelerator), the space-like hyperbolic and spherical angular velocities with accelerations are the following [see also in (165A)–(168A)]:
$v^* = c\cdot\sinh\gamma,\ w_\alpha^* = v^*/r,\ v = c\cdot\tanh\gamma,\ w_\alpha = v/r;\ [w_\theta^* = d\theta/d\tau = -(\cosh\gamma - 1)\cdot w_\alpha^*]$;
$\overset{\perp}{g}_K = g_\nu = c\cdot\sinh\gamma\cdot w_\alpha^* = v^*\cdot w_\alpha^* = (v^*)^2/r = c^2\mathcal{K}_\nu = c^2/R_K$,
$\overset{\perp}{g}{}^{(1)} = \overset{\perp}{g}\cdot\text{sech}\ \gamma = v\cdot w_\alpha^* = v^*\cdot w_\alpha,\ (\overline{\overline{g}} = 0)$.

And for the time-like part of (251A) there hold:
$c^* = \cosh\gamma\cdot c$ is the proper velocity of the coordinate time t stream for a world line,
$\overset{\perp}{g}_Y = c^2\mathcal{Y} = c^2/R_Y = c^2\mathcal{C}_R\cosh\gamma = c\cdot\cosh\gamma\cdot w_\alpha^* = c^*\cdot w_\alpha^* = (c^*)^2/s$.

The main peculiarity of screwed curves (without hyperbolic or spherical curvature) is such, that all they are produced not only by rotation of the binormal, because there is else an inherent operation of the progressive orthoprocession along the frame axis. The latter is bonded one-to-one with this rotation. A pseudoscrew consists from the complementary space (here sine) and time (here cosine) pseudoorthogonal parts. Mathematically the vector orthoprocession is mixed with the tangent $\mathbf{i}_\alpha$ to a curve. But physically independence of the vector orthoprocession and the Euclidean rotation of $\mathbf{i}_\alpha$ may be inferred by their different types of motions. Indeed, such orthoprocessions do not relate to the group of rotation, they relate to the independent group of translations as progressive motions in the enveloping binary space! In the Euclidean space, it is a progressive torsion due to the Frenet–Serret theory. In the pseudo- and quasi-Euclidean binary spaces, it is a progressive orthoprocession.

* * *

Next we construct the spherical type of 3D screw in $\langle \mathcal{Q}^{2+1}\rangle$ by differentiation-rotation of the tangent $\mathbf{i}$ in $d\alpha$, as the *uniform* absolute orthospherical motion $d\alpha$ along a regular curve having our artificial, but true trihedron $\hat{E}_m^{(3)} = \langle \mathbf{b}_\nu,\ \mathbf{b}_\alpha,\ \mathbf{t}_1\rangle$ with two bonded curvatures! Here rotational driving alone is not enough to form this full curve, otherwise it will be mixed with itself. To avoid this, we'll use the progressive orthoprocession of the point M along the rotated axis $\overrightarrow{y}$ as in (247A), (248A). Since this curve as a whole only rotates through angle $d\alpha$, its general pseudocurvature $\mathcal{C}_\mathcal{R}$ and its radius $R_C = 1/\mathcal{C}_\mathcal{R}$ remain constant. When the curve makes a turn through an angle $d\alpha = 1rad$, the point M passes along it arc R_C. The Euclidean projection of this segment is opposite to the angle of motion $\varphi_\imath$, therefore its length is $r = \sin\varphi_\imath \cdot R_C$, the projection of this segment onto $\overrightarrow{y}$ is $s = \cos\varphi_\imath \cdot R_C$. Under $d\alpha = 1rad$, we have a *parametric Euclidean right triangle* $\mathbf{A}$ with hypotenuse R_C (curve arc length) and legs: r (Euclidean radius of rotation) and s (step) with its Pythagorean theorem $s^2 - r^2 = R_C^2 \to\ r/s = \tan\varphi_\imath = \cot\xi_\imath$, where $r = R$. We may use here as argument also $\xi_\imath = \pi/2 - \varphi_\imath$ and translate by analogy (323) hyperbolic formulae (255A) in spherical variant! By the abstract hyperbolic–spherical analogy with (251A), (255A), or under differentiation in $d\alpha$ along a regular curve at $\varphi_\imath = \mathrm{const}$, we get the screw *also with the orthoprocession* $\mathcal{Y}$ *and the normal curvature* $\mathcal{K}_\nu$ giving the spherical Absolute Euclidean Pythagorean theorem:

$$\left.\begin{aligned} &\left\{\frac{d\mathbf{t}_\alpha(l)}{R_C d\alpha}\right\}_\varphi = \cos\varphi_\imath \cdot \frac{1}{R_C}\cdot \mathbf{t}_1 + \sin\varphi_\imath \cdot \frac{1}{R_C}\cdot \mathbf{b}_\nu = \mathbf{y} + \mathbf{k}_\nu = \mathbf{h}_\nu = \\ &= \mathcal{Y}\cdot \mathbf{t}_1 + \mathcal{K}_\nu \cdot \mathbf{b}_\nu = \cos\varphi_\imath \cdot \mathcal{C}_\mathcal{R}\cdot \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} + \sin\varphi_\imath \cdot \mathcal{C}_\mathcal{R}\cdot \begin{bmatrix} \mathbf{e}_\nu \\ 0 \end{bmatrix} = \mathcal{C}_R \cdot \mathbf{t}_\nu = \\ &= \mathcal{C}_R \cdot \begin{bmatrix} \sin\varphi_\imath \cdot \mathbf{e}_\nu \\ \cos\varphi_\imath \end{bmatrix} \Rightarrow \boxed{\mathcal{C}_\mathcal{R}{}^2 = \mathcal{Y}^2 + \mathcal{K}_\nu^2 = (w_\alpha/v)^2 > 0;\ \varphi_\imath \in [0 \div \pi/2]}\,. \end{aligned}\right\} \quad (256A)$$

Here: $\mathcal{C}_R = 1/R_C = \mathrm{const}$ is the general curvature of this screwed curve,
$R_C = \mathrm{const}$ is the radius of this curvature and the length of the curve arc at $d\alpha = 1rad$,
$\mathbf{h}_\nu$ is the 3-vector of the general curvature, as the normal tangent to the curve;
$\mathcal{K}_\nu = 1/R_K = \sin\varphi_\imath \cdot \mathcal{C}_\mathcal{R}$ is a normal curvature of the curve with sine binormal $\mathbf{b}_\nu$.
$\mathcal{Y} = 1/R_Y = \cos\varphi_\imath \cdot \mathcal{C}_\mathcal{R}$ is the *orthoprocession* of the curve for a *progressive part of a screw*,
$s^2 - r^2 = R_C^2$, $r/s = \tan\varphi_\imath = \cot\xi_\imath$ for the screw Euclidean radius r and step s,
$r = R$ is the radius of the concomitant hyperspheroid, tangent to this screw in its Equator.

In the usual 1-st trihedron, its principal tangent $\mathbf{t}_\alpha$ is an impotent vector – without curvature ($\mathcal{K}_\alpha = 0$ at $d\varphi_\imath = 0$, but $\varphi_\imath \neq 0$), although tangent $\mathbf{t}_\alpha(l)$ exists.

Thus (!), in the case of simultaneous double motion with right and left'$\pm d\alpha$, we obtain the *double screw*, which in Molecular Biology, for some reason, is called by the *double helix* for describing the DNA structure, although in Geometry they are different curves.

Note, that in the *usual Euclidean space* $\langle \mathcal{E}^3\rangle$, the frame axis can be selected in three ways. Any each of them will have its own angle of motion as: $\Phi_1,\ \Phi_2,\ \Phi_3$ ($\sum_{s=1}^{3}\cos^2\varphi_k = 1$. In particular, if we choose some coordinate x_3 as a frame axis, then there are two variants of the complete base with x_1, x_2 or with x_2, x_1: ($\cos\varphi_1 = \sin\varphi_2 \sim \cos\varphi = \sin\xi$).

In $\langle \mathcal{Q}^{3+1} \rangle$ with the frame axis $\overrightarrow{y}$ and the *concomitant movable unity hyperspheroid* (see in Chs. 5, 8A), we can realize simultaneously particular differentiation-rotations of the tangent $\mathbf{i}$ and the quasinormal $\mathbf{i}$ along a regular curves at $v =$ const, under abstract analogy (323) with relations (243A), (244A), Then we obtain:

$$\left.\begin{array}{l} d\varphi \cdot \mathbf{n} = d\varphi_i \cdot \mathbf{n}_\alpha + \sin\varphi_i \; d\alpha_1 \cdot \mathbf{b}_\nu + \cos\varphi_i \; d\alpha_2 \cdot \mathbf{b}_\mu = \\ = \mathcal{K}_\alpha \cdot \mathbf{n}_\alpha + \mathcal{K}_\nu \cdot \mathbf{b}_\nu + \mathcal{K}_\mu \cdot \mathbf{b}_\mu = \mathcal{Y}_{cos} \cdot \mathbf{t}_1 + \mathcal{X}_{sin} \cdot \mathbf{b}_\alpha + \mathcal{K}_\nu \cdot \mathbf{b}_\nu + \mathcal{K}_\mu \cdot \mathbf{b}_\mu; \\ \{dl/R\}^2 = d\varphi^2 = d\varphi_i^2 + (\sin^2\varphi_i \; d\alpha_1{}^2 + \cos^2\varphi_i \; d\alpha_2{}^2) = \\ = \cos^2\varphi_i \; d\varphi_i^2 + \sin^2\varphi_i \; d\varphi_i^2 + \sin^2\varphi_i \; d\alpha_1{}^2 + \cos^2\varphi_i \; d\alpha_2{}^2 > 0 \Rightarrow \mathcal{C}_\mathcal{R}{}^2 = \\ = w_{\varphi_i}{}^2/v^2 + \sin^2\varphi_i \cdot w_{\alpha 1}{}^2/v^2 + \cos^2\varphi_i \cdot w_{\alpha 2}{}^2/v^2 = \mathcal{K}_\alpha^2 + \mathcal{K}_\nu^2 + \mathcal{K}_\mu^2 > 0. \end{array}\right\} \quad (257A, 258A)$$

Two principal spherical arcs $d\varphi_i$ – primary and mutual are situated in quasiplane $\langle \mathcal{Q}^{1+1} \rangle_S^{(m)} \equiv \langle \mathbf{n}_\alpha, \mathbf{t}_\alpha \rangle$ of entire $\langle \mathcal{Q}^{3+1} \rangle$, presented with two bonded primary and mutual spherical arcs $d\varphi$ at Figure 3. The first is the spherical osculating quasiplane of spherical curvature $\mathcal{K}_\alpha$. Principal spherical angles and differentials act as binary ones too. And both binary differentials $d\varphi$ act also symmetrically with respect to as if specific cone in the middle between them (Ch. 5). Here they express the simultaneous spherical identical, but contrary differential motions-rotations of $\mathbf{t}_\alpha$ and $\mathbf{n}_\alpha$, *according to the binary structure of our spherical tensor of motion* (313), with their permanent symmetry relative to this middle cone. The sine $\mathbf{b}_\nu$ and cosine $\mathbf{b}_\mu$ vectorial binormal with their sine and cosine normal curvatures act in the own sine and cosine Euclidean normal planes. Both planes are not really divided and even are bonded, thank to the simple connection of two complementary spherical angles, in that number, on the common here concomitant 3D hyperspheroid. The third independent arc $d\alpha_3$ expresses free complete orthospherical rotations in the binormal's Euclidean plane $\langle \mathcal{E}^2 \rangle_B^{(1)} \equiv \langle \mathbf{b}_\nu, \mathbf{b}_\mu \rangle$ as the Cardano gimbal in the Euclidean space $\langle \mathcal{E}^3 \rangle$. It is this creates an own trigonometric harmony of the 4D binary quasi-Euclidean space.

* * *

Let's go back to motions *with variable two parameters of roth* $\Gamma_i = F(\gamma_i, \mathbf{e}_\alpha)$ in $\langle \mathcal{P}^{3+1} \rangle$. Pay essential attention to the fact that simultaneously with an instantaneous point M of a world line and of accompanied movable unity hyperboloid I with their common time-like tangent $\mathbf{i}_\alpha{}^{(I)}$ and space-like pseudonormal $\mathbf{p}_\alpha{}^{(I)}$, moving all at 4-velocity $\mathbf{c}$, there is the point N on the conjugate hyperboloid II with its also conjugate space-like tangent $\mathbf{i}_\alpha{}^{(II)}$ and time-like pseudonormal $\mathbf{p}_\alpha{}^{(II)}$. In Ch. 12 we denoted such conjugate points of two hyperboloids as $\mathbf{v}$ and $\mathbf{u}$ in a textual part and also at Figure 4. Between all six basis vectors at the point M of a world line at $d\gamma_i \neq 0$, $d\alpha \neq 0$ and at points M and N of hyperboloids, there are such one-to-one correspondences with differential relations in $\langle \mathcal{P}^{3+1} \rangle$ under $\{I^\pm\}$:

$$\left.\begin{array}{l} \mathbf{i}_\alpha = \mathbf{i}_{(I)} = \mathbf{p}_{(II)} = \mathbf{r}_{(II)} = [\mathbf{p}_\alpha]'_\alpha - \text{of Poincaré 4-velocity in (218A) at a world line,} \\ \mathbf{p}_\alpha = \mathbf{p}_{(I)} = \mathbf{r}_{(I)} = \mathbf{i}_{(II)} = [\mathbf{i}_\alpha]'_\alpha - \text{of 4-acceleration in (228A) at a world line;} \\ \mathbf{b}_\nu = \mathbf{b}_{\nu(II)} \sim \mathbf{e}_{\nu(II)} - \text{of normal 3-shift } \sinh\gamma_i \; d\alpha_1 \text{ on II or of sine 3-acceleration,} \\ \mathbf{b}_\mu = \mathbf{b}_{\mu(I)} \sim \mathbf{e}_{\mu(I)} - \text{of normal 3-shift } \cosh\gamma_i \; d\alpha_2 \text{ on I or of cosine 3-acceleration;} \\ \{\mathbf{b}_\mu\}'_\gamma = \{\mathbf{b}_\nu\}; \;\; \{\mathbf{b}_\nu\}'_{\gamma,\alpha} = 0. \end{array}\right\} \quad (259A)$$

We obtain in 5-th raw last our third and forth formulae along a world line, in addition, to our previous first (228A) and second (238A) hyperbolic ones !!!

Thus, let us assume that, in the neighborhood of the world point M on the world line, there is such a branch of the time-like hyperbola as in its osculating pseudoplane at point M. We choose the point M also as the instantaneous point M of the concomitant hyperboloid I. From this point M we mentally draw the principal unity tangent $\mathbf{i}_\alpha$ to this hyperboloid I and to this world line. Its length and direction coincide with the directed segment ON till the conjugate point N of the hyperboloid II. This directed segment ON for the hyperboloid II is its principal pseudonormal $\mathbf{p}_\alpha$ at N and the tangent $\mathbf{i}_\alpha$ at M of these world line and hyperboloid I under the instantaneous motion angle γ_i. Accordingly, $\mathbf{e}_\alpha$, $\mathbf{e}_\nu$, $\mathbf{e}_\mu$ are three possible unity Euclidean perpendiculars in $\langle \mathcal{E}^3 \rangle \subset \langle \mathcal{P}^{3+1} \rangle$, with respect to points O, M, N.

Correspondences (259A) make it possible to better see the reason, why motions along a world-line are displayed on both accompanying hyperboloids. It was presented at Figure 4. So, the first two-steps differentiations-rotations of tangent $\mathbf{i}_{\alpha(I)}$ in (225A) are executed in the 1-st step by space-like hyperbolic motion $d\gamma_i$ on the osculating pseudoplane $\langle\mathcal{P}^{1+1}\rangle_{(N)}$, which we see at Figure 4 on the hyperboloid II, and in the 2-nd step by perpendicularly to the latter orthospherical motion with its sine slope in the Euclidean plane of the sine normal curvature $\langle\mathcal{E}^2\rangle^{(m)}_{Ns} \equiv \langle\mathbf{p}_\alpha, \mathbf{b}_\nu\rangle^{(m)}$ under its sine slope. This plane is tangent to the hyperboloid II at the point N. Hence the motion is transferred mathematically from the hyperboloid I onto the hyperboloid II, according to the second and third bonds in relations (259A). The second two-steps differentiations-rotations of the pseudonormal $\mathbf{p}_{\alpha(I)}$ in (235A) are executed in the 1-st step by time-like hyperbolic motion $d\gamma_i$ on the osculating pseudoplane $\langle\mathcal{P}^{1+1}\rangle_{(M)}$, which we see at Figure 4 on the hyperboloid I, and in the 2-nd step by perpendicularly to the latter orthospherical motion with its cosine slope in the Euclidean plane of the cosine normal curvature $\langle\mathcal{E}^2\rangle^{(m)}_{Nc} \equiv \langle\mathbf{i}_\alpha, \mathbf{b}_\mu\rangle^{(m)}$ under its cosine slope. This plane is tangent to the hyperboloid I at the point M. Hence the motion is displayed mathematically on the unity hyperboloid I and along a world line, according to the first and fourth bonds in (259A). Both space-like and time-like hyperbolic motions are realized in the common pseudoplane!

As the final result, we obtain in entire $\langle\mathcal{P}^{3+1}\rangle$ all absolute parameters of a world line in $\tilde{E}_1 = \{I\}$ and $\hat{E}_m$ under permanent action of the current motion tensor $roth\ \Gamma_i = F(\gamma_i, \mathbf{e}_\alpha)$:

$$\mathbf{p}_\alpha = \begin{bmatrix} \cosh\gamma_i \cdot \mathbf{e}_\alpha \\ \sinh\gamma_i \end{bmatrix}, \quad \mathbf{p}_\nu = \begin{bmatrix} \mathbf{e}_\nu \\ 0 \end{bmatrix}, \quad \mathbf{p}_\mu = \begin{bmatrix} \mathbf{e}_\mu \\ 0 \end{bmatrix}, \quad \mathbf{i}_\alpha = \begin{bmatrix} \sinh\gamma_i \cdot \mathbf{e}_\alpha \\ \cosh\gamma_i \end{bmatrix}. \qquad (260A)$$

$$\mathcal{K}_\alpha = \eta^*_\gamma/c, \ \mathbf{k}_\alpha = \mathcal{K}_\alpha\ \mathbf{p}_\alpha; \ \mathcal{K}_\nu = \sinh\gamma_i \cdot w^*_{\alpha(1)}/c, \ \mathbf{k}_\nu = \mathcal{K}_\nu\ \mathbf{p}_\nu; \ \mathcal{Q}_\mu = \cosh\gamma_i \cdot w^*_{\alpha(2)}/c, \ \ \mathbf{q}_\mu = \mathcal{Q}_\mu\ \mathbf{p}_\mu.$$

$$\text{At } \gamma_i = 0: \ \mathbf{p}_\alpha = \mathbf{b}_\alpha = \begin{bmatrix} \mathbf{e}_\alpha \\ 0 \end{bmatrix}, \ \mathbf{p}_\nu = \mathbf{b}_\nu = \begin{bmatrix} \mathbf{e}_\nu \\ 0 \end{bmatrix}, \ \mathbf{p}_\mu = \mathbf{b}_\nu = \begin{bmatrix} \mathbf{e}_\mu \\ 0 \end{bmatrix}, \ \mathbf{i}_\alpha = \mathbf{i}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

By differential tensor trigonometry approach, such a model, with movable tetrahedron and with involvement of the two accompanying hyperboloids, describes pretty accurately, clarity and unequivocally the kinematic and dynamic of matter relativistic movement with any degree of complexity in the Minkowski space-time in their tensor-vector-scalar (tvs) forms. For the simplest understanding the model, it is enough to refer to one-to-one correspondence (259A) between four unity basis vectors. In $\langle\mathcal{P}^{3+1}\rangle$ we obtain exactly the maximal order of the absolute motions curvatures $\zeta_{max} - 1 = 3$ and all one-valued results. What is more, on the basis of executed analysis of various relative and absolute time-like and space-time motions with their tensor trigonometric models in Chs. 2A÷7A and 9A with current 10A, even with peculiarities, we showed in parallel that the well-understandable, non-contradance and clear arrangement of the Universe may be displayed from the nearest astronomical stellar environment in our Galaxy with the use of the 4D space-time by Poincaré–Minkowski, i. e., either with metric tensor $\{I^+\}$ in $\langle\mathcal{Q}^{3+1}\rangle_c$ or metric tensor $\{I^\pm\}$ in $\langle\mathcal{P}^{3+1}\rangle$. However, for the more far astronomical picture with entire Megagalaxy, if necessary, it is possible to use the BMT space-time, *with two metric tensors* (Ch. 9A), namely for lensed by gravity observable space-time. Both these theoretical arrangements of the Universe does not violate the sacred Principles and Laws of Nature with the material Higgs field and the Quantum Mechanics.

Quadruple $\hat{E}^{(4)}_m = \{\mathbf{p}_\alpha(c\tau), \mathbf{b}_\nu(c\tau), \mathbf{b}_\mu(c\tau), \mathbf{i}_\alpha(c\tau)\}$ as the *movable tetrahedron* to current world line in entire $\langle\mathcal{P}^{3+1}\rangle$ complements *both unity accompanying movable hyperboloids* and gives the asymmetric *pseudoorthogonal tensor* $U\{\mathbf{p}_\alpha, \mathbf{b}_\nu, \mathbf{b}_\mu, \mathbf{i}_\alpha\}(c\tau)$ which is connected as one-to-one with the motion tensor $roth\ \Gamma_i = F(\gamma_i, \mathbf{e}_\alpha)$ and determines completely both orientation and configuration of a world line at its point M. We accompany them below with four measureless trigonometric tensors of absolute motions for applications in pseudo-Euclidean binary spaces of dimensions $n+1$, $3+1$, $2+1$. See for $\langle\mathcal{Q}^{3+1}\rangle$ in (295A).

In entire $\langle\mathcal{P}^{3+1}\rangle$ with tetrahedron for a world line, we have tensors (261A)

$$roth\ \Gamma_i = F_h(\gamma_i, \mathbf{e}_\alpha),\ \ (F = F'),\ \langle\mathcal{P}^{n+1}\rangle \qquad U(\gamma_i, \mathbf{e}_\alpha, \mathbf{e}_\nu, \mathbf{e}_\mu),\ \ (U \neq U'),\ \langle\mathcal{P}^{3+1}\rangle$$

$\cosh\gamma_i \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}'$	$\sinh\gamma_i \cdot \mathbf{e}_\alpha$
$\sinh\gamma_i \cdot \mathbf{e}'_\alpha$	$\cosh\gamma_i$

....

$\cosh\gamma_i \cdot \mathbf{e}_\alpha$	$\mathbf{e}_\nu$	$\mathbf{e}_\mu$	$\sinh\gamma_i \cdot \mathbf{e}_\alpha$
$\sinh\gamma_i$	0	0	$\cosh\gamma_i$

.

and with two trthedrons both possible tensors (262A)

$$U(\gamma_i, \mathbf{e}_\alpha, \mathbf{e}_\nu,\ \ (U \neq U'),\ \langle\mathcal{P}^{2+1}\rangle_{(II)} \qquad U(\gamma_i, \mathbf{e}_\alpha, \mathbf{e}_\mu),\ \ (U \neq U'),\ \langle\mathcal{P}^{2+1}\rangle_{(I)}$$

$\cosh\gamma_i \cdot \mathbf{e}_\alpha$	$\mathbf{e}_\nu$	$\sinh\gamma_i \cdot \mathbf{e}_\alpha$
$\sinh\gamma_i$	0	$\cosh\gamma_i$

....

$\cosh\gamma_i \cdot \mathbf{e}_\alpha$	$\mathbf{e}_\mu$	$\sinh\gamma_i \cdot \mathbf{e}_\alpha$
$\sinh\gamma_i$	0	$\cosh\gamma_i$

.

$$(U = roth\ \Gamma_i \cdot rot\ \Theta_i\ \Rightarrow\ roth\ \Gamma_i = \sqrt{UU'},\ rot\ \Theta = \sqrt{UU'}^{-1} \cdot U = roth\ (-\Gamma_i) \cdot U.)$$

The 3D pseudoscrewed motion is described without an exception in its true trihedron (254) with its time-arrow, space-like sine and time-like cosine binormals plus impotent $\mathbf{b}_\alpha$:

$$U(\gamma_i, w^*{}_\alpha; \mathbf{e}_\alpha, \mathbf{e}_\nu) = \begin{array}{|c|c|c|}\hline \sinh\gamma_i \cdot \mathbf{e}_\nu & \mathbf{e}_\alpha & 0 \\ \hline 0 & 0 & \cosh\gamma_i \cdot 1 \\ \hline \end{array},\ \ (\gamma_i = \text{const},\ w^*{}_\alpha = \text{const}). \qquad (263A)$$

Here U gives in $\tilde{E}_m$ the movable asymmetric tensor of motion along a world line, which may be decomposed polary into hyperbolic and orthospherical parts in $\langle\mathcal{P}^{3+1}\rangle$ as in (111A). Non-collinear motion with the sine binormal induces in (172A) the dependent Thomas orthospherical precession around the cosine binormal. In addition, the pseudoscrewed motion, generated by Euclidean rotation $d\alpha$ of the space-like sine binormal, induces in (251A) the independent progressive orthoprocession parallel to rotated $\overrightarrow{ct}^{(1)} = s \cdot \alpha \mathbf{i}_1$. In the quasi-Euclidean space, for the screwed motion, it is even more obviously – see above in (258A).

Hyperbolic tensors F_h in (261A) and arbitrary or induced orthospherical tensors $\langle rot\ \Theta\rangle$ produce also full set of the homogeneous Lorentzian pseudo-Euclidean transformations in clear trigonometric forms, according to unambiguous polar decompositions of the latters. In (153A) and (202A), we expressed such mixed motion tensors by canonical forms in $\tilde{E}_1$.

* * *

Further we'll fill all the remaining "blind spots" related to the Appendix.

We will start as before with aspects related to geometric (as more general and abstract) motions and physical movements in the real-valued 3D and 4D pseudo-Euclidean spaces. Obviously, quasi-Euclidean binary spaces have the unity metric tensor and the Euclidean metric (according to their definitions in sects. 5.7 and 6.5). However their objects and transformations must correspond to the reflector tensor, as in (460), and the spaces themselves must have a structure defined by the same tensor as in (500). For pseudo-Euclidean spaces, the metric tensor and reflector tensor are equal, according to their definitions in Ch. 6.

We shall complete a part of our tensor trigonometry concerning to the *immediate summation* of two-steps motions on both Minkowskian hyperboloids II and I, and also on the hyperspheroid. Moreover we must take into account the inverse order (485) of the motion matrices, expressed in the initial unity base $\tilde{E}_1$, with respect to the order of motions! But in any case, we shall use the tensor of motion (100A) as the first summand in the given order of summation. Such a procedure with inferred formulae are true also for two-steps rotations in $\langle\mathcal{P}^{3+1}\rangle$ and STR, according to isomorphism of these motions on the embedded perfect hypersurface and rotations in its enveloping binary space!

On the hyperboloid II, with its affine topology and the Lobachevsky–Bolyai geometry, we apply time-like unity vectors $\mathbf{i}_{12}, \mathbf{i}_{23}$ (146A) and for them their directions $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$ as we did preliminary in two-steps transformation scheme of type (148A). In result, we get immediately the cosine and sine laws of two-steps summation of motions on it or rotations in $\langle \mathcal{P}^{3+1} \rangle$, united below in the general law by summary unity 4-radius-vector, applied to the unity hyperboloid II with Euclidean and time-arrow projections, or as the summary 4-velocity by Poincaré from two 4-velocities, applied to the hyperboloid II of radius "c":

$$roth\ \Gamma_{12} \cdot \mathbf{i}_{23} = \begin{array}{|c|c|} \hline I_{3\times 3} + (\cosh\gamma_{12} - 1) \cdot \mathbf{e}_\alpha \mathbf{e}'_\alpha & +\sinh\gamma_{12} \cdot \mathbf{e}_\alpha \\ \hline +\sinh\gamma_{12} \cdot \mathbf{e}'_\alpha & \cosh\gamma_{12} \\ \hline \end{array} \cdot \left\{ \frac{\sinh\gamma_{23} \cdot \mathbf{e}_\beta}{\cosh\gamma_{23}} \right\} =$$

$$= \left\{ \frac{[\sinh\ \gamma_{12} \cdot \cosh\gamma_{23} + \cos\varepsilon \cdot \sinh\gamma_{23} \cdot (\cosh\gamma_{12} - 1)] \cdot \mathbf{e}_\alpha + \sinh\gamma_{23} \cdot \mathbf{e}_\beta}{\cosh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\varepsilon \cdot \sinh\gamma_{12} \cdot \sinh\gamma_{23}} \right\} =$$

$$= \left\{ \frac{[\sinh\ \gamma_{12} \cdot \cosh\gamma_{23} + \cos\varepsilon \cdot \sinh\gamma_{23} \cdot \cosh\gamma_{12}] \cdot \mathbf{e}_\alpha + \sin\varepsilon \cdot \sinh\gamma_{23} \cdot \mathbf{e}_\nu}{\cosh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\varepsilon \cdot \sinh\gamma_{12} \cdot \sinh\gamma_{23}} \right\} =$$

$$= \left\{ \frac{\sinh\gamma_{13} \cdot \mathbf{e}_\sigma}{\cosh\gamma_{13}} \right\} = \mathbf{i}_{13} = rot\ \Theta_{13} \cdot \overset{\angle}{\mathbf{i}}_{13} \qquad (\mathbf{i}'_{13} \cdot \{I^{\pm}\} \cdot \mathbf{i}_{13} = i^2 = -1). \qquad (264A)$$

This only one operation summarizes immediately all scalar and vector formulae (122A), (124A), (135A) and obviously (125A), (138A) for hyperboloid II and Lobachevsky–Bolyai hyperplane, gotten earlier through the calculation of two sequential modal transformations of the initial unity base $\tilde{E}_1$ by the same tensor of motions (100A) for first and second steps.

For the reverse two-steps motions as $\mathbf{i}_{23}, \mathbf{i}_{12} \to \overset{\angle}{\mathbf{i}}_{13}$ on the same perfect surface, we have:

$$roth\ \Gamma_{23} \cdot \mathbf{i}_{12} = \begin{array}{|c|c|} \hline I_{3\times 3} + (\cosh\gamma_{23} - 1) \cdot \mathbf{e}_\beta \mathbf{e}'_\beta & +\sinh\gamma_{23} \cdot \mathbf{e}_\beta \\ \hline +\sinh\gamma_{23} \cdot \mathbf{e}'_\beta & \cosh\gamma_{23} \\ \hline \end{array} \cdot \left\{ \frac{\sinh\gamma_{12} \cdot \mathbf{e}_\alpha}{\cosh\gamma_{12}} \right\} =$$

$$= \left\{ \frac{[\sinh\ \gamma_{23} \cdot \cosh\gamma_{12} + \cos\varepsilon \cdot \sinh\gamma_{12} \cdot (\cosh\gamma_{23} - 1)] \cdot \mathbf{e}_\beta + \sinh\gamma_{12} \cdot \mathbf{e}_\alpha}{\cosh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\varepsilon \cdot \sinh\gamma_{12} \cdot \sinh\gamma_{23}} \right\} =$$

$$= \left\{ \frac{[\sinh\ \gamma_{23} \cdot \cosh\gamma_{12} + \cos\varepsilon \cdot \sinh\gamma_{12} \cdot \cosh\gamma_{23}] \cdot \mathbf{e}_\beta + \sin\varepsilon \cdot \sinh\gamma_{12} \cdot \mathbf{e}_{\overset{\angle}{\nu}}}{\cosh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\varepsilon \cdot \sinh\gamma_{12} \cdot \sinh\gamma_{23}} \right\} =$$

$$= \left\{ \frac{\sinh\gamma_{13} \cdot \mathbf{e}_{\overset{\angle}{\sigma}}}{\cosh\gamma_{13}} = \right\} = \overset{\angle}{\mathbf{i}}_{13} = rot'\ \Theta_{13} \cdot \mathbf{i}_{13} \qquad (\overset{\angle}{\mathbf{i}}{}'_{13} \cdot \{I^{\pm}\} \cdot \overset{\angle}{\mathbf{i}}_{13} = i^2 = -1). \qquad (265A)$$

We see that the direct and reverse summations are connected by the orthospherical rotation $rot\ \Theta_{13}$ from (112A), just as is the case of the general formulae for summation of polysteps motions in (153A).

Further, with known $\mathbf{e}_\sigma$ and $\mathbf{e}_{\overset{\angle}{\sigma}}$, using (141A), we obtain tensor trigonometric formulae for the accompanied induced secondary orthospherical shift with inference also for two-steps motion.

We can express the most complete general law of two-steps summation, combined with the induced orthospherical shift in compact clear trigonometric form as follows:

$$\left\{\begin{array}{l} \overrightarrow{\mathbf{r}_N}(\theta_{13}) = \mathbf{e}_{\angle\sigma} \otimes \mathbf{e}_\sigma = -\sin\theta_{13}\cdot\overrightarrow{\mathbf{e}_N} = \\ = -\frac{\sin\theta_{13}}{\sin\varepsilon}\cdot\mathbf{e}_\alpha^{(1)}\otimes\mathbf{e}_\beta^{(2)} = -\sin\theta_{13}\cdot\mathbf{e}_\alpha^{(1)}\otimes\mathbf{e}_\nu^{(2)}, \\ (\mathbf{e}_{\angle\sigma} = rot'\ \Theta_{13}\cdot\mathbf{e}_\sigma,\ \mathbf{e}'_{\angle\sigma}\cdot\mathbf{e}_\sigma = \cos\theta_{13}) \end{array}\right\} \qquad (266A)$$

where $\overrightarrow{\mathbf{e}_N} \equiv \mathbf{e}_\mu$ is a directed third normal vector.

In that time, with (111A) and (153A), summary two-steps hyperbolic transformation with its polar decomposition has the very clear kind in our tensor trigonometry approach with simplest interpretation:

$$T_{13} = roth\ \Gamma_{12}\cdot roth\ \Gamma_{23} = roth\Gamma_{13}\cdot rot\ \Theta_{13} = rot\ \Theta_{13}\cdot roth\ \overset{\angle}{\Gamma}_{13} = \qquad (267A).$$

$$= \left[\begin{array}{c|c} (\cosh\gamma_{13}-1)\cdot\mathbf{e}_\sigma\mathbf{e}'_{\angle\sigma} + [rot\ \Theta_{13}]_{3\times3} & \sinh\gamma_{13}\cdot\mathbf{e}_\sigma \\ \hline \sinh\gamma_{13}\cdot\mathbf{e}'_{\angle\sigma} & \cosh\gamma_{13} \end{array}\right].$$

$$= \left[\begin{array}{c|c} (\cosh\gamma_{13}-1)\cdot\mathbf{e}_\sigma\mathbf{e}'_\sigma + I_{3\times3} & \sinh\gamma_{13}\cdot\mathbf{e}_\sigma \\ \hline \sinh\gamma_{13}\cdot\mathbf{e}'_\sigma & \cosh\gamma_{13} \end{array}\right]\cdot\left[\begin{array}{c|c} [rot\ \Theta_{13}]_{3\times3} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \end{array}\right] =$$

$$= \left[\begin{array}{c|c} (\cosh\gamma_{13}-1)\cdot\mathbf{e}_\sigma\mathbf{e}'_{\angle\sigma} + [rot\ \Theta_{13}]_{3\times3} & \sinh\gamma_{13}\cdot\mathbf{e}_\sigma \\ \hline \sinh\gamma_{13}\cdot\mathbf{e}'_{\angle\sigma} & \cosh\gamma_{13} \end{array}\right].$$

Note (!), that in (264A)–(267A) all used matrices are given in their canonical forms, set in the original unity base $\tilde{E}_1$, according to their tensor trigonometric representations.

<u>**On the hyperboloid I**</u>, constrained by its cylindrical topology and with the cylindrical hyperbolic–elliptical geometry, we use space-like unity radius-vector $\mathbf{p}_{12}, \mathbf{p}_{23}$ (149A) and for them their directions $\mathbf{e}_\alpha$ and $\mathbf{e}_\kappa$ as we did preliminary in two-steps transformation scheme of type (152A). In result, we get now immediately the cosine and sine laws of two-steps summation of motions on it or rotations in the Looking Glass of complete $\langle\mathcal{P}^{3+1}\rangle$ in the right direction (see above), united below in the general law by the summary unity 4-vector $\mathbf{p}_{13}$, applied to the unity hyperboloid I with the Euclidean and scalar time-like projections:

$$roth\ \Gamma_{12}\cdot\mathbf{p}_{23} = \left[\begin{array}{c|c} I_{3\times3} + (\cosh\gamma_{12}-1)\cdot\mathbf{e}_\alpha\mathbf{e}'_\alpha & +\sinh\gamma_{12}\cdot\mathbf{e}_\alpha \\ \hline +\sinh\gamma_{12}\cdot\mathbf{e}'_\alpha & \cosh\gamma_{12} \end{array}\right]\cdot\left\{\begin{array}{c} \cosh\gamma_{23}\cdot\mathbf{e}_\kappa \\ \hline \sinh\gamma_{23} \end{array}\right\} =$$

$$= \left\{\begin{array}{c} \sinh\ \gamma_{12}\cdot\sinh\gamma_{23} + \cos\epsilon\cdot\cosh\gamma_{23}\cdot(\cosh\gamma_{12}-1)\cdot\mathbf{e}_\alpha + \cosh\gamma_{23}\cdot\mathbf{e}_\kappa \\ \hline \sinh\gamma_{23}\cdot\cosh\gamma_{12} + \cos\epsilon\cdot\sinh\gamma_{12}\cdot\cosh\gamma_{23} \end{array}\right\} =$$

$$= \left\{\begin{array}{c} \sinh\ \gamma_{12}\cdot\sinh\gamma_{23} + \cos\epsilon\cdot\cosh\gamma_{12}\cdot\cosh\gamma_{23}\cdot\mathbf{e}_\alpha + \sin\epsilon\cdot\cosh\gamma_{23}\cdot\mathbf{e}_\mu \\ \hline \sinh\gamma_{23}\cdot\cosh\gamma_{12} + \cos\epsilon\cdot\sinh\gamma_{12}\cdot\cosh\gamma_{23} \end{array}\right\} =$$

$$= \left\{\begin{array}{c} \cosh\gamma^*_{13}\cdot\mathbf{e}^*_\sigma \\ \hline \sinh\gamma^*_{13} \end{array}\right\} = \mathbf{p}_{13} \qquad (\mathbf{p}'_{13}\cdot\{I^\pm\}\cdot\mathbf{p}_{13} = +1). \qquad (268A)$$

Thus, only one operation above summarizes immediately all additional scalar and vector formulae of two-steps summations for the cylindrical hyperbolic–elliptical hypersurface.

For the reverse two-steps motions as $\mathbf{p}_{23}, \mathbf{p}_{12} \to \overset{\angle}{\mathbf{p}}_{13}$ on the same perfect surface, we get:

$$roth\ \Gamma_{23} \cdot \mathbf{p}_{12} = \begin{array}{|c|c|} \hline I_{3\times3} + (\cosh\gamma_{23} - 1) \cdot \mathbf{e}_\kappa \mathbf{e}'_\kappa & + \sinh\gamma_{23} \cdot \mathbf{e}_\kappa \\ \hline + \sinh\gamma_{23} \cdot \mathbf{e}'_\kappa & \cosh\gamma_{23} \\ \hline \end{array} \cdot \left\{ \frac{\cosh\gamma_{12} \cdot \mathbf{e}_\alpha}{\sinh\gamma_{12}} \right\} =$$

$$= \left\{ \frac{\sinh\ \gamma_{12} \cdot \sinh\gamma_{23} + \cos\epsilon \cdot \cosh\gamma_{12} \cdot (\cosh\gamma_{23} - 1) \cdot \mathbf{e}_\kappa + \cosh\gamma_{12} \cdot \mathbf{e}_\alpha}{\sinh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\epsilon \cdot \sinh\gamma_{23} \cdot \cosh\gamma_{12}} \right\} =$$

$$= \left\{ \frac{\sinh\ \gamma_{12} \cdot \sinh\gamma_{23} + \cos\epsilon \cdot \cosh\gamma_{12} \cdot \cosh\gamma_{23} \cdot \mathbf{e}_\kappa + \sin\epsilon \cdot \cosh\gamma_{12} \cdot \mathbf{e}_{\overset{\angle}{\mu}}}{\sinh\gamma_{12} \cdot \cosh\gamma_{23} + \cos\epsilon \cdot \sinh\gamma_{23} \cdot \cosh\gamma_{12}} \right\} =$$

$$= \left\{ \frac{\cosh \overset{\angle}{\gamma}{}^{*}_{13} \cdot \mathbf{e}^{*}_{\overset{\angle}{\sigma}}}{\sinh \overset{\angle}{\gamma}{}^{*}_{13}} \right\} = \overset{\angle}{\mathbf{p}}_{13} \qquad (\overset{\angle}{\mathbf{p}}{}'_{13} \cdot \{I^{\pm}\} \cdot \overset{\angle}{\mathbf{p}}_{13} = +1). \tag{269A}$$

We see that here the direct and reverse summations in their 3×1 Euclidean parts are not connected by $rot\ \Theta_{13}$ and they are generally non-commutative. In (268A), (269A) both matrices and cosine 4-vectors are given in canonical forms, set in $\tilde{E}_1$. However, since both elements (268A) and (269A) as the radius-vectors remain on the same hyperboloid I and belong to it as its invariants, then the first and the second are connected by a certain mixed pseudo-Euclidean rotation:

$$T = \{roth\ \Gamma_{32} \cdot roth\ \Gamma_{21} \cdot rot\ \Theta\ (\mathbf{e}_\kappa \to \mathbf{e}_\alpha) \cdot roth\ \Gamma_{21} \cdot roth\ \Gamma_{32}\}.$$

Let's pay attention to the fact that immediate summation (264A) and (265A) from two 4-vectors $\mathbf{r}_{12}$ and $\mathbf{r}_{23}$ are actually implemented by instituting the zero element $\mathbf{r}_1$, introduced initially in (146A), from the right after the products $roth\ \Gamma_{12} \cdot roth\ \Gamma_{23}$ with reverse one in the modal formula (111A) as we did in (148A) for the construction of a hyperbolic triangle

$$roth\ \Gamma_{12} \cdot \mathbf{r}_{23} = roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdot \mathbf{r}_1 = \mathbf{r}_{13} = \tag{270A}$$

$$= roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdot \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = roth\ \Gamma_{13} \cdot rot\ \Theta_{13} \cdot \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \equiv roth\ \Gamma_{13} \cdot \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = \begin{bmatrix} \sinh\gamma_{13} \cdot \mathbf{e}_\sigma \\ \cosh\gamma_{13} \end{bmatrix}.$$

$$roth\ \Gamma_{23} \cdot \mathbf{r}_{12} = roth\ \Gamma_{23} \cdot roth\ \Gamma_{12} \cdot \mathbf{r}_1 = \overset{\angle}{\mathbf{r}}_{13} = \tag{271A}$$

$$roth\ \Gamma_{23} \cdot roth\ \Gamma_{12} \cdot \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = roth\overset{\angle}{\Gamma}_{13} \cdot rot'\ \Theta_{13} \cdot \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \equiv roth\overset{\angle}{\Gamma}_{13} \cdot \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = \begin{bmatrix} \sinh\gamma_{13} \cdot \mathbf{e}_{\overset{\angle}{\sigma}} \\ \cosh\gamma_{13} \end{bmatrix}.$$

Let's try to apply for (268A) and (269A) analogous procedure (152A) on the hyperboloid I

$$roth\ \Gamma_{12} \cdot \mathbf{p}_{23} = roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdot \mathbf{p}_\kappa = \mathbf{p}_{13} = \tag{272A}$$

$$= roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdot \begin{bmatrix} \mathbf{e}_\kappa \\ 0 \end{bmatrix} = roth\ \Gamma_{13} \cdot rot\ \Theta_{13} \cdot \begin{bmatrix} \mathbf{e}_\kappa \\ 0 \end{bmatrix} \equiv roth\ \Gamma_{13} \cdot \begin{bmatrix} \mathbf{e}^{*}_\kappa \\ 0 \end{bmatrix} = \begin{bmatrix} \cosh\gamma^{*}_{13} \cdot \mathbf{e}^{*}_\sigma \\ \sinh\gamma^{*}_{13} \end{bmatrix}.$$

$$roth\ \Gamma_{23} \cdot \mathbf{p}_{12} = roth\ \Gamma_{23} \cdot roth\ \Gamma_{12} \cdot \mathbf{p}_\alpha = \overset{\angle}{\mathbf{p}}_{13} = \tag{273A}$$

$$= roth\ \Gamma_{23} \cdot roth\ \Gamma_{12} \cdot \begin{bmatrix} \mathbf{e}_\alpha \\ 0 \end{bmatrix} = roth\overset{\angle}{\Gamma}_{13} \cdot rot'\ \Theta_{13} \cdot \begin{bmatrix} \mathbf{e}_\alpha \\ 0 \end{bmatrix} \equiv roth\overset{\angle}{\Gamma}_{13} \cdot \begin{bmatrix} \mathbf{e}^{*}_\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \cosh \overset{\angle}{\gamma}{}^{*}_{13} \cdot \mathbf{e}^{*}_{\overset{\angle}{\sigma}} \\ \sinh \overset{\angle}{\gamma}{}^{*}_{13} \end{bmatrix}.$$

Both summary elements are situated on the hyperboloid II and are connected by the orthospherical rotation $rot\ \Theta_{13}$ as in (264A), (265A). However both summary elements on the hyperboloid I are connected only by these contrary mixed pseudo-Euclidean rotations:

$$T\ (\mathbf{p}_{13} \to \overset{\angle}{\mathbf{p}}_{13}) = \{roth\ \Gamma_{23} \cdot roth\ \Gamma_{12} \cdot rot\ \Theta\ (\mathbf{e}_\kappa \to \mathbf{e}_\alpha) \cdot roth\ \Gamma_{32} \cdot roth\ \Gamma_{21}\}. \qquad (274A)$$

$$T\ (\overset{\angle}{\mathbf{p}}_{13} \to \mathbf{p}_{13}) = \{roth\ \Gamma_{12} \cdot roth\ \Gamma_{23} \cdot rot\ \Theta\ (\mathbf{e}_\alpha \to \mathbf{e}_\kappa) \cdot roth\ \Gamma_{21} \cdot roth\ \Gamma_{32}\}. \qquad (275A)$$

For the hyperboloid I, we have:

$$\mathbf{e}^{*\prime}_\sigma \cdot \mathbf{e}^*_{\overset{\angle}{\sigma}} = \cos\theta^*, \quad \mathbf{e}^*_{\overset{\angle}{\sigma}} = \{rot'\ \Theta^*_{13}\} \cdot \mathbf{e}^*_\sigma; \quad \mathbf{e}^*_\alpha = \{rot\ \Theta^*\}_{3\times 3} \cdot \mathbf{e}_\alpha, \quad \mathbf{e}^*_\kappa = \{rot\ \Theta^*\}_{3\times 3} \cdot \mathbf{e}_\kappa.$$

And for different initial summarized angles (segments) there hold $\gamma^*_{13} \neq \gamma_{13}$, $\overset{\angle}{\gamma}{}^*_{13} \neq \overset{\angle}{\gamma}_{13}$.

The pseudo-Euclidean angle between two last is calculated as follows:

$$\rho_\sigma = \overset{\angle}{\mathbf{p}}{}'_{13} \cdot I^\pm \cdot \mathbf{p}_{13} = \mathbf{p}'_{13} \cdot I^\pm \cdot \overset{\angle}{\mathbf{p}}_{13} = [\cosh\overset{\angle}{\gamma}_{13} \cdot \cosh\gamma_{13} \cdot \mathbf{e}'_{\overset{\angle}{\sigma}}\mathbf{e}_\sigma] - [\sinh\overset{\angle}{\gamma}_{13} \cdot \sinh\gamma_{13}] =$$

$$= \sinh\gamma_{12}\sinh\gamma_{23} \cdot (\cosh\gamma_{12} + \cosh\gamma_{23} - \cosh\gamma_{12}\cosh\gamma_{23}) +$$
$$+ \cos\epsilon \cdot (\cosh\gamma_{12}\cosh\gamma_{23} + \cosh^2\gamma_{12}\cosh\gamma_{23} + \cosh\gamma_{12}\cosh^2\gamma_{23} - \sinh^2\gamma_{12} - \sinh^2\gamma_{23} - \sinh^2\gamma_{23} \cdot \sinh^2\gamma_{12}) +$$
$$+ \cos^2\epsilon \cdot (\sinh\gamma_{12}\sinh\gamma_{23}\cosh\gamma_{12}\cosh\gamma_{23} - \sinh\gamma_{12}\sinh\gamma_{23}\cosh\gamma_{12} - \sinh\gamma_{12}\sinh\gamma_{23}\cosh\gamma_{23}) +$$
$$+ \cos^3\epsilon \cdot (\cosh^2\gamma_{12}\cosh^2\gamma_{23} - \cosh^2\gamma_{12}\cosh\gamma_{23} - \cosh\gamma_{12}\cosh^2\gamma_{23} + \cosh\gamma_{12}\cosh\gamma_{23}).$$

If $\epsilon = 0$ $(\cos\epsilon = \pm 1)$, then γ_{12} and γ_{23} are trigonometrically compatible and both motions are commutative in direct and inverse ordering, then $\rho_\sigma = \cosh^2\gamma_{13} + \sinh^2\gamma_{13} = 1$. If $\epsilon = \pm\pi/2$ $(\sin\epsilon = \pm 1, \cos\epsilon =$, both motions are conveniently orthogonal in direct and inverse ordering, then $\rho_\sigma = \sinh\gamma_{12}\sinh\gamma_{23} \cdot (\cosh\gamma_{12} + \cosh\gamma_{23} - \cosh\gamma_{12} \cdot \cosh\gamma_{23})$.

Next put $\gamma_{12} = \gamma$, $\gamma_{23} = d\gamma$ and use also $\mathbf{e}^{(1)}_\alpha$ and $\mathbf{e}^{(m)}_\mu$ for them. We obtain:

$$\rho_\sigma = \overset{\angle}{\mathbf{p}}{}'_{13} \cdot I^\pm \cdot \mathbf{p}_{13} = \mathbf{p}'_{13} \cdot I^\pm \cdot \overset{\angle}{\mathbf{p}}_{13} = [\cosh\overset{\angle}{\gamma}_{13} \cdot \cosh\gamma_{13} \cdot \mathbf{e}'_{\overset{\angle}{\sigma}}\mathbf{e}_\sigma]_E - [\sinh\overset{\angle}{\gamma}_{13} \cdot \sinh\gamma_{13}]_P =$$

$$= \{+\cos\epsilon \cdot \cosh^2\gamma + [(\sinh\gamma\cosh\gamma + \cos^2\epsilon \cdot \sinh\gamma\cosh\gamma) + \sin^2\epsilon \cdot \sinh\gamma)\ d\gamma]\}_E -$$
$$-[\cos\epsilon \cdot \sinh^2\gamma + (\sinh\gamma\cosh\gamma + \cos^2\epsilon \cdot \sinh\gamma\cosh\gamma)\ d\gamma]_P. \Rightarrow$$
$$\Rightarrow \rho_\sigma = \cos\epsilon, \quad [\rho_\sigma]_E = \mathbf{e}^{*\prime}_\sigma \cdot \mathbf{e}^*_{\overset{\angle}{\sigma}} = \cos\theta^* = \cos\epsilon \cdot \cosh^2\gamma, \ [\rho_\sigma]_P = \cos\epsilon \cdot \sinh^2\gamma.$$

We decompose the pseudo-Euclidean product in current orthospherical and hyperbolic parts. The differential shift $d\theta$ has place, it can be fixed with respect to variable principal parts.

Further we represent all corresponding similar and concomitant formulae for the tensor trigonometry of the hyperboloid I earlier absent in Ch. 7A. For the beginning, we give the scalar time-like sine expressions for summing two-steps non-collinear motions or segments, both set by radius (149A), in direct and inverse orders:

$$\sinh\gamma_{13} = \cosh\gamma_{12} \cdot \sinh\gamma_{23} + \cos\epsilon \cdot \sinh\gamma_{12} \cdot \cosh\gamma_{23}, \qquad (276A)$$

$$\sinh\overset{\angle}{\gamma}_{13} = \cosh\gamma_{23} \cdot \sinh\gamma_{12} + \cos\epsilon \cdot \sinh\gamma_{23} \cdot \cosh\gamma_{12}. \qquad (277A)$$

They are non-commutative in contrast to common scalar cosine formula (122A) for II.

For the scalar space-like cosine formulae we have accordingly these two quadric expressions:

$$\cosh^2\gamma_{13} = (\sinh\gamma_{12}\cdot\sinh\gamma_{23} + \cos\epsilon\cdot\cosh\gamma_{12}\cdot\cosh\gamma_{23})^2 + (\sin\epsilon\cdot\cosh\gamma_{23})^2, \quad (278A)$$

$$\cosh^2\overset{\angle}{\gamma}_{13} = (\sinh\gamma_{12}\cdot\sinh\gamma_{23} + \cos\epsilon\cdot\cosh\gamma_{12}\cdot\cosh\gamma_{23})^2 + (\sin\epsilon\cdot\cosh\gamma_{12})^2. \quad (279A)$$

In addition to special angular relations (137A), (140A) for the hyperboloid II, we give the corresponding relations, in particular, with the unity vectors of orthogonal increments, for the considered hyperboloid I:

$$\left.\begin{aligned}&\mathbf{e}_\kappa = \cos\epsilon\cdot\mathbf{e}_\alpha + \sin\epsilon\cdot\mathbf{e}_\mu, \mathbf{e}_\alpha = \cos\epsilon\cdot\mathbf{e}_\kappa + \sin\epsilon\cdot\mathbf{e}_{\overset{\angle}{\mu}}, \cos\epsilon = \mathbf{e}'_\kappa\cdot\mathbf{e}_\alpha = \mathbf{e}'_\alpha\cdot\mathbf{e}_\kappa;\\ &\mathbf{e}'_\kappa\cdot\mathbf{e}_\mu = \mathbf{e}'_\alpha\cdot\mathbf{e}_{\overset{\angle}{\mu}} = \sin\epsilon,\ \mathbf{e}'_\kappa\cdot\mathbf{e}_{\overset{\angle}{\mu}} = \mathbf{e}'_\alpha\cdot\mathbf{e}_\mu = 0,\ \mathbf{e}'_\mu\cdot\mathbf{e}_{\overset{\angle}{\mu}} = -\cos\epsilon; \quad (\epsilon\in[0;\pi])\end{aligned}\right\} \quad (280A)$$

Vectors *for direct summing* lie in the Euclidean *normal plane of cosine normal curvature* $\langle\mathcal{E}^2\rangle_{Nc} \equiv \langle\mathbf{e}_\alpha, \mathbf{e}_\mu\rangle$ with cosine binormal $\mathbf{b}_\mu$ and acceleration $j_\mu = c^*\ w^*_{\alpha(2)}$ – see in (236A). Thus, now we can give more understandably the two vector space-like cosine expressions and also non-commutative in direct and inverse orders:

$$\mathbf{cosh}\gamma_{13} = \cosh\gamma_{13}\cdot\mathbf{e}_\sigma =$$

$$= (\sinh\gamma_{12}\cdot\sinh\gamma_{23} + \cos\epsilon\cdot\cosh\gamma_{12}\cdot\cosh\gamma_{23})\cdot\mathbf{e}_\alpha + \sin\epsilon\cdot\cosh\gamma_{23}\cdot\mathbf{e}_\mu, \quad (281A)$$

$$\mathbf{cosh}\overset{\angle}{\gamma}_{13} = \cosh\overset{\angle}{\gamma}_{13}\cdot\mathbf{e}_{\overset{\angle}{\sigma}} =$$

$$(\sinh\gamma_{23}\cdot\sinh\gamma_{12} + \cos\epsilon\cdot\cosh\gamma_{23}\cdot\cosh\gamma_{12})\cdot\mathbf{e}_\kappa + \sin\epsilon\cdot\cosh\gamma_{12}\cdot\mathbf{e}_{\overset{\angle}{\mu}}. \quad (282A)$$

Our readers can easily verify that expressions (276A), (278A) and (277A), (279A) for sine and cosine both in the direct order and in the reverse order, form two own quadratic sine-cosine hyperbolic invariants, but with a small difference between quadrics of these functions:

$$\cosh^2\gamma_{13} - \sinh^2\gamma_{13} = 1 = \cosh^2\overset{\angle}{\gamma}_{13} - \sinh^2\overset{\angle}{\gamma}_{13} \Rightarrow \cosh^2\gamma_{13} - \cosh^2\overset{\angle}{\gamma}_{13} = \sinh^2\gamma_{13} - \sinh^2\overset{\angle}{\gamma}_{13} \Rightarrow$$

$$\cosh^2\gamma_{13} - \cosh^2\overset{\angle}{\gamma}_{13} = \sin^2\epsilon\cdot(\cosh^2\gamma_{12} - \cosh^2\gamma_{23}) = \quad (283A)$$

$$= \sinh^2\gamma_{13} - \sinh^2\overset{\angle}{\gamma}_{13} = \sin^2\epsilon\cdot(\sinh^2\gamma_{12} - \sinh^2\gamma_{23}). \quad (284A)$$

From vectorial (281A) and scalar (276A) formulae, the additional vectorial and scalar non-commutative formulae for cotangent summation, so in ordering γ_{12}, γ_{23}, are inferred:

$$\coth\gamma_{13}\cdot\mathbf{e}_\sigma = \frac{1 + \cos\epsilon\cdot\coth\gamma_{12}\cdot\coth\gamma_{23}}{\coth\gamma_{12} + \cos\epsilon\cdot\coth\gamma_{23}}\cdot\mathbf{e}_\alpha + \frac{\sin\epsilon\cdot\operatorname{csch}\ \gamma_{12}\cdot\coth\gamma_{23}}{\coth\gamma_{12} + \cos\epsilon\cdot\coth\gamma_{23}}\cdot\mathbf{e}_\mu. \quad (285A-I)$$

$$\coth^2\gamma_{13} = \left[\frac{1 + \cos\epsilon\cdot\coth\gamma_{12}\cdot\coth\gamma_{23}}{\coth\gamma_{12} + \cos\epsilon\cdot\coth\gamma_{23}}\right]^2 + \left[\frac{\sin\epsilon\cdot\operatorname{csch}\ \gamma_{12}\cdot\coth\gamma_{23}}{\coth\gamma_{12} + \cos\epsilon\cdot\coth\gamma_{23}}\right]^2. \quad (285A-II)$$

We note repeatedly, that the very wonderful in STR and non-Euclidean geometries – see in Ch. 7A at (135A)-(138A) and above in (280A): for summing motions (rotations) ones may combine correctly their directive vectors in own Euclidean planes of acting, for example, in $\langle\mathcal{E}^2\rangle^{(1)}$ and $\langle\mathcal{E}^2\rangle^{(m)}$. Since unity vectors $\mathbf{e}_\sigma$ and $\mathbf{e}_{\overset{\angle}{\sigma}}$ of direct and inverse summary cosines in (281A) and (282A) are linear combinations of $\mathbf{e}_\alpha$, $\mathbf{e}_\mu$ and $\mathbf{e}_\kappa$, $\mathbf{e}_{\overset{\angle}{\mu}}$, then they lie in the direct and reverse normal planes of the normal cosine curvature under the angle of current hyperbolic inclination. It is a very unusual property of STR and non-Euclidean geometries.

Motions from $\mathbf{p}_2$ to $\mathbf{p}_3$ in (268A), (269A) are possible iff flat cotangent or cylindrical tangent projections of $\mathbf{p}_2$ and $\mathbf{p}_3$ outside Cayley oval can be connected by straight cotangent **coth** γ_{23} or tangent **tanh** γ_{23} segments without the having topological obstacles. Compare cotangent formula for the two-steps summation at the hyperboloid I with tangent formula (138A) for the same goal at the hyperboloid II. There is full correspondence!

Recall, that earlier in Chs. 6 and 7A, in addition to the well-known sine-cosine invariant in hyperbolic geometries, we installed else *cosecant-cotangent hyperbolic invariant.* We have

$$\mathbf{coth}^2\,\gamma_{13} - \mathrm{csch}^2\gamma_{13} = +1 = \mathrm{coth}^2\,\overset{\angle}{\gamma}_{13} - \mathrm{csch}^2\overset{\angle}{\gamma}_{13}. \qquad (286A)$$

Finally, the scalar reverse cosecant and also non-commutative variant of two non-collinear segments summation is expressed, so in ordering γ_{12}, γ_{23}, from (285A-II):

$$\mathrm{csch}\gamma_{13} = \sqrt{\mathbf{coth}^2\,\gamma_{13} - 1} = \frac{\mathrm{csch}\gamma_{12}\cdot\mathrm{csch}\gamma_{23}}{\mathbf{coth}\,\gamma_{12} + \cos\varepsilon\cdot\mathbf{coth}\,\gamma_{23}}. \qquad (287A)$$

Hence, the cotangent–cosecant two-steps summations of space-like and time-like motions on the hyperboloid I are possible also (as we did in Ch. 7A for the vector direct tangent and scalar reverse secant summations of necessary space-like motions on the hyperboloid II).

What is more, with such an abundance of invariants and quasi-invariants in our subject Tensor Trigonometry, it would be very useful to formulate such a clear **Mnemonic Rule**, that connects all similar ones of spherical and hyperbolic types.

In the *complete scalar, vector and tensor trigonometry of any kinds*, we obtain the very important universal correspondence useful for its users memory:

"*Each trigonometric quadratic invariant (quasi-invariant) for paired spherical functions is in one-to-one correspondence, in that number by its form, with the quadratic quasi-invariant (invariant) for paired hyperbolic functions, when both of them are sparring between each to another by complete functional specific spherical–hyperbolic analogy of type (331)!* "

Natural paired cotangent-cosecant bond takes place also for the 1-st metric forms on both hyperboloids using for motions the complementary angle $\upsilon(\gamma)$ as their argument, with own two $3D$ Relative Euclidean, $4D$ Absolute Euclidean and $4D$ Absolute pseudo-Euclidean Pythagorean Theorems, under the correspondences between the complementary hyperbolic angles namely for the motions on the hyperboloid I, according to all relations (235A)–(238A) and with executing analogous operations. Recall from Chs. 6, 12, that in any admitted base:

$$\left\{\begin{array}{l} \mathbf{sinh}(\Gamma,\Upsilon) = \mathrm{csch}\ (\Upsilon,\Gamma) \ \Leftrightarrow\ \mathbf{sinh}(\Gamma,\Upsilon)\cdot\mathbf{sinh}(\Upsilon,\Gamma) = I, \\ \mathbf{cosh}(\Gamma,\Upsilon) = \mathbf{coth}\ (\pm\Upsilon,\Gamma) \ \Leftrightarrow\ \mathbf{tanh}(\pm\Gamma,\Upsilon) = \mathrm{sech}(\Upsilon,\Gamma); \end{array}\right\} \qquad (288A)$$

$$\mathbf{cosh}'\gamma\cdot\mathbf{cosh}\ \gamma - \sinh^2\gamma \equiv \mathbf{coth}'\upsilon\cdot\mathbf{coth}\ \upsilon - \mathrm{csch}^2\upsilon = 1 \text{ (for I)} \qquad (289A-I)$$

$$\cosh\gamma^2 - \mathbf{sinh}'\gamma\cdot\mathbf{sinh}\ \gamma \equiv \coth^2\upsilon - \mathbf{csch}'\upsilon\cdot\mathbf{csch}\ \upsilon = 1 \text{ (for II)} \qquad (289A-II)$$

* * *

Concerning to the hyperspheroid, by analogous way, we complete Tensor Trigonometry also by the immediate summation of finite two-steps motions on the oriented hyperspheroid, introduced by us in Ch. 8A, with its spherical geometry at the set frame axis, both under admissible quasi-Euclidean transformations, defined by the simplest reflector tensor $\{I^{\pm}\}$, so, as homogeneous motions on it and identical rotations in its enveloping binary space – principal spherical and secondary or induced orthospherical. We'll use the canonical tensors of motion from (179A), expressed in the initial unity base $\tilde{E}_1$ and in the given orders of summations, according to inverse formula (485), Ch. 11, how this was on both hyperboloids. See before tvs presentations of different motions on the hyperspheroid in (199A)–(204A).

In Ch. 8A, on the base of the abstract hyperbolic–spherical analogy (323) with two-steps hyperbolic summation (148A) in $\langle \mathcal{Q}^{2+1}\rangle$, we did preliminary scheme (201A) for two-steps spherical summation, but without final formulae with corresponding Pythagorean theorems. We'll infer them for motions on the hyperspheroid and identical rotations along regular curves in $\langle \mathcal{Q}^{2+1}\rangle$ and $\langle \mathcal{Q}^{3+1}\rangle$. In $\tilde{E}_1$, we have the element $\mathbf{t}(\varphi)$ for motions from North Pole as its first radius-vector and also the principal tangent to a curve, and the element $\mathbf{n}(\varphi)$ for motions from its Equator as its second radius-vector (orthogonally to the former) and also the principal quasinormal to the same curve at motion along it with principal *clockwise* angle $\varphi \in [0; +\pi/2]$ and complementary *counterclockwise* angle $\xi \in [+\pi/2; 0]$, $\varphi + \xi = \pi/2$

$$\mathbf{t}(\varphi) = \begin{bmatrix} \sin\varphi \\ \cos\varphi \end{bmatrix} = \begin{bmatrix} \sin\varphi \cdot \mathbf{e}_\alpha \\ \cos\varphi \end{bmatrix}, \quad \mathbf{n}(\varphi) = \begin{bmatrix} \cos\varphi \\ -\sin\varphi \end{bmatrix} = \begin{bmatrix} \cos\varphi \cdot \mathbf{e}_\alpha \\ -\sin\varphi \end{bmatrix}. \qquad (290A)$$

On the hyperspheroid, for the direct order of summation from its North Pole in $\langle \mathcal{Q}^{3+1}\rangle$ of two segments, we apply the tangent $\mathbf{t}_{23}$ as the second summand with direction $\mathbf{e}_\beta$ how we did preliminary in spherical two-steps transformations (201A) in Ch. 8A. In result, we get immediately the cosine and sine laws of two-steps summation of motions on it or rotations in $\langle \mathcal{Q}^{3+1}\rangle$, united in the general law by summary unity vector-radius $\mathbf{t}_{13}$, applied to the hyperspheroid with its vector cosine and scalar sine projections in this quasi-Euclidean space under conservation of relations (137A), (140A) for three basis unity vectors and $\varepsilon = \pi - A_{123}$:

$$rot\ \Phi_{12} \cdot \mathbf{t}_{23} = \left|\begin{array}{c|c} I_{2\times 2} - (1-\cos\varphi_{12}) \cdot \mathbf{e}_\alpha \mathbf{e}'_\alpha & +\sin\varphi_{12} \cdot \mathbf{e}_\alpha \\ \hline -\sin\varphi_{12} \cdot \mathbf{e}'_\alpha & \cos\varphi_{12} \end{array}\right| \cdot \left\{\frac{\sin\varphi_{23} \cdot \mathbf{e}_\beta}{\cos\varphi_{23}}\right\} =$$

$$= \left\{\frac{[\sin\ \varphi_{12} \cdot \cos\varphi_{23} - \cos\varepsilon \cdot \sin\varphi_{23} \cdot (1-\cos\varphi_{12})] \cdot \mathbf{e}_\alpha + \sin\varphi_{23} \cdot \mathbf{e}_\beta}{\cos\varphi_{12} \cdot \cos\varphi_{23} - \cos\varepsilon \cdot \sin\varphi_{12} \cdot \sin\varphi_{23}}\right\} =$$

$$= \left\{\frac{[\sin\ \varphi_{12} \cdot \cos\varphi_{23} + \cos\varepsilon \cdot \sin\varphi_{23} \cdot \cos\varphi_{12}] \cdot \mathbf{e}_\alpha + \sin\varepsilon \cdot \sin\varphi_{23} \cdot \mathbf{e}_\nu}{\cos\varphi_{12} \cdot \cos\varphi_{23} - \cos\varepsilon \cdot \sin\varphi_{12} \cdot \sin\varphi_{23}}\right\} =$$

$$= \left\{\frac{\sin\varphi_{13} \cdot \mathbf{e}_\sigma}{\cos\varphi_{13}}\right\} = \mathbf{t}_{13} = rot\ \Theta_{13} \cdot \overset{\angle}{\mathbf{t}}_{13} \qquad (\mathbf{t}'_{13} \cdot \mathbf{t}_{13} = 1). \qquad (291A)$$

This operation summarizes immediately all scalar and vector formulae (189A), (190A), (192A), (194A) and obviously (191A), (195A) for the hyperspheroid, gotten earlier in Ch. 8A, with the same tensor of motions (179A) for first and second steps. For summation in $\langle \mathcal{Q}^{3+1}\rangle$ of two-steps motions with the inverse order as $\mathbf{t}_{23}, \mathbf{t}_{12}$, we obtain these contrary relations:

$$rot\ \Phi_{23} \cdot \mathbf{t}_{12} = \left|\begin{array}{c|c} I_{2\times 2} - (1-\cos\varphi_{23}) \cdot \mathbf{e}_\beta \mathbf{e}'_\beta & +\sin\varphi_{23} \cdot \mathbf{e}_\beta \\ \hline -\sin\varphi_{23} \cdot \mathbf{e}'_\beta & \cos\varphi_{23} \end{array}\right| \cdot \left\{\frac{\sin\varphi_{12} \cdot \mathbf{e}_\alpha}{\cos\varphi_{12}}\right\} =$$

$$= \left\{\frac{[\sin\ \varphi_{23} \cdot \cos\varphi_{12} - \cos\varepsilon \cdot \sin\varphi_{12} \cdot (1-\cos\varphi_{23})] \cdot \mathbf{e}_\beta + \sin\varphi_{12} \cdot \mathbf{e}_\alpha}{\cos\varphi_{12} \cdot \cos\varphi_{23} - \cos\varepsilon \cdot \sin\varphi_{12} \cdot \sin\varphi_{23}}\right\} =$$

$$= \left\{\frac{[\sin\ \varphi_{23} \cdot \cos\varphi_{12} + \cos\varepsilon \cdot \sin\varphi_{12} \cdot \cos\varphi_{23}] \cdot \mathbf{e}_\beta + \sin\varepsilon \cdot \sin\varphi_{12} \cdot \mathbf{e}_{\overset{\angle}{\nu}}}{\cos\varphi_{12} \cdot \cos\varphi_{23} - \cos\varepsilon \cdot \sin\varphi_{12} \cdot \sin\varphi_{23}}\right\} =$$

$$= \left\{\frac{\sin\varphi_{13} \cdot \mathbf{e}_{\overset{\angle}{\sigma}}}{\cos\varphi_{13}} = \right\} = \overset{\angle}{\mathbf{t}}_{13} = rot'\ \Theta_{13} \cdot \mathbf{t}_{13} \qquad (\overset{\angle}{\mathbf{t}}{}'_{13} \cdot \overset{\angle}{\mathbf{t}}_{13} = 1). \qquad (292A)$$

Here is: $\mathbf{e}_\beta = \cos\varepsilon \cdot \mathbf{e}_\alpha + \sin\varepsilon \cdot \mathbf{e}_\nu$, $\mathbf{e}_\alpha = \cos\varepsilon \cdot \mathbf{e}_\beta + \sin\varepsilon \cdot \mathbf{e}_{\overset{\angle}{\nu}}$.

The direct and inverse summations are connected by the orthospherical rotation $rot\ \Theta_{13}$ from (181A), but scalarly commutative, just as for polysteps summation of motions in (202A). It is similar to rotation $rot\ \Theta_{13}$ on as if hyperboloid II – analog with contrary signs.

Let us translate to two-steps summation on the hyperspheroid in $\langle Q^{3+1}\rangle$ in the direction from its Equator. Now for the direct order of summation on it of two segments, we apply the quasinormal $\mathbf{n}_{23}$ as the second summand with its direction $\mathbf{e}_\kappa$ as we did for the hyperboloid I. In result, we get immediately the vector cosine and scalar sine laws of two-steps summation of motions on it in the direction from its Equator or of identical rotations in $\langle Q^{3+1}\rangle$, united in the general law by summary unity vector-radius $\mathbf{n}_{13}$, applied to the unity hyperspheroid with its scalar sine and vector cosine projections under conservation of added relations (280A) for the three basis unity vectors for such a type of presentations:

$$rot\ \Phi_{12}\cdot\mathbf{n}_{23} = \begin{array}{|c|c|}\hline I_{2\times 2}-(1-\cos\varphi_{12})\cdot\mathbf{e}_\alpha\mathbf{e}'_\alpha & +\sin\varphi_{12}\cdot\mathbf{e}_\alpha \\ \hline -\sin\varphi_{12}\cdot\mathbf{e}'_\alpha & \cos\varphi_{12} \\ \hline\end{array}\ \cdot\ \left\{\frac{\cos\varphi_{23}\cdot\mathbf{e}_\kappa}{-\sin\varphi_{23}}\right\} =$$

$$= \left\{\frac{[-\sin\ \varphi_{12}\cdot\sin\varphi_{23}-\cos\epsilon\cdot\cos\varphi_{23}\cdot(1-\cos\varphi_{12})]\cdot\mathbf{e}_\alpha+\cos\varphi_{23}\cdot\mathbf{e}_\kappa}{-\cos\varphi_{12}\cdot\sin\varphi_{23}-\cos\epsilon\cdot\sin\varphi_{12}\cdot\cos\varphi_{23}}\right\} =$$

$$= \left\{\frac{[-\sin\ \varphi_{12}\cdot\sin\varphi_{23}+\cos\epsilon\cdot\cos\varphi_{23}\cdot\cos\varphi_{12}]\cdot\mathbf{e}_\alpha+\sin\epsilon\cdot\cos\varphi_{23}\cdot\mathbf{e}_\mu}{-\cos\varphi_{12}\cdot\sin\varphi_{23}-\cos\epsilon\cdot\sin\varphi_{12}\cdot\cos\varphi_{23}}\right\} =$$

$$= \left\{\frac{\cos\varphi^*_{13}\cdot\mathbf{e}^*_\sigma}{-\sin\varphi^*_{13}}\right\} = \mathbf{n}_{13} \qquad (\mathbf{n}'_{13}\cdot\mathbf{n}_{13}=1). \tag{293A}$$

Now this operation summarizes immediately all scalar sine and vector cosine formulae for two-steps summation of motions on the hyperspheroid from its Equator, non-gotten in Ch. 8A, but gotten above for the hyperboloid I as the abstract analog of presentation $\mathbf{n}(\varphi)$. And for summation of such two-steps motions with the inverse order as $\mathbf{n}_{23},\mathbf{n}_{12}$, we obtain:

$$rot\ \Phi_{23}\cdot\mathbf{n}_{12} = \begin{array}{|c|c|}\hline I_{2\times 2}-(1-\cos\varphi_{23})\cdot\mathbf{e}_\kappa\mathbf{e}'_\kappa & +\sin\varphi_{23}\cdot\mathbf{e}_\kappa \\ \hline -\sin\varphi_{23}\cdot\mathbf{e}'_\kappa & \cos\varphi_{23} \\ \hline\end{array}\ \cdot\ \left\{\frac{\sin\varphi_{12}\cdot\mathbf{e}_\alpha}{-\cos\varphi_{12}}\right\} =$$

$$= \left\{\frac{[-\sin\ \varphi_{23}\cdot\cos\varphi_{12}-\cos\epsilon\cdot\sin\varphi_{12}\cdot(1-\cos\varphi_{23})]\cdot\mathbf{e}_\kappa+\sin\varphi_{12}\cdot\mathbf{e}_\alpha}{\cos\varphi_{12}\cdot\cos\varphi_{23}-\cos\epsilon\cdot\sin\varphi_{12}\cdot\sin\varphi_{23}}\right\} =$$

$$= \left\{\frac{(-\sin\ \varphi_{23}\cdot\cos\varphi_{12}+\cos\epsilon\cdot\sin\varphi_{12}\cdot\cos\varphi_{23})\cdot\mathbf{e}_\kappa+\sin\epsilon\cdot\sin\varphi_{12}\cdot\mathbf{e}_{\overset{\angle}{\nu}}}{\cos\varphi_{12}\cdot\cos\varphi_{23}-\cos\epsilon\cdot\sin\varphi_{12}\cdot\sin\varphi_{23}}\right\} =$$

$$= \left\{\frac{\sin\overset{\angle}{\varphi}{}^{*}_{13}\cdot\mathbf{e}^*_{\overset{\angle}{\sigma}}}{-\cos\overset{\angle}{\varphi}{}^{*}_{13}}\right\} = \overset{\angle}{\mathbf{n}}_{13} \qquad (\overset{\angle}{\mathbf{n}}{}'_{13}\cdot\overset{\angle}{\mathbf{n}}_{13}=+1). \tag{294A}$$

Here is: $\mathbf{e}_\kappa=\cos\epsilon\cdot\mathbf{e}_\alpha+\sin\epsilon\cdot\mathbf{e}_\mu,\ \ \mathbf{e}_\alpha=\cos\epsilon\cdot\mathbf{e}_\kappa+\sin\epsilon\cdot\mathbf{e}_{\overset{\angle}{\mu}}$.

Note, that for 3D hyperspheroid in complete $\langle Q^{3+1}\rangle$, all relations (291A)–(294) are united with all its four unity basis vectors, by abstract analogy (322, 323) with both hyperboloids.

Just as for summation of two-steps motions on the hyperboloid I, in more complex case of motions from the Equator in (293A, 294A), their direct and inverse sums are not connected by the orthospherical rotation $rot\ \Theta_{13}$, and these sums are generally non-commutative. Though also the summary vectors are situated always as if on the same hyperspheroid I – analog of the radius R with conservation of their Euclidean module R. But for realization of such losed properties and only in the spherical case, it is enough to add in the beginning the motion from the Pole till the Equator and further to move from the Equator.

Let's go back to differential motions in $\langle Q^{3+1}\rangle$ after (257A), (258A) with our tensor of motion $roth\ \Phi_i = F(\varphi_i, \mathbf{e}_\alpha)$. Pay attention to the fact that simultaneously with point M of a curve and the concomitant hyperspheroid with their moving common tangent $\mathbf{t}_\alpha = \mathbf{t}_\alpha{}^{(I)}$ and quasinormal $\mathbf{n}_\alpha = \mathbf{n}_\alpha{}^{(I)}$, there is the point N on the hyperspheroid with its also conjugate tangent $\mathbf{t}_\alpha{}^{(II)}$ and quasinormal $\mathbf{n}_\alpha{}^{(II)}$. Between $\mathbf{n}_\alpha{}^{(I)}$ and $\mathbf{n}_\alpha{}^{(II)}$ and also $\mathbf{t}_\alpha{}^{(I)}$ and $\mathbf{t}_\alpha{}^{(II)}$, there is rotation right angle $\pm\pi/2$. Hence, $\mathbf{n}_\alpha = \mathbf{n}_\alpha{}^{(I)} = \mathbf{t}_\alpha{}^{(II)}$, $\mathbf{t}_\alpha \equiv \mathbf{t}_\alpha{}^{(I)} \equiv \ \mathbf{n}_\alpha{}^{(II)}$. These features have place in $\langle Q^{3+1}\rangle$ for double differentials $d\varphi_i$ for simultaneous one–side spherical rotations of tangent $\mathbf{t}_\alpha$ and quasinormal $\mathbf{n}_\alpha$ under our spherical tensor of motion (313). Between all six basis vectors at a point M of a curve at $d\varphi_i \neq 0$, $d\alpha \neq 0$ and of concomitant unity hypersperoid there are the next one-to-one correspondences and relations in $\langle Q^{3+1}\rangle$:

$$\left.\begin{array}{l}
\mathbf{t}_\alpha = \mathbf{t}_{(I)} = \mathbf{n}_{(II)} = \mathbf{r}_{(II)} = -[\mathbf{n}_\alpha]'_\alpha - \text{of } \mathcal{K}_\alpha \text{ and 4-velocity } \mathbf{u}_\alpha \text{ below along a curve,}\\
\mathbf{n}_\alpha = \mathbf{n}_{(I)} = \mathbf{r}_{(I)} = \mathbf{t}_{(II)} = [\mathbf{t}_\alpha]'_\alpha - \text{of } \mathcal{K}_\alpha \text{ and 4-acceleration } [\mathbf{u}_\alpha]'_\alpha \text{ along a curve;}\\
\mathbf{b}_\nu = \mathbf{b}_{\nu\,(II)} \sim \mathbf{e}_{\nu\,(II)} - \text{of } \mathcal{K}_\nu \text{ and normal 3-shift } \sin\varphi_i\ d\alpha_1 \text{ or sine 3-acceleration,}\\
\mathbf{b}_\mu = \mathbf{b}_{\mu\,(I)} \sim \mathbf{e}_{\mu\,(I)} - \text{of } \mathcal{K}_\mu \text{ and normal 3-shift } -\cos\varphi_i\ d\alpha_2 \text{ or cosine 3-acceleration;}\\
\{\mathbf{b}_\mu\}'_\gamma = \{\mathbf{b}_\nu\};\ \{\mathbf{b}_\nu\}'_{\gamma,\alpha} = 0.
\end{array}\right\} \quad (295A)$$

We obtain in 5-th raw our last third and forth formulae along a curve, in addition, to both spherical analogs of our previous first (228A) and second (238A) for hyperbolic ones !!! Both particular cases in two $\langle Q^{2+1}\rangle$ are realized in normal rotation either only of tangent with sine binormal $\mathbf{b}_\nu$ or only of quasinormal with cosine binormal $\mathbf{b}_\mu$. They are no connected here through the spherical relation $\sin\varphi_i = \cos\xi_i$ (i. e., in $\langle Q^{3+1}\rangle$ and on the 3D hyperspheroid)!

Further, using our tensor trigonometric approach to consideration of the complete theory of motions along a world line in the Minkowski space-time, we'll state the theory of motions along a regular curve also generally in the quasi-Euclidean space $\langle Q^{3+1}\rangle_s$ under acting our spherical tensor of motion in (313), (314) with the concomitant unity hyperspheroid, but for illustration with the sine binormal, in addition, to the previous considerations in Ch. 8A. The principal and free-valued characteristics $\mathbf{k}_\alpha$ and $\mathbf{k}_\beta$ are produced with the 1-st differentiations in dl along a curve with one and two degrees of freedom (at $\zeta = 3$), logically accompanied by the concomitant hyperspheroid from zero point in its North Pole as if in $\langle Q^{2+1}\rangle_s$:

$$\left.\begin{array}{l}
\left\{\dfrac{d\mathbf{t}_\alpha(l)}{dl}\right\}_\alpha = \mathcal{K}_\alpha(l)\cdot\begin{bmatrix}\cos\varphi_i\cdot\mathbf{e}_\alpha\\ \sin\varphi_i\end{bmatrix} = \mathcal{K}_\alpha(l)\cdot\mathbf{n}_\alpha(l) = \mathbf{k}_\alpha(l),\\[2ex]
\dfrac{d\mathbf{t}_\alpha(l)}{dl} = \mathcal{K}_\beta(l)\cdot\begin{bmatrix}\cos\varphi_p\cdot\mathbf{e}_\beta\\ \sin\varphi_p\end{bmatrix} = \mathcal{K}_\beta(l)\cdot\mathbf{n}_\beta(l) = \mathbf{k}_\beta(l).
\end{array}\right\}$$

First expression is the tensor trigonometric *quasianalog of the 1-st Frenet–Serret formula* in the usual 3D Euclidean space. But with our two-steps approach, second expression must reveal the sine binormal in the sine normal plane. Unity 4-vectors $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$ are principal and free quasinormals to a curve. Derivatives in φ_i – $\mathbf{t}_\alpha$ and $\mathbf{n}_\alpha$ at the change of a curve slope to $\overrightarrow{y}$ are rotated in one side at $d\varphi_i$. In the binary quasi-Euclidean space $\langle Q^{2+1}\rangle_s$ with metric tensor $\{I^+\}$ at $\zeta \geq 3$, due to (295A) with the use of (137A), we execute the first two-steps differentiation along a curve with revealing all relative and absolute characteristics:

$$\mathbf{k}_\beta(l) = \frac{d\mathbf{t}_\alpha(l)}{dl} = \frac{d\varphi_p}{dl}\cdot\begin{bmatrix}\cos\varphi_p\cdot\mathbf{e}_\beta\\ \sin\varphi_p\end{bmatrix} = \frac{d\varphi_p}{dl}\cdot\mathbf{n}_\beta(l) = \mathcal{K}_\beta(l)\cdot\mathbf{n}_\beta(l) \equiv \qquad (296A)$$

$$\equiv \frac{d\varphi_i}{dl}\cdot\begin{bmatrix}\cos\varphi_i\cdot\mathbf{e}_\alpha\\ \sin\varphi_i\end{bmatrix}_\alpha + \begin{bmatrix}\sin\varphi_i\cdot\dfrac{d\mathbf{e}_\alpha}{dl}\\ 0\end{bmatrix}_\varphi^{(1)} = \frac{d\varphi_i}{dl}\cdot\begin{bmatrix}\cos\varphi_i\cdot\mathbf{e}_\alpha\\ \sin\varphi_i\end{bmatrix}_\alpha + \begin{bmatrix}\sin\varphi_i\cdot\dfrac{d\alpha_1}{dl}\cdot\mathbf{e}_\nu\\ 0\end{bmatrix}_\varphi^{(1)} =$$

$$= K_\alpha(l)\cdot\begin{bmatrix}\cos\varphi_i\cdot\mathbf{e}_\alpha\\ \sin\varphi_i\end{bmatrix}_\alpha + K_\nu(l)\cdot\begin{bmatrix}\mathbf{e}_\nu\\ 0\end{bmatrix}_\varphi^{(1)} = K_\alpha(l)\cdot\mathbf{n}_\alpha(l) + K_\nu(l)\cdot\mathbf{b}_\nu(l) \equiv$$

$$\equiv \frac{d\varphi_p}{d(l)}\cdot\begin{bmatrix}\cos\varphi_p\cdot\mathbf{e}_\beta\\ \sin\varphi_p\end{bmatrix} = \frac{d\varphi_p}{dl}\cdot\left\{\begin{bmatrix}\cos\varepsilon\cdot\cos\varphi_p\cdot\mathbf{e}_\alpha\\ \sin\varphi_p\end{bmatrix} + \begin{bmatrix}\sin\varepsilon\cdot\cos\varphi_p\cdot\mathbf{e}_\nu\\ 0\end{bmatrix}^{(1)}\right\} =$$

$$= \mathcal{K}_\beta(l)\cdot\begin{bmatrix}\cos\varphi_p\cdot\mathbf{e}_\beta\\ \sin\varphi_p\end{bmatrix} = \mathcal{K}_\beta(l)\cdot\mathbf{n}_\beta(l) = \overline{\overline{\mathcal{K}_\beta^\times}}\cdot\mathbf{n}_\alpha(l) + \overset{\perp}{\mathcal{K}_\beta^\star}\cdot\mathbf{b}_\nu(l) = \overline{\overline{\mathbf{k}_\beta^\times}}(l) + \overset{\perp}{\mathbf{k}_\beta^\star}(l) = \mathbf{k}_\alpha + \mathbf{k}_\nu.$$

Equaling under I^+ paired summands, we get relations above with $\varrho > \varepsilon$: $d\varphi_p^2 = \cos^2\varphi_p\, d\varphi_p^2 + \sin^2\varphi_p\, d\varphi_p^2 =$
$= \underline{(\cos^2\varepsilon\cdot\cos^2\varphi_p\, d\varphi_p^2 + \sin^2\varepsilon\cdot\cos^2\varphi_p\, d\varphi_p^2)} + \sin^2\varphi_p\, d\varphi_p^2 = \underline{(\cos^2\varphi_i\, d\varphi_i^2 + \sin^2\varphi_i\, d\alpha_1{}^2)} + \sin^2\varphi_i\, d\varphi_i^2 =$
$= \underline{d\varphi_i^2 + \sin^2\varphi_i\, d\alpha_1{}^2} = \underline{(\cos^2\varepsilon\cdot\cos^2\varphi_p + \sin^2\varphi_p)\, d\varphi_p^2 + (\sin^2\varepsilon\cdot\cos^2\varphi_p)\, d\varphi_p^2} = \underline{\cos^2\varrho\, d\varphi_p^2 + \sin^2\varrho\, d\varphi_p^2} > 0,$
Surprisingly, but we get two identical decompositions of $d\varphi_i$ – quasi-Euclidean and Euclidean (with underline for Relative and Absolute Theorems), the latter corresponds to 1-st metric form of hyperspheroid (109A-II) in Ch.6A ! *This paradox* relates to hypotenuses of right triangles only for moving from its Pole at $n \geq 2$.

For a dynamic of regular curves in $\langle\mathcal{Q}^{3+1}\rangle$, let up, that the point M of a curve moves with the constant quasi-Euclidean 4-velocity $\mathbf{u}$, as analogue of 4-velocity by Poincaré in $\langle\mathcal{P}^{3+1}\rangle$. *The Relative Pythagorean theorem* below follows from the Euclidean part of (296A) in vector and scalar quadric forms. It acts really in the sine normal plane $\langle\mathcal{E}^2\rangle^{(m)}_{Ns} \equiv \langle\mathbf{e}^{(m)}_\alpha, \mathbf{e}^{(1)}_\nu\rangle$ for all proportional geometric characteristics as orthoprojections into the Cartesian subbase $\tilde{E}^{(3)}_1$ at the motion angle $\varphi \in [0, \pi/2]$ and the angle of normal deviation $\varepsilon \in [0;\pi]$, using (296A) with (137A) and confirming preliminary valid two-steps decompositions (194A) in Ch. 8A:

$$\left\{\begin{array}{l} \cos\varphi_p\, d\varphi_p\cdot\mathbf{e}_\beta = \cos\varphi_p\,(\cos\varepsilon\, d\varphi_p\cdot\mathbf{e}_\alpha + \sin\varepsilon\, d\varphi_p\cdot\mathbf{e}_\nu) = \cos\varphi_i\, d\varphi_i\cdot\mathbf{e}_\alpha + \sin\varphi_i\, d\alpha\cdot\mathbf{e}_\nu,\\ \cos^2\varphi_p\, d\varphi_p^2 = \cos^2\varphi_p\,(\cos^2\varepsilon\, d\varphi_p^2 + \sin^2\varepsilon\, d\varphi_p^2) = \cos^2\varphi_p\,[(\overline{\overline{d\varphi_p}})^2_E + (\overset{\perp}{d\varphi_p})^2_E] =\\ = \cos^2\varphi_i\, d\varphi_i^2 + \sin^2\varphi_i\, d\alpha^2 = \sin^2\xi_p\, d\xi_p^2 = \sin^2\xi_p\,[(\overline{\overline{d\xi_p}})^2_E + (\overset{\perp}{d\xi_p})^2_E] = \sin^2\xi_i\, d\xi_i^2 + \cos^2\xi_i\, d\alpha^2; \end{array}\right\} \Rightarrow$$

$$\Rightarrow\left\{\begin{array}{l} \mathcal{K}_\beta\cdot\cos\varphi_p\cdot\mathbf{e}_\beta = \mathcal{K}^*_\beta\cdot\mathbf{e}_\beta = \cos\varepsilon\cdot\mathcal{K}^*_\beta\cdot\mathbf{e}_\alpha + \sin\varepsilon\cdot\mathcal{K}^*_\beta\cdot\ \mathbf{e}_\nu = \ \overline{\overline{\mathcal{K}^*}}_\beta\cdot\mathbf{e}_\alpha + \ \overset{\perp}{\mathcal{K}^*_\beta}\cdot\mathbf{e}_\nu =\\ = \mathcal{K}_\alpha\cdot\cos\varphi_i\cdot\mathbf{e}_\alpha + \sin\varphi_i\cdot\dfrac{d\alpha}{dl} = \mathcal{K}^*_\alpha\cdot\mathbf{e}_\alpha + \dfrac{v^*_i\cdot w^*_\alpha}{v^2}\cdot\mathbf{e}_\nu = \mathcal{K}^*_\alpha\cdot\mathbf{e}_\alpha + \mathcal{K}_\nu\cdot\mathbf{e}_\nu = \end{array}\right\}\Rightarrow$$

$$\Rightarrow\left\{\begin{array}{l} = \mathbf{k}^*_\beta = \overline{\overline{\mathbf{k}^*_\beta}} + \overset{\perp}{\mathbf{k}^*_\beta} = \mathbf{k}^*_\alpha + \mathbf{k}_\nu,\\ (\mathcal{K}^*_\beta)^2 = (\overline{\overline{\mathcal{K}^*_\beta}})^2 + (\overset{\perp}{\mathcal{K}^*_\beta})^2 = (\mathcal{K}^*_\alpha)^2\ + (\mathcal{K}_\nu)^2; \end{array}\right\} \qquad (297A)$$

$$\mathcal{K}_\beta\cdot\sin\varphi_p = \mathcal{K}_\alpha\cdot\sin\varphi_i \Leftrightarrow \boxed{\sin\varphi_p\, d\varphi_p = \sin\varphi_i\, d\varphi_i} \to d\varphi_p/d\varphi_i > 1. \qquad (298A)$$

$$\Rightarrow\ \cos\varphi_p\cdot\cos\varepsilon\, d\varphi_p = \cos\varphi_p\,\overline{\overline{d\varphi_p}} = \cos\varphi_i\, d\varphi_i \Rightarrow\ \cos\varepsilon = 1 \leftrightarrow \varphi_p = \varphi_i,\ \cos\varepsilon = 0 \leftrightarrow \gamma_p = 0;$$
$$\sin\varphi_i = v^*_i/u \leq 1,\ \tan\varphi_i = v_i/u,\ \varphi_i \leq \pi/2;\ \{\tan\varphi_p = \cos\varepsilon\cdot\tan\varphi_i\} \to \varphi_p < \varphi_i\ \{\varepsilon\in[0;\pi]\}$$
$$\varphi_p/\varphi_i < 1 - \text{see above}\ \Rightarrow\ v_p < v_i,\ \varphi_i = 0 \to \varphi_p = 0;\ d\varphi_p > \overline{\overline{d\varphi_p}} > d\varphi_i\ \{\varphi\in[0,\pi/2]\}.$$

From (296A)–(298A), we obtain the *Absolute Euclidean Pythagorean theorem* with the *1-st mobile trihedron* $\hat{E}^{(3)}_m = \langle\mathbf{n}_\alpha,\ \mathbf{b}_\nu,\ \mathbf{i}_\alpha\rangle$ in $\langle\mathcal{Q}^{2+1}\rangle$ **under metric tensor** I^+. It acts on the Euclidean sine normal plane $\langle\mathcal{E}^2\rangle^{(m)}_{Ns\ II} \equiv \langle\mathbf{n}^{(m)}_\alpha, \mathbf{b}^{(1)}_\nu\rangle$ in 3D $\langle\mathcal{Q}^{2+1}\rangle_s \equiv \{\langle\mathcal{E}^2\rangle^{(m)}_{Ns} \boxplus \overrightarrow{y}\}$ $(\zeta = 3)$. In the right triangle of $\mathbf{t}_\alpha$ rotations, it corresponds to the *angular normal* 1-st metric form (109A-II) for the concomitant hyperspheroid (!!!), as a *perfect hypersurface* of $\langle\mathcal{Q}^{2+1}\rangle$. It is expressed in the universal complete *tensor-vector-scalar ("tvs") form* with own trigonometric and proportional geometric items:

$$\left\{\begin{array}{l} \mathbf{k}_\beta = \mathcal{K}_\beta\ \mathbf{n}_\beta = \ \overline{\overline{\mathcal{K}^\times_\beta}}\ \mathbf{n}_\alpha + \ \overset{\perp}{\mathcal{K}^*_\beta}\ \ \mathbf{b}_\nu = \mathcal{K}_\alpha\ \mathbf{n}_\alpha + \mathcal{K}_\nu\ \mathbf{b}_\nu = \mathbf{k}_\alpha + \mathbf{k}_\nu,\\ \mathcal{K}^2_\beta = (\mathcal{K}^*_\beta)^2 - (\mathcal{K}^\circ_\beta)^2 = (\ \overline{\overline{\mathcal{K}^\times_\beta}})^2 + (\overset{\perp}{\mathcal{K}^*_\beta})^2 = \mathcal{K}^2_\alpha + \mathcal{K}^2_\nu, \end{array}\right\}\Rightarrow$$

$$\Rightarrow\left\{\begin{array}{l} d\varphi_p\cdot\mathbf{n}_\beta = d\varphi_i\cdot\mathbf{n}_\alpha + \sin\varphi_i\, d\alpha\cdot\mathbf{b}_\nu,\quad (\mathbf{n}'_\alpha\cdot I^+\cdot\mathbf{n}_\alpha = +1,\quad \mathbf{b}'_\nu\cdot I^+\cdot\mathbf{b}_\nu = +1)\\ d\varphi_p^2 = d\varphi_i^2 + \sin^2\varphi_i\, d\alpha^2 = \cos^2\varrho\, d\varphi_p{}^2 + \sin^2\varrho\, d\varphi_p{}^2 = \left(\overline{\overline{d\varphi_p}}\right)^2_Q + \left(\overset{\perp}{d\varphi_p}\right)^2_E > 0. \end{array}\right\} \qquad (299A)$$

Here $d\varphi_p = dl_R/R,\ \varrho > \varepsilon$. By this Egregium Theorem of Differential Tensor Trigonometry (1-st from two spherical), we reduce this mixed motion in initial $\tilde{E}_1$ along a curve and on the hyperspheroid as a perfect surface to purely spherical one along hypotenuse $d\varphi_p$ in $\tilde{E}_m$. This Theorem in tvs-form corresponds to analog (109A-II), Ch. 6A, in vs-form, where φ_i is the angle of motion namely from the Pole of the hyperspheroid and it is calculated at $d\alpha_1$ off its frame axis, as in analogical hyperbolic cases (132A) and (228A) γ_i on the hyperboloid II.

In (109A-I) φ_i was the angle of motion namely from the Equator (Euclidean subspace or axis) of the hyperspheroid, φ_i is calculated at $d\alpha_2$ off it, as γ_i in analogous hyperbolic cases (133A) and (238A) on the hyperboloid I. Since both complementary spherical angles are bonded by simplest formula $\xi = \pi/2 - \varphi$, then in metric forms (109A-II) and (109A-I), they are simply exchanged, however their nature at the Euclidean normal part with $d\alpha$ must correspond to those, indicated above for these two types of motions on the hyperspheroid. Then our readers may test understanding the problem, inferring the spherical 2-nd Egregium Theorem (which is closer by its cosine sense to the 2-nd Frenet-Serret formula in the Euclidean space $\langle\mathcal{E}^3\rangle$), with revealing its cosine binormal $\mathbf{b}_\mu$ at $d\alpha_2$ in $\langle\mathcal{Q}^{2+1}\rangle_c \equiv \{\langle\mathcal{E}^2\rangle^{(m)}_{Nc} \boxplus \overrightarrow{y}\}$ $(\zeta = 3)$.

The hyperbolic complementary angle υ_i (non of motions) is calculated at α contrary to motion angle γ_i for (228A) – *namely from the isotropic cone or diagonal of* $\langle\mathcal{P}^{3+1}\rangle$, but also to the frame axis, and for (238A) – *namely from the isotropic cone or diagonal of* $\langle\mathcal{P}^{3+1}\rangle$, but also to the Euclidean subspace or axis. See these latter facts descriptively on the front and back covers of this book, and our readers may present this peculiarity at Figure 2A(1) with its isotropic diagonal. That is why, the complementary angles γ and υ with their $d\gamma$ and $d\upsilon$ are connected by complex formulae (360-II), (360-IY), (360-Y), inferred in Ch. 6 of the main Part-II. As a consequence, both concomitant hyperboloids are divided, and the metric forms of a world line and both of them are expressed only through the motion angle γ_i, and they are calculated at α for II from the frame axis and for I from the Equator !!!

Let us represent tensor analogs of hyperbolic (261A), (262A) in the quasi-Euclidean space.

In entire $\langle\mathcal{Q}^{3+1}\rangle$ with tetrahedron for a regular curve, we get orthogonal tensors $\quad (300A - I)$

$$rot\ \Phi_i = F_s(\varphi_i, \mathbf{e}_\alpha),\ \ (F \neq F'),\ \langle\mathcal{Q}^{3+1}\rangle \qquad V(\varphi_i, \mathbf{e}_\alpha, \mathbf{e}_\nu, \mathbf{e}_\mu),\ \ (V \neq V'),\ \langle\mathcal{Q}^{3+1}\rangle$$

$$\begin{array}{|c|c|}\hline \cos\varphi_i \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' + \overrightarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha}' & \sin\varphi_i \cdot \mathbf{e}_\alpha \\ \hline -\sin\varphi_i \cdot \mathbf{e}'_\alpha & \cos\varphi_i \\ \hline \end{array} \cdots\cdots \begin{array}{|c|c|c|c|}\hline \cos\varphi_i \cdot \mathbf{e}_\alpha & \mathbf{e}_\nu & \mathbf{e}_\mu & \sin\varphi_i \cdot \mathbf{e}_\alpha \\ \hline -\sin\varphi_i & 0 & 0 & \cos\varphi_i \\ \hline \end{array},$$

and else with two trihedrons both possible orthogonal tensors $\quad (300A - II)$

$$V(\varphi_i, \mathbf{e}_\alpha, \mathbf{e}_\nu),\ \ (V \neq V'),\ \langle\mathcal{Q}^{2+1}\rangle_{(t)} \qquad V(\varphi_i, \mathbf{e}_\alpha, \mathbf{e}_\mu),\ \ (V \neq V'),\ \langle\mathcal{Q}^{2+1}\rangle_{(n)}$$

$$\begin{array}{|c|c|c|}\hline \cos\varphi_i \cdot \mathbf{e}_\alpha & \mathbf{e}_\nu & \sin\varphi_i \cdot \mathbf{e}_\alpha \\ \hline -\sin\varphi_i & 0 & \cos\varphi_i \\ \hline \end{array} \cdots\cdots \begin{array}{|c|c|c|}\hline \cos\varphi_i \cdot \mathbf{e}_\alpha & \mathbf{e}_\mu & \sin\varphi_i \cdot \mathbf{e}_\alpha \\ \hline -\sin\varphi_i & 0 & \cos\varphi_i \\ \hline \end{array}.$$

$$(V = rot\ \Phi_i \cdot rot\ \Theta_i\ \Rightarrow\ rot\ \Phi_i = \sqrt{VV^*},\ rot\ \Theta = rot\ (-\Phi_i) \cdot V = \sqrt{VV^*}^{-1} \cdot V)$$

* * *

Tensor Trigonometry, may be, is the most musical subject of the Mathematical Science. This is stated due to the clear harmony of all its tensor angles and trigonometric functions as each to others in tensor, vector and scalar forms. If such harmony does not work, then the results contain an error. That is why, the greatest mathematician and man Leonard Euler, who created the logarithmic theory of the musical scale with explanation of its harmony, elegantly presented and gave also a modern look to the Scalar Trigonometry! It is such a true golden rule, which consists in observing this high harmony in formulae and theorems of the Tensor Trigonometry, allowed the author in its third edition to achieve the most correct and complete presentation of Theory of world lines in the Poincaré–Minkowski space-time with all their geometric characteristics and with interpretation of their physical senses. Besides, this golden rule allowed the author to give tensor-trigonometric explanations of all well-known and new relativistic effects, including such in the gravitational field without GTR-bending of the most perfect space-time $\langle\mathcal{P}^{3+1}\rangle$ of the Universe, which is still used really in astronomy. It is space-time bending, without its true necessity, has caused the non-compatibility of GTR with the Quantum Mechanics. Though, according to the Noether's Theorems, it is space-time $\langle\mathcal{P}^{3+1}\rangle$ ensures a strict compliance of the Theory of Relativity with the fundamental Law of Energy-Momentum Conservation as the accompanied physical harmony.

If Henri Poincaré life had not ended so early – at the age of 58, he, may be, continuing own pioneering relativistic works, would develop further his trigonometric approach till its tensor level with more general concepts of the binary spaces and their perfect hypersurfaces. The very spirit of his unwavering striving for novelty and grandiose generalization of existing particular theories speaks about this! Unfortunately, some mathematicians and physicists have blackened their names by borrowing brilliant ideas from his works in own publications without references to their author. So, the honest and eminent mathematician V. I. Arnol'd wrote about such facts in his Essay [110], as author's Russian version without English edits.

In our monograph, there are many examples of applications of Tensor Trigonometry in geometric and physical fields. Thus, it reveals clarity the true cause of angular deviations in convex figures on non-Euclidean surfaces of spherical and hyperbolic types as the cosine orthospherical shifts in their apexes (Chs. 7A, 8A). In the hyperbolic case, Identity of this negative orthospherical shift, but in time (!), with the Thomas precession is established.

The Integral Laws of Energy and Momentum conservation in the Minkowski space-time $\langle \mathcal{P}^{3+1} \rangle$ can be simply inferred by the *4D Absolute pseudo-Euclidean Pythagorean Theorem of three momenta*, gotten by us in (99A), Ch. 5A, with the use of the absolute *own* 4×1-*momentum* $\mathbf{P}_0 = P_0 \cdot \mathbf{i}_\alpha$ as a right column of the 4×4-tensor of momentum $\mathcal{T}_P = P_0 \cdot roth\Gamma_t$. I. e., the last is proportional to our measureless trigonometric tensor of motion $roth\Gamma_t$ (100A). They have four independent scalar arguments, as hyperbolic angle of motion γ_t and its unity 3-vector of three directional cosines $\mathbf{e}_\alpha$. Then, in initial pseudo-Cartesian base $\tilde{E}_1 = \langle \mathbf{x}, \overrightarrow{ct} \rangle$ of $\langle \mathcal{P}^{3+1} \rangle$, the tensors $\mathcal{T}_P$ and $\mathcal{T}_E$ have proportional to (100A) canonical physical structures with the 4×1-momentum $\mathbf{P}_0$ as 4-th column of $\mathcal{T}_P$ (under $c = \text{const}$ and $\gamma > 0$ as $\Delta ct > 0$):

$$\mathcal{T}_P == P_0 \cdot \begin{array}{|c|c|} \hline I_{3\times3} + (\cosh\gamma - 1) \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} & \sinh\gamma \cdot \mathbf{e}_\alpha \\ \hline \sinh\gamma \cdot \mathbf{e}_\alpha' & \cosh\gamma \\ \hline \end{array} = \begin{array}{|c|c|} \hline P_0 \cdot I_{3\times3} + \Delta P \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} & \mathbf{p} \\ \hline \mathbf{p}' & P \\ \hline \end{array} = m_0 c \cdot roth\Gamma,$$

$$\mathcal{T}_E = E_0 \cdot \begin{array}{|c|c|} \hline I_{3\times3} + (\cosh\gamma - 1) \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} & \sinh\gamma \cdot \mathbf{e}_\alpha \\ \hline \sinh\gamma \cdot \mathbf{e}_\alpha' & \cosh\gamma \\ \hline \end{array} = \begin{array}{|c|c|} \hline E_0 \cdot I_{3\times3} + A \cdot \overleftarrow{\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha{}'} & \mathbf{p}c \\ \hline \mathbf{p}'c & E \\ \hline \end{array} = m_0 c^2 \cdot roth\Gamma;$$

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{p} \\ P \end{bmatrix} = P_0 \cdot \mathbf{i}_\alpha = m_0 \cdot \mathbf{c} = P_0 \cdot \begin{bmatrix} \mathbf{sinh}\,\gamma \\ \cosh\gamma \end{bmatrix} = P_0 \cdot \begin{bmatrix} \sinh\gamma \cdot \mathbf{e}_\alpha \\ \cosh\gamma \end{bmatrix} = \begin{bmatrix} m_0 \mathbf{v}^* \\ m_0 c^* \end{bmatrix} = \begin{bmatrix} m\mathbf{v} \\ mc \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ E/c \end{bmatrix},$$

$$E = \cosh\gamma \cdot E_0 = E_0 + (\cosh\gamma - 1) \cdot E_0 = E_0 + k_E \cdot E_0 = E_0 + A, \text{ where } k_E = \cosh\gamma - 1 = \Delta E / E_0.$$

$\mathcal{T}_E$ includes the total 4×1-momentum $\mathbf{P}$ and the Hamiltonian as scalar E and in Euclidean 3×3-tensor form, where the work $A = \Delta E$ acts logically in the direction $\mathbf{e}_\alpha$ as in (173A)! We have the trigonometric and proportional physical concepts staying on a world line in absolute $\langle \mathcal{P}^{3+1} \rangle$ with tensors of momentum and energy in $\tilde{E}_1$ under 4-velocity $\mathbf{c}$ of Poincaré.

The vectorial own 4×1-momentum $\mathbf{P}_0 = m_0 \cdot \mathbf{c}$, directed along a world line, has its invariant scalar value $P_0 = m_0 c$ (proportional to $E_0 = m_0 c^2$). Therefore $\mathbf{P}_0$, without the inner force F, can change only its direction in the internal (light) cone under constant P_0. Mass $m_0 \neq 0$ (as P_0/c or E_0/c^2) is used for massive objects. The relative projections are:

$P = P_0 \cdot \cosh\gamma = mc$ as the scalar cosine orthoprojection onto the time-arrow $\overrightarrow{ct^{(1)}}$,
$\mathbf{p} = P_0 \cdot \sinh\gamma \cdot \mathbf{e}_\alpha = m\mathbf{v}$ as the sine orthoprojection into the Euclidean subspace $\langle \mathcal{E}^3 \rangle^{(1)}$.

In insulated systems, there is the absolute preserving characteristic under passive Lorentz transformations: $\mathbf{P}_0 = P_0 \cdot \mathbf{i}_\alpha = \mathbf{const}$ as the *invariant hypotenuse* of the Pythagorean right triangle of 3 momenta. Its relative cathetuses are preserved under next own conditions.
The mechanical energy $E = c \cdot P_0 \cdot \cosh\gamma = cP = \text{const}$ under γ is constant.
The real momentum $\mathbf{p} = P_0 \cdot \sinh\gamma \cdot \mathbf{e}_\alpha = \mathbf{const}$ under γ and $\mathbf{e}_\alpha$ are constant together.

Similar approach is applicable at arbitrary quantity of moving and no interacting *massive* material points also in the insulated for them system, with its various adopted bases $\tilde{E}_t$:

$$\Sigma \mathbf{P}_{0(k)} = \Sigma[P_{0(k)} \cdot \mathbf{i}_{(k)}] = c \cdot \Sigma[m_{0(k)} \cdot \mathbf{i}_{(k)}] = \mathbf{const}.$$

Our inferences are in complete and one-to-one correspondences of Tensor Trigonometry with fundamental concepts as the Noether Theorems, the Higgs Theory, the Mach Principle and the isomorphic mathematical and physical Principles of Relativity in sect. 12.3 and Ch. 1A.

Besides, for the author, in these scrupulously studied by him areas of the exact science, one of the most surprises, revealed by Tensor Trigonometry in this Appendix among many others miracles, was not only the presence, but the abundance of Absolute and Relative Pythagorean theorems in their quasi-Euclidean and pseudo-Euclidean versions in $\langle \mathcal{Q}^{3+1} \rangle$, $\langle \mathcal{Q}^{3+1} \rangle_c$ and $\langle \mathcal{P}^{3+1} \rangle$, including their sine and cosine 3D binary subspaces; and even (!) on the embedded *curvilinear perfect hypersurfaces* with their three non-Euclidean geometries and the spherical, hyperbolic and mixed hyperbolic–elliptical principal motions on them. So, Relative Euclidean Big and Small Pythagorean theorems relate to summing motion angles in trigonometric functions with their reduction to the original Cartesian subbase; 4D Absolute and 3D Relative Pythagorean theorems relate to summing motions' angular differentials. *The pseudo-Euclidean Tensor Trigonometry is an isomorphic progenitor of all formulae and theorems of the non-Euclidean geometry on the three sheets perfect hypersurface in* $\langle \mathcal{P}^{3+1} \rangle$, *formulae and laws of the Theory of Relativity with using corresponding constant factors!!!*

For applications of the Tensor Trigonometry to relativistic calculations of ultra-long-distance space travel, at least to the star systems closest to us (which has become now the subject of intense interest), we, in principle, covered this question both at the end of Chapter 5A, having obtained for this purpose a relativistic version of the Ziolkovsky cosmic formula with examination of its application to our specific extreme example of space travel, and at the end of Chapter 7A with coordinate representation of the travel itself under its kinematics and dynamics. As became clarity from results in Chapter 5A, in the foreseeable future such ultra-long-distance travel is possible only for robotic ships, moreover of miniature sizes and equipped with highly advanced artificial intelligence, and, of course, with the use of maximum possible acceleration for them to achieve near-light speeds. For now, their task can only consist of identifying in the promising star systems the presence of the most suitable planet for the implementation of Earthly life on it and communicating this with a power laser signal towards Earth. And only after this will it be possible to send astronauts-settlers there in one direction, and even then, most likely, with their long-term freezing! Otherwise, nothing will remain of our civilization with its living and culture worlds – especially since the fanatical politicians only accelerate its destruction and death with no entrance to space!

Of course, no one can force for mathematicians and physicists else in the Past to stop operating in relativistic transformations with the relativistic factors "γ" (as our $\mathbf{cosh}\,\gamma$) and "β" (as our $\mathbf{tanh}\,\gamma$), and they continue to suffer with them in numerous operations and doubt whether they have been correctly performed, instead of switching to application of simple and well understandable tensor trigonometric operations in their scalar, vector and tensor forms. From the point of view of Tensor Trigonometry, using such factors as "γ" and "β", i. e., only really of the cosine and tangent functions, for development of Theory of Relativity and its numerous applications is a really pseudo-scientific sadomasochismus absurdity. These factors do not provide any visual theoretical representation. It is a great pity for the relativists, who from the beginning doomed themselves to torments of "creativity" with them, instead using simplest and descriptive trigonometric approach in tvs-forms above. Without this approach, up to now such enthusiasts do not understand difference in senses of factors "γ" and "$\mathbf{1}/\gamma$" in cosine and secant formulae of STR (see in Chs. 3A, 4A), though they relate to different transformations: sine-cosine rotations as group and tangent-secant deformations as no group!

Let's hope that progress and useful scientific renovations cannot be stopped!

The author of Tensor Trigonometry wishes creative success to all those researchers and readers, who will continue to apply and further develop all these new geometric directions established in this book since 2004 and hopes for high notions of scientific ethics from all its users and readers. The great mathematician, physicist and man Henri Poincaré stated the highest ethical bar for scientists in the past 20th century. By our opinion, the most terrible crime in the Science is deliberate and camouflaged plagiarism. Besides, the author is opposed to any, especially hidden, manifestation of mossy nationalism in the Saint scientific sphere!

⋆⋆⋆ The End ⋆⋆⋆

Mathematical–Physical Kunstkammer

1. Consider an algebraic equation of power n with real positive coefficients in its alternating-sign form. Represent Cardano's ($n = 3$) and Ferrari's ($n = 4$) formulae in terms of small and large medians.

Prove that, if the roots of the algebraic equation in such form are real-valued numbers, then at any "n" there hold:

$$0 < k_2 < [(n-1)/2n]k_1^2.$$

Give the similar chain for all the coefficients.

2. Explain why each of the following equations has complex conjugate roots with positive real parts.

$$y(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 90x - 64 = 0,$$

$$y(x) = x^5 - 10x^4 + 40x^3 - 70x^2 + 80x - 64 = 0,$$

$$y(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 75x - 60 = 0,$$

$$y(x) = x^5 - 25x^4 + 90x^3 - 640x^2 + 80x - 1 = 0,$$

$$y(x) = x^5 - 25x^4 + 160x^3 + 80x - 1 = 0.$$

General conditions to coefficients of an algebraic equation for its roots to be real-valued see in our other monograph [17].

3. Equation $y = ||\mathbf{z}(\mathbf{x})|| = ||A\mathbf{x} - \mathbf{a}|| = \min$, where A is a $m \times n$-matrix, $\mathbf{a}$ is a n-vector, has a unique solution $\mathbf{x} = \mathbf{b}$. Express $\mathbf{b}$, $\mathbf{z}(\mathbf{b})$, and $y(\mathbf{b})$ as formulae only with A and $\mathbf{a}$. Find the spherical angle between the vector $\mathbf{b}$ and the plane $\langle im\ A\rangle$. Find condition for it be zero, be right. What is the geometric nature of the vector $\mathbf{z}(\mathbf{b})$ in the m-dimensional Euclidean space? How does geometry of solutions depend on relations between m and n?

4. For a pair of conjugate complex numbers with operations over them, give their real-valued representations without the imaginary unit. What is the main distinction between complex-valued representations of such numbers with operations and these real-valued ones?

Prove that a *real-valued* algebraic equation of power n has a complete *real-valued* general solution unique up to admitted permutations.

5. In the first half of the 19-th century Urbain Le Verrier "discovered on tip of a pen" (by the words of F.-J. Arago) the new planet Neptune (1846). He used his own algorithm for inverting a square matrix B with evaluating scalar characteristic coefficients of the matrix B in terms of traces of powers B^t. Prove the following statements for these characteristic coefficients of a $n \times n$-matrix B and its powers B^t, $1 \le t \le n$.

a. If $tr\ B = tr\ B^2 = \cdots = tr\ B^j = \cdots = tr\ B^t = +1$, then $k(B,t) = 0$. In particular, $det\ B = 0$ if $t = n$.

b. If $tr\ B = tr\ B^2 = \cdots = tr\ B^j = \cdots = tr\ B^t = -1$, then $k(B,t) = (-1)^t$. In particular, $det\ B = (-1)^n$.

c. If $tr\ B = tr\ B^2 = \cdots = tr\ B^j = \cdots = tr\ B^t = +t$, then $k(B,t) = +1$.

d. If $-tr\ B = +tr\ B^2 = \cdots = (-1)^j tr\ B^j = \cdots = (-1)^t tr\ B^t = t$, then $k(B,t) = (-1)^t$.

e. If $tr\ B = tr\ B^2 = \cdots = tr\ B^j = \cdots = trB^t = +n$, then $k(B,t) = +C_n^t$.

f. If $-tr\ B = +tr\ B^2 = \cdots = (-1)^j tr\ B^j = \cdots = (-1)^t tr\ B^t = n$, then $k(B,t) = (-1)^n C_n^t$.

6. For $n \times m$-matrices A_1 and A_2, prove equalities for the scalar coefficients of any order t:

$$k(A_1 \cdot A_2', t) = k(A_1' \cdot A_2, t) = k(A_2 \cdot A_1', t) = k(A_2' \cdot A_1, t).$$

7. Integer-number $n \times n$-matrices generalize the notion of number. They keep also a lot of mysteries and phenomena. Prove the following formulae (they are connected with these characteristic coefficients too).

$$det \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 2 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & t-2 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & t-1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix} = 0. \qquad (1)$$

$$det \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & -(t-2) & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & -(t-1) \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix} = t!. \qquad (2)$$

$$det \begin{bmatrix} t & 1 & 0 & \cdots & 0 & 0 & 0 \\ t & t & 2 & \cdots & 0 & 0 & 0 \\ t & t & t & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ t & t & t & \cdots & t & t-2 & 0 \\ t & t & t & \cdots & t & t & t-1 \\ t & t & t & \cdots & t & t & t \end{bmatrix} = t!. \qquad (3)$$

$$det \begin{bmatrix} -t & 1 & 0 & \cdots & 0 & 0 & 0 \\ +t & -t & 2 & \cdots & 0 & 0 & 0 \\ -t & +t & -t & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (-1)^{t-2}t & (-1)^{t-3}t & (-1)^{t-4}t & \cdots & -t & t-2 & 0 \\ (-1)^{t-1}t & (-1)^{t-2}t & (-1)^{t-3}t & \cdots & +t & -t & t-1 \\ (-1)^{t}t & (-1)^{t-1}t & (-1)^{t-2}t & \cdots & -t & +t & -t \end{bmatrix} = (-1)^t t!. \qquad (4)$$

$$det \begin{bmatrix} n & 1 & 0 & \cdots & 0 & 0 & 0 \\ n & n & 2 & \cdots & 0 & 0 & 0 \\ n & n & n & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n & n & n & \cdots & n & t-2 & 0 \\ n & n & n & \cdots & n & n & t-1 \\ n & n & n & \cdots & n & n & n \end{bmatrix} = t! C_n^t. \qquad (5)$$

$$det \begin{bmatrix} -n & 1 & 0 & \cdots & 0 & 0 & 0 \\ +n & -n & 2 & \cdots & 0 & 0 & 0 \\ -n & +n & -n & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (-1)^{t-2}n & (-1)^{t-3}n & (-1)^{t-4}n & \cdots & -n & t-2 & 0 \\ (-1)^{t-1}n & (-1)^{t-2}n & (-1)^{t-3}n & \cdots & +n & -n & t-1 \\ (-1)^{t}n & (-1)^{t-1}n & (-1)^{t-2}n & \cdots & -n & +n & -n \end{bmatrix} = (-1)^t t! C_n^t. \qquad (6)$$

Note. For (5) and (6) there holds, if $t > n$, then the determinant is 0.

8. For $r \times r$-matrices B and C of rank r, give the matrix interpretation of simple relations:

$$\frac{det\ B_{11}}{det\ B_{21}} = \frac{det\ B_{11}}{det\ (C_{21} \cdot B_{11})} = \frac{det\ (B_{11} \cdot C_{12})}{det\ (C_{21} \cdot B_{11} \cdot C_{12})} = \frac{det\ B_{12}}{det\ B_{22}} \Rightarrow \frac{det\ B_{11}}{det\ B_{21}} = \frac{det\ B_{12}}{det\ B_{22}} \Leftrightarrow$$

$$\Leftrightarrow det\ B_{11} \cdot det\ B_{22} = det\ B_{12} \cdot det\ B_{21}.$$

For example, with the use of this relation, infer exact formula for the spherically orthogonal quasi-inverse matrix A^+ in sect.2.5 through elements a_{ij} of singular matrix A ($r \leq [m, n]$).

9. For singular matrices determining *planars* or *lineors*, write down in our unified notation all characteristic eigenprojectors, *orthogonal* and *oblique* ones. Their quantities are:

- 8 and 12 for real-number and complex-number square matrices,
- 4 and 6 for real-number and complex-number rectangular matrices,
- 8 for a pair of real-number rectangular matrices,
- 12 for a pair of complex-number rectangular matrices.

Why paired orthogonal and oblique eigenprojectors mutually change their nature under translations from quasi-Euclidean space into pseudo-Euclidean one and vice versa?

Are there any geometric distinctions between orthogonal and symmetric eigenprojectors, oblique and nonsymmetric ones in the spaces with quadratic metrics?

10. In a geometry with its binary space and quadratic metric, a reflector tensor and the mid-reflector of the tensor angle have similar expressions. What is the principal distinction between these notions?

11. For such "circles" and "hyperbolae" draw on computer graphs of the functions $y(x)$:

$$|y|^n + |x|^n = |R|^n, \quad |y|^n - |x|^n = |R|^n, \quad n = 0, 1/4, 1/3, 1/2, 1, 3/2, 2, 3, 4, \infty.$$

Why the value $n = 2$ is chosen just for Euclidean, quasi- and pseudo-Euclidean spaces? Does the parameter n have any geometric sense for affine planes and spaces?

These questions are connected with justification of the Pythagorean Theorem, as well as the quadratic types metrics in Euclidean, quasi-Euclidean and non-Euclidean geometries, the theory of relativity, the Gaussian method of least squares and quadratic regression, etc.

Whether it is possible to consider that the mathematical condition $n = 2$ follows from the nature of our real space and space-time or it is used as an axiom for them?

Give comparative analysis of the *generalized* trigonometric functions for integer $n \geq 1$:

$$y/R = Sin\ \varphi, \ x/R = Cos\ \varphi; \quad y/R = Sinh\ \gamma, \ \ x/R = Cosh\ \gamma;$$

Why angles in quadratic geometries (i. e., Euclidean, quasi-Euclidean, and pseudo-Euclidean), as well as their trigonometric functions have the nature of bivalent tensors?

When the tensors are orthogonal, either spherically, or quasi-Euclidean, or hyperbolically, or pseudo-Euclidean, and when they are affine ones?

What kinds of invariants and quasi-invariants take place for functions of spherical and hyperbolic angles? What distinction is between invariants and quasi-invariants? What distinction is between spherical and hyperbolic ones?

How a choice of $n = 2$ for the relativistic space-time is connected with Einsteinian physical definition of events' simultaneity?

12. In the process of construction and development of fundamental and applications of the subject "Tensor Trigonometry", we revealed in parallel some very brief and clear infers of some classic theorems, corollaries and connections between as if different classic concepts.

Connect by general inequality all classic means of positive numbers filling "blind spots".

Give one-line infer of the classic Hamilton-Cayley Theorem.

Give one-line infer of the classic Kronecker–Capelli Theorem.

Connect the cosine and sine general inequalities for squared and rectangular matrices with the classic algebraic Inequalities of Cauchy and Hadamard. Give the single condition of the first former's intersection with the classic algebraic Inequality by Hermann Weyl.

Connect by one-line simplest trigonometric relations the Harriot's excess (from 1603), the Lambert's defect (from 1763) in spherical and hyperbolic right triangles and the Thomas precession (from 1926) in STR. Give the direct connection of the latter and the Coriolis relativistic acceleration in the rested and rotated bases.

13. The *sine-tangent analogy* leads to the hyperbolic analog ω of spherical number $\pi/4$: $\sinh\omega = 1 = \tan\pi/4 \Rightarrow \omega = \text{arsinh } 1 \approx 0.881$ rad; $\pi/4 = \arctan\ 1 \approx 0.785$ rad. Moreover:

$$\pi/4 = \arctan\ 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots \quad \text{(the Leibnitz series)},$$
$$\omega = \text{arsinh}\ 1 = 1 - \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{1\cdot 3}{2\cdot 4} - \frac{1}{7}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \cdots + \frac{(-1)^n}{2n+1}\cdot\frac{(2n-1)!!}{(2n)!!} + \cdots .$$

Why ω as well as $\pi/4$ is a transcendental number? What is the geometric sense of ω?

14. What common geometric feature do have – the circle and sphere, the equilateral hyperbola and hyperboloids, the catenary and catenoids, the tractrix and tractricoids with the Beltrami pseudosphere? Why a tractrix is a hyperbolic analog of a one-step cycloid?

How do a quadrohyperbola in a pseudoplane lead isomorphically to the emergence of four catenaries and tractrices in their Special quasiplanes (two time-like and two space-like) with common determining parameter R? Give bond of their hyperbolic and spherical angles.

Describe geometrically and by equations catenoids obtained with rotation of these parametric double time-like and space-like catenaries around the single normal axis.

Describe geometrically and by equations tractricoids obtained with rotation of these parametric double space-like and time-like tractrices around the single normal axis.

Why a catenary (evolute) and a tractrix (involute) are connected trigonometrically by the countervariant spherical-hyperbolic specific analogy and as time-like and space-like curves?

What is a main distinction in 1-st metric forms of a Beltrami pseudosphere, hyperboloids I and II and a hyperspheroid given in their quasi-Euclidean and pseudo-Euclidean spaces? Why all they are parametric? Which of them are "perfect surfaces" and why it is?

15. Which roles do play the angles γ and υ in pseudo-Euclidean and in non-Euclidean geometries and in theory of relativity? How they are connected to each other and correspond to the purely spherical and countervariant Lobachevsky parallel angle Π?

How do the angles of orthospherical rotation θ (as scalar) and Θ (as tensor) appear in non-collinear principal motions (1), in 1-st metric forms (2), in Thomas precession (3), in angular deviations inside concave closed figures from geodesic segments (4), in astronomical data (5)? Give simplest trigonometric explanation of the induced shifting and precession.

16. What tensor trigonometric distinctions does exist in the mathematical description and interpretation of these well-known relativistic effects: Minkowski dilation of time and Lorentzian contraction of extent? Describe concomitant to them other relativistic effects.

17. What does the *mathematical principle of relativity* in some geometries consist in? How does it lead to the *physical Principle of Relativity* in the Nature?

18. Which kinds of curvatures do take place for world lines in Minkowskian space–time? How do they correspond trigonometrically to main types of physical movement of a particle with its kinematic and dynamic characteristics?

19. What distinctions does exist between the classical differential theory of regular curves by Frenet–Serret in the usual 3D Euclidean space and the differential trigonometric theory of regular curves in the 3D and 4D quasi-Euclidean binary spaces? Why the latters may be realized with two different trihedrons and one tetrahedron? How easy and correctly to construct the trihedron of screwed motions in the tensor trigonometric form?

20. Describe the trigonometric tensor of motion in space-time $\langle\mathcal{P}^{3+1}\rangle$ and its isomorphic bond with the physical tensor of momentum-energy. How does it lead to the 4D pseudo-Euclidean Pythagorean Theorem of three momenta and to the Law of momentum–energy conservation in insulated systems? What is it a gravitational cosine? How and when does it manifest in GR effects?

21. Explain: why the interpretation of 4D observed space-time, i. e., either it is real one (as positivist point of view) or it is mapping of real one lensed by gravitation, is associated with the adoption or not of the Law of momentum–energy conservation in the Universe?

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Scientific Publication

Ninúl Anatoly Sergeevich

"TENSOR TRIGONOMETRY"

The third edition (in English)

ISBN-13: 978-5-89155-429-0 (for this paper book)

(Its electronic form has ISBN-13: 978-5-89155-430-6)

Layout in La TEX by *Ninul A. S.*

Design of the color cover by *Lunin A. S.*

TEX format B5: $70 \times 100/16$ (176×250 mm)

ISO 216 B5 (GOST 5773-90)

with hard cover of the type "7BC", GOST 22240-76

Издательство ООО "Физматкнига"

Publisher "Fizmatkniga Ltd"

Гл. редактор Розанов Алексей Константинович

Ch. editor Rosanov Alexey Konstantinovich

Адрес: 141701, Россия, Московская обл., г. Долгопрудный, ул. Первомайская, д. 5.

Address: 141701, Russia, Moscow region, Dolgoprudny city, Pervomayskaya st., h. 5.

Tel. +7-499-3905138, e-mail: fizmatkniga@mail.ru

Signed to print 17.11.2024.

Printed in the typography of Publisher "Triumph" 31.12.2024.

Адрес: 141701, Россия, Москва, ул. Михайловская, д. 63В, строение 3.

Address: 141701 Russia, Moscow, st. Mikhalkovskaya, h. 63B, building 3.

Tel. +7-499-4040781, e-mail: books@triumph.ru , izd.triumph@gmail.com

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"...The manuscript contains many of fresh ideas, which open new posibilities! ..." -
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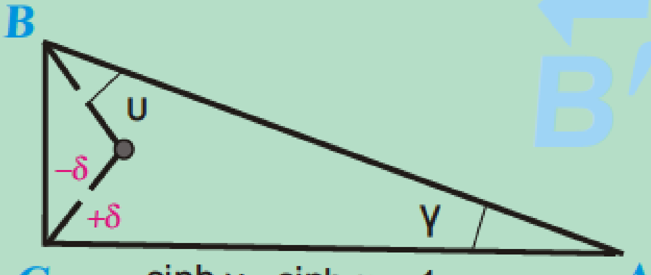

$\sinh\gamma \cdot \sinh\upsilon = 1$

(with Einsteinian rays in exterior right triangle)